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THE BASIS PROPERTY OF EIGENFUNCTIONS IN THE PROBLEM OF A NONHOMOGENEOUS DAMPED STRING

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Abstract. The equation which describes the small vibrations of a nonhomogeneous damped string can be rewritten as an abstract Cauchy problem for the densely defined closed operator iA. We prove that the set of root vectors of the operator A forms a basis of subspaces in a certain Hilbert space H. Furthermore, we give the rate of convergence for the decomposition with respect to this basis. In the second main result we show that with additional assumptions the set of root vectors of the operator A is a Riesz basis for H.

Keywords: nonhomogeneous damped string, Hilbert space, Riesz basis, modulus of continuity, basis with parentheses, basis of subspaces, string equation.

Mathematics Subject Classification: 34L10, 34B08.

1. INTRODUCTION

We focus on a nonhomogeneous one-dimensional string of length one. Its density is denoted by $\rho\colon [0,1] \to (0,\infty)$, modulus of elasticity by $p\colon [0,1] \to (0,\infty)$. We assume the presence of a damping coefficient $2d$ and a potential q, where $d\colon [0,1] \to \mathbb{R}$ and $q\colon [0,1] \to \mathbb{R}$. In our case the string is fixed at the left end and the right one is damped with coefficient $h \in \mathbb{C}$.

Set $v := v(x,t)$ to be a vertical position of a string on the interval $[0,1]$ in time $t \in [0,\infty)$. Then small vibrations of our string are described by the string equation

$$v_{tt}(x,t) - \rho(x)^{-1}\big(p(x)v_x(x,t)\big)_x + 2d(x)v_t(x,t) + q(x)v(x,t) = 0 \tag{1.1}$$

with the boundary conditions

$$v(0,t) = 0, \quad v_x(1,t) + hv_t(1,t) = 0,$$

and initial conditions

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x). \tag{1.2}$$

Here v_0 and v_1 are the initial position and velocity of the string, respectively. v is considered as a function $v\colon [0,1] \times [0,\infty) \to \mathbb{C}$.

In what follows, the symbol $W_p^1[0,1]$, $p \geq 1$, stands for the Sobolev space with the first derivative in $L_p[0,1]$. For convenience, we introduce the notation

$$\widehat{W}_p^1[0,1] := \{y \in W_p^1[0,1] : \ y(0) = 0\},$$

where the scalar product $\langle \cdot, \cdot \rangle_1$ on $\widehat{W}_2^1[0,1]$ is given by

$$\langle u_1, u_2 \rangle_1 := \int_0^1 u_1'(x)\overline{u}_2'(x)dx, \ \ u_j \in \widehat{W}_2^1[0,1], \ j = 1,2.$$

We assume that

$$\rho \in W_2^1[0,1] \ \text{ and } \ 0 < m_\rho \leq \rho(x), \quad x \in [0,1], \tag{1.3}$$

$$p \in W_2^1[0,1] \ \text{ and } \ 0 < m_p \leq p(x), \quad x \in [0,1], \tag{1.4}$$

where m_ρ, $m_p > 0$ are constants independent of x, and

$$d \in L_2[0,1], \tag{1.5}$$

$$q \in L_2[0,1]. \tag{1.6}$$

As $L_2([0,1];\rho)$ we understand the space $L_2[0,1]$ equipped with the norm induced by the scalar product

$$\langle v_1, v_2 \rangle_2 := \int_0^1 \rho(x)v_1(x)\overline{v}_2(x)dx, \ \ v_j \in L_2([0,1];\rho).$$

When the domain of an arbitrary function is not indicated, it is assumed to be $[0,1]$.

The problem of a damped string can be transformed into an abstract Cauchy problem in a suitable Hilbert space H. We take

$$H := \widehat{W}_2^1[0,1] \times L_2([0,1];\rho),$$

where the scalar product on H is

$$\Big\langle (u_1,u_2),(v_1,v_2) \Big\rangle := \int_0^1 u_1'(x)\overline{v}_1'(x)dx + \int_0^1 \rho(x)u_2(x)\overline{v_2}(x)dx.$$

If $V(t) := [v(\cdot,t), v_t(\cdot,t)]^T$, then the new representation of the problem (1.1)–(1.2) is:

$$\begin{aligned} V'(t) &= iA_hV(t), \quad t > 0,\\ V(0) &= [v_0, v_1]^T, \end{aligned}$$

where the linear operator $A_h\colon \mathcal{D}(A_h) \to H$ is defined by

$$A_h = -i\begin{bmatrix} 0 & I \\ 1/\rho\frac{d}{dx}\big(p\frac{d}{dx}\,\cdot\,\big) - q & -2d \end{bmatrix}$$

and I is the identity operator on $\widehat{W}_2^1[0,1]$. The domain of A_h is

$$\mathcal{D}(A_h) := \big\{(u,v) \in W_2^2[0,1] \times \widehat{W}_2^1[0,1] :\ u(0) = 0,\ u'(1) + hv(1) = 0\big\}.$$

We show in the next section that A_h is densely defined and closed.

If d, $q \geq 0$ and $\operatorname{Re} h \geq 0$ then A_h generates the C_0-semigroup of contractions. See [6] and [7], for the proofs and some estimation for the exponential decay of energy in this problem.

The eigenfunction $(u,v) \in \mathcal{D}(A_h)$ of A_h associated with the eigenvalue μ, satisfies $v = i\mu u$, where u is the solution of the equation:

$$\big(p(x)u'(x)\big)' + \Big(\mu^2 - 2i\mu d(x) - q(x)\Big)\rho(x)u(x) = 0, \quad x \in [0,1], \tag{1.7}$$

with the boundary conditions

$$u(0) = 0, \quad U[u](\mu) := u'(1) + i\mu hu(1) = 0.$$

Our first aim is to study spectral properties of A_h. We prove that under assumptions (1.3)–(1.6) and

$$\sqrt{\frac{\rho(1)}{p(1)}} \neq |h|, \tag{1.8}$$

all eigenvalues of A_h lie in a finite stripe $|\operatorname{Im}\mu| < a$, $a > 0$ and almost all eigenvalues are simple. The main result states that for every $w \in H$ the following spectral decomposition is true

$$w = \sum_{n=0}^{\infty} x_n, \quad x_n \in H_n,\ n \geq 1, \tag{1.9}$$

where H_0 is the finite-dimensional space spanned by eigen- and associated functions and H_n, $n \geq 1$ are two-dimensional spaces spanned only by eigenfunctions associated with simple eigenvalues. It means that the set of root vectors (i.e. eigen- and associated functions) of the operator A_h forms a basis of subspaces for H.

Furthermore, with the additional claim

$$\int_0^1 \frac{\omega_1^2(\rho',\tau)}{\tau^2}\,d\tau < \infty, \quad \int_0^1 \frