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ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF HIGHER-ORDER INTEGRO-DYNAMIC EQUATIONS

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Abstract. In this paper, we establish some new criteria on the asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations on time scales.

Keywords: dynamic equations, time scales, nonoscillation, asymptotics.

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1. INTRODUCTION

In this paper, we are concerned with the asymptotic behavior of nonoscillatory solutions of the higher-order integro-dynamic equation on time scales

$$x^{\Delta^n}(t) + \int_0^t a(t,s)F(s,x(s))\Delta s = 0. \tag{1.1}$$

We take $\mathbb{T} \subseteq \mathbb{R}$ to be an arbitrary time scale with $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Whenever we write $t \geq s$, we mean $t \in [s,\infty) \cap \mathbb{T}$. We assume throughout that:

(H_1) $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is rd-continuous such that $a(t,s) \geq 0$ for $t > s$ and

$$\sup_{t \geq T} \int_0^T a(t,s)\Delta s =: k_T < \infty \quad \text{for all} \quad T \geq 0; \tag{1.2}$$

(H_2) $F : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $f_1, f_2 : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$, such that $F(t,x) = f_1(t,x) - f_2(t,x)$ for $t \geq 0$;

(H_3) there exist constants β and γ of ratios of positive odd integers and $p_i \in \mathrm{C_{rd}}(\mathbb{T},(0,\infty))$, $i \in \{1,2\}$, such that

$$f_1(t,x) \geq p_1(t)x^\beta \quad \text{and} \quad f_2(t,x) \leq p_2(t)x^\gamma \quad \text{for} \quad x>0 \quad \text{and} \quad t \geq 0,$$

$$f_1(t,x) \leq p_1(t)x^\beta \quad \text{and} \quad f_2(t,x) \geq p_2(t)x^\gamma \quad \text{for} \quad x<0 \quad \text{and} \quad t \geq 0.$$

We only consider those solutions of equation (1.1) which are nontrivial and differentiable on $[0,\infty)$. The term solution henceforth applies to such solutions of equation (1.1). A solution x of equation (1.1) is said to be oscillatory if for every $t_0>0$, we have $\inf_{t\geq t_0} x(t) < 0 < \sup_{t \geq t_0} x(t)$ and nonoscillatory otherwise. Dynamic equations on time scales are fairly new objects of study and for the general basic ideas and background, we refer to [1,2].

Oscillation results for integral equations of Volterra type are scant and only a few references exist on this subject. Related studies can be found in [4,6–8]. To the best of our knowledge, there appear to be no such results on the asymptotic behavior of nonoscillatory solutions of equations (1.1). Our aim here is to initiate such a study by establishing some new criteria for the asymptotic behavior of nonoscillatory solutions of equation (1.1) and some related equations.

2. AUXILIARY RESULTS

We shall employ the following auxiliary results.

Lemma 2.1 ([3])**.** *If $X, Y \geq 0$, then*

$$X^\lambda + (\lambda-1)Y^\lambda - \lambda X Y^{\lambda-1} \geq 0 \quad \textit{for} \quad \lambda > 1 \tag{2.1}$$

and

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda X Y^{\lambda-1} \leq 0 \quad \textit{for} \quad \lambda < 1, \tag{2.2}$$

and equality holds if and only if $X=Y$.

Lemma 2.2 ([5, Corollary 1])**.** *Assume that $n \in \mathbb{N}$, $s,t \in \mathbb{T}$, and $f \in \mathrm{C_{rd}}(\mathbb{T},\mathbb{R})$. Then*

$$\int_s^t \int_{\eta_n}^t \cdots \int_{\eta_2}^t f(\eta_1)\Delta\eta_1\Delta\eta_2\cdots\Delta\eta_n = (-1)^{n-1}\int_s^t h_{n-1}(s,\sigma(\eta))f(\eta)\Delta\eta.$$

Remark 2.1. Under the conditions of Lemma 2.2, we may reverse all occurring integrals to obtain

$$\int_t^s \int_t^{\eta_n} \cdots \int_t^{\eta_2} f(\eta_1)\Delta\eta_1\Delta\eta_2\cdots\Delta\eta_n = \int_t^s h_{n-1}(s,\sigma(\eta))f(\eta)\Delta\eta$$

and then replace t by t_0 and s by t to arrive at

$$\int_{t_0}^{t}\int_{t_0}^{\eta_n}\cdots\int_{t_0}^{\eta_2} f(\eta_1)\Delta\eta_1\Delta\eta_2\cdots\Delta\eta_n = \int_{t_0}^{t} h_{n-1}(t,\sigma(\eta))f(\eta)\Delta\eta, \tag{2.3}$$

which is the formula that will be needed in the proofs of our main results in Section 3 below.

In Lemma 2.2 above, the h_n stand for the Taylor monomials (see [1, Section 1.6]) which are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_{s}^{t} h_n(\tau,s)\Delta\tau \quad \text{for} \quad t,s\in\mathbb{T} \quad \text{and} \quad n\in\mathbb{N}.$$

It follows that $h_1(t,s) = t - s$ for any time scale, but simple formulas, in general, do not hold for $n \geq 2$. We define

$$H_n(t) = h_0(t,0) + h_1(t,0) + \ldots + h_n(t,0). \tag{2.4}$$

Remark 2.2. Note that the properties of the Taylor monomials imply that

$$h_0(t,t_0) + h_1(t,t_0) + \ldots + h_n(t,t_0) \leq H_n(t) \quad \text{for all} \quad t_0 \geq 0. \tag{2.5}$$

3. MAIN RESULTS

In this section, we give the following main results.

Theorem 3.1. *Let conditions* (H$_1$)–(H$_3$) *hold with* $\beta > 1$, $\gamma = 1$ *and suppose*

$$\lim_{t\to\infty} \frac{1}{H_n(t)} \int_{t_0}^{t} h_{n-1}(t,\sigma(u)) \int_{t_0}^{u} a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\Delta u < \infty \tag{3.1}$$

for all $t_0 \geq 0$. *If* x *is a nonoscillatory solution of equation* (1.1), *then*

$$x(t) = \mathrm{O}\left(H_n(t)\right) \quad \textit{as} \quad t\to\infty. \tag{3.2}$$

Proof. Let x be a nonoscillatory solution of equation (1.1). Hence x is either eventually positive or x is eventually negative.

First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H$_2$) and (H$_3$) with $\beta > 1$ and $\gamma = 1$ in equation (1.1), we have

$$x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s)\left[p_2(s)x(s) - p_1(s)x^{\beta}(s)\right]\Delta s - \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s \tag{3.3}$$

for $t \geq t_0$. Let

$$m := \max_{0 \leq t \leq t_0} |F(t, x(t))| < \infty.$$

By assumption (H_1), we have

$$\left| -\int_0^{t_0} a(t,s) F(s, x(s)) \Delta s \right| \leq \int_0^{t_0} a(t,s) |F(s, x(s))| \Delta s \leq$$
$$\leq m \int_0^{t_0} a(t,s) \Delta s \leq m k_T =: b$$

for all $t \geq t_0$. Hence from (3.3), we get

$$x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s) \left[p_2(s) x(s) - p_1(s) x^{\beta}(s) \right] \Delta s + b \quad \text{for} \quad t \geq t_0. \tag{3.4}$$

By applying (2.1) with

$$\lambda = \beta, \quad X = p_1^{\frac{1}{\beta}}(t) x(t), \quad Y = \left(\frac{1}{\beta} p_2(t) p_1^{-\frac{1}{\beta}}(t) \right)^{\frac{1}{\beta - 1}},$$

we obtain

$$p_2(t) x(t) - p_1(t) x^{\beta}(t) \leq (\beta - 1) \beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(t) p_2^{\frac{\beta}{\beta-1}}(t) \quad \text{for} \quad t \geq t_0. \tag{3.5}$$

Using (3.5) in (3.4), we find

$$x^{\Delta^n}(t) \leq A(t) + b \quad \text{for} \quad t \geq t_0, \tag{3.6}$$

where

$$A(t) = (\beta - 1) \beta^{\frac{\beta}{1-\beta}} \int_{t_0}^{t} a(t,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) \Delta s.$$

Integrating (3.6) n times from t_0 to t and then using (2.3), we obtain

$$\begin{aligned} x(t) &\leq \int_{t_0}^{t} \int_{t_0}^{\xi_n} \cdots \int_{t_0}^{\xi_2} A(\xi_1) \Delta \xi_1 \cdots \Delta \xi_n + b h_n(t, t_0) + \sum_{k=0}^{n-1} x^{\Delta^k}(t_0) h_k(t, t_0) = \\ &= \int_{t_0}^{t} h_{n-1}(t, \sigma(u)) A(u) \Delta u + b h_n(t, t_0) + \sum_{k=0}^{n-1} x^{\Delta^k}(t_0) h_k(t, t_0). \end{aligned} \tag{3.7}$$

From (3.7), using (2.5), we get

$$|x(t)| \leq \int_{t_0}^{t} h_{n-1}(t, \sigma(u)) A(u) \Delta u + c H_n(t), \tag{3.8}$$

where

$$c := \max\left\{b, \max_{0\le k\le n-1}\left|x^{\Delta^k}(t_0)\right|\right\}.$$

Dividing (3.8) by $H_n(t)$ and using (3.1) shows that (3.2) is valid.

Now assume x is eventually negative, say $x(t) < 0$ for $t \ge t_0$ for some $t_0 \ge 0$. Using conditions (H_2) and (H_3) with $\beta > 1$ and $\gamma = 1$ in equation (1.1), we now have

$$x^{\Delta^n}(t) \ge \int_{t_0}^{t} a(t,s)\left[p_2(s)x(s) - p_1(s)x^{\beta}(s)\right]\Delta s - \int_0^{t_0} a(t,s)F(s,x(s))\Delta s \tag{3.9}$$

for $t \ge t_0$. With m defined as before and by assumption (H_1), we have

$$\left|\int_0^{t_0} a(t,s)F(s,x(s))\Delta s\right| \le \int_0^{t_0} a(t,s)|F(s,x(s))|\Delta s \le m\int_0^{t_0} a(t,s)\Delta s \le mk_T =: b$$

for all $t \ge t_0$. Hence from (3.9), we get

$$x^{\Delta^n}(t) \ge \int_{t_0}^{t} a(t,s)\left[p_2(s)x(s) - p_1(s)x^{\beta}(s)\right]\Delta s - b \quad \text{for} \quad t \ge t_0. \tag{3.10}$$

By applying (2.1) with

$$\lambda = \beta, \quad X = -p_1^{\frac{1}{\beta}}(t)x(t), \quad Y = \left(\frac{1}{\beta}p_2(t)p_1^{-\frac{1}{\beta}}(t)\right)^{\frac{1}{\beta-1}},$$

we obtain

$$p_2(t)x(t) - p_1(t)x^{\beta}(t) \ge -(\beta-1)\beta^{\frac{\beta}{1-\beta}}p_1^{\frac{1}{1-\beta}}(t)p_2^{\frac{\beta}{\beta-1}}(t) \quad \text{for} \quad t \ge t_0. \tag{3.11}$$

Using (3.11) in (3.10), we find

$$x^{\Delta^n}(t) \ge -A(t) - b \quad \text{for} \quad t \ge t_0, \tag{3.12}$$

where A is defined as before. Integrating (3.12) n times from t_0 to t and then using (2.3), we obtain

$$\begin{aligned} x(t) &\ge -\int_{t_0}^{t}\int_{t_0}^{\xi_n}\cdots\int_{t_0}^{\xi_2} A(\xi_1)\Delta\xi_1\cdots\Delta\xi_n - bh_n(t,t_0) - \sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0) = \\ &= -\left(\int_{t_0}^{t} h_{n-1}(t,\sigma(u))A(u)\Delta u + bh_n(t,t_0) + \sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0)\right). \end{aligned} \tag{3.13}$$

From (2.5), we get

$$x(t) \geq -\left[\int_{t_0}^{t} h_{n-1}(t,\sigma(u))A(u)\Delta u + cH_n(t)\right],$$

where c is defined as before. This implies (3.8), and thus (3.2) follows as before. □

Theorem 3.2. *Let conditions* (H_1)–(H_3) *hold with* $\beta = 1$, $\gamma < 1$ *and suppose*

$$\lim_{t\to\infty} \frac{1}{H_n(t)} \int_{t_0}^{t} h_{n-1}(t,\sigma(u)) \int_{t_0}^{u} a(u,s) p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s)\Delta s\Delta u < \infty \tag{3.14}$$

for all $t_0 \geq 0$. *If* x *is a nonoscillatory solution of equation* (1.1)*, then* (3.2) *holds.*

Proof. Let x be a nonoscillatory solution of equation (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H_2) and (H_3) with $\beta = 1$ and $\gamma < 1$ in equation (1.1), we have

$$x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s)\left[p_2(s)x^{\gamma}(s) - p_1(s)x(s)\right]\Delta s - \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s$$

for $t \geq t_0$. Hence

$$x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s)\left[p_2(s)x^{\gamma}(s) - p_1(s)x(s)\right]\Delta s + b \quad \text{for} \quad t \geq t_0, \tag{3.15}$$

where b is defined as in the proof of Theorem 3.1. By applying (2.2) with

$$\lambda = \gamma, \quad X = p_2^{\frac{1}{\gamma}}(t)x(t), \quad Y = \left(\frac{1}{\gamma}p_1(t)p_2^{-\frac{1}{\gamma}}(t)\right)^{\frac{1}{\gamma-1}},$$

we obtain

$$p_2(t)x^{\gamma}(t) - p_1(t)x(t) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} p_1^{\frac{\gamma}{\gamma-1}}(t) p_2^{\frac{1}{1-\gamma}}(t) \quad \text{for} \quad t \geq t_0. \tag{3.16}$$

Using (3.16) in (3.15), we find

$$x^{\Delta^n}(t) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_0}^{t} a(t,s) p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s)\Delta s + b.$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence is omitted. □

Finally, we present the following result with different nonlinearities, i.e., with $\beta > 1$ and $\gamma < 1$.

Theorem 3.3. *Let conditions* (H_1)–(H_3) *hold with* $\beta > 1$, $\gamma < 1$ *and suppose that there exists a positive rd-continuous function* $\xi : \mathbb{T} \to \mathbb{R}$ *such that*

$$\lim_{t\to\infty} \frac{1}{H_n(t)} \int_{t_0}^{t} h_{n-1}(t, \sigma(u)) \int_{t_0}^{u} a(u, s) \Big[c_1 \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) + + c_2 \xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \Big] \Delta s \Delta u < \infty \tag{3.17}$$

for all $t_0 \geq 0$, *where* $c_1 = (\beta - 1)\beta^{\frac{\beta}{1-\beta}}$ *and* $c_2 = (1 - \gamma)\gamma^{\frac{\gamma}{1-\gamma}}$. *If* x *is a nonoscillatory solution of equation* (1.1), *then* (3.2) *holds.*

Proof. Let x be a nonoscillatory solution of equation (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H_2) and (H_3) in equation (1.1), we obtain

$$\begin{aligned} x^{\Delta^n}(t) \leq & \int_{t_0}^{t} a(t, s) \left[\xi(s)x(s) - p_1(s)x^{\beta}(s) \right] \Delta s + \\ & + \int_{t_0}^{t} a(t, s) \left[p_2(s)x^{\gamma}(s) - \xi(s)x(s) \right] \Delta s - \int_{0}^{t_0} a(t, s)F(s, x(s))\Delta s \quad \text{for } t \geq t_0. \end{aligned} \tag{3.18}$$

As in the proofs of Theorems 3.1 and 3.2, one can easily find

$$\begin{aligned} x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t, s) \Big[& (\beta - 1)\beta^{\frac{\beta}{1-\beta}} \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) + \\ & + (1 - \gamma)\gamma^{\frac{\gamma}{1-\gamma}} \xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \Big] \Delta s + b. \end{aligned} \tag{3.19}$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence is omitted. □

4. REMARKS AND EXTENSIONS

We conclude by presenting several remarks and extensions of the results given in Section 3.

Remark 4.1. The results presented in this paper are new for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Let us therefore rewrite the crucial condition in Theorem 3.1 (this can be done similarly for Theorem 3.2 and Theorem 3.3) for the two special time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = Z$. If $\mathbb{T} = \mathbb{R}$, then (1.1) becomes

$$x^{(n)}(t) + \int_{0}^{t} a(t, s)F(s, x(s))\mathrm{d}s = 0$$

and condition (3.1) turns into

$$\lim_{t\to\infty}\frac{1}{\sum_{k=0}^{n}\frac{t^k}{k!}}\int_{t_0}^{t}\frac{(t-u)^{n-1}}{(n-1)!}\int_{t_0}^{u}a(u,s)p_1^{\frac{1}{1-\beta}}(s)p_2^{\frac{\beta}{\beta-1}}(s)\mathrm{d}s\mathrm{d}u<\infty.$$

If $\mathbb{T}=\mathbb{Z}$, then (1.1) becomes

$$\Delta^n x(t)+\sum_{s=0}^{t-1}a(t,s)F(s,x(s))=0$$

and condition (3.1) turns into

$$\lim_{t\to\infty}\frac{1}{\sum_{k=0}^{n}\frac{t^{\underline{k}}}{k!}}\sum_{u=t_0}^{t-1}\frac{(t-u-1)^{\underline{n-1}}}{(n-1)!}\sum_{s=t_0}^{u-1}a(u,s)p_1^{\frac{1}{1-\beta}}(s)p_2^{\frac{\beta}{\beta-1}}(s)<\infty.$$

Remark 4.2. The results of this paper are presented in a form which is essentially new for equation (1.1) with different nonlinearities.

Remark 4.3. The results of this paper will remain the same if we replace (1.2) of assumption (H_1) by

$$\sup_{0\le s\le T\le t}a(t,s)=:K_T<\infty\quad\text{for all}\quad T\ge 0,$$

since then (1.2) is satisfied with $k_T=TK_T$.

Remark 4.4. The results of this paper will remain the same if we replace (1.2) of assumption (H_1) by the assumption that there exist rd-continuous functions $\alpha,\beta:\mathbb{T}\to\mathbb{R}^+$ such that $a(t,s)<\alpha(t)\beta(s)$ for all $t\ge s$,

$$\sup_{t\ge 0}\alpha(t)=:K_\alpha<\infty;$$

and

$$\sup_{t\ge 0}\int_0^t\beta(s)\Delta s=:K_\beta<\infty,$$

since then (1.2) is satisfied with $k_T=K_\alpha K_\beta$.

Remark 4.5. If we skip (1.2) of assumption (H_1) and pick $t_0=0$ in Theorem 3.1, Theorem 3.2 and Theorem 3.3, then the results of this paper will remain true for an eventually positive and eventually negative solution.

Remark 4.6. The techniques described in this paper can be employed to Volterra integral equations on time scales of the form

$$x(t)+\int_0^t a(t,s)F(s,x(s))\Delta s=0. \tag{4.1}$$

As an example illustrating Remark 4.5 and 4.6, we reformulate Theorem 3.1 as follows.

Theorem 4.7. *Let conditions* (H$_1$)–(H$_3$) *hold with* $\beta > 1$, $\gamma = 1$ *and assume*

$$\int_0^\infty a(t,s) p_2^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) \Delta s < \infty.$$

Then any positive solution of equation (4.1) *is bounded.*

Remark 4.8. The results of this paper can be extended easily to delay integro-dynamic equations of the form

$$x^{\Delta^n}(t) + \int_0^t a(t,s) F(s, x(g(s))) \Delta s = 0,$$

where $g : \mathbb{T} \to \mathbb{T}$ is rd-continuous such that $g(t) \leq t$ and $g^\Delta(t) \geq 0$ for $t \geq 0$ and $\lim_{t\to\infty} g(t) = \infty$.

Remark 4.9. We note that we can reformulate the obtained results for the time scales $\mathbb{T} = \mathbb{R}$ (the continuous case), $\mathbb{T} = \mathbb{Z}$ (the discrete case), $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ (the quantum calculus case), $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = \mathbb{N}_0^2$ etc.; see [1, 2].

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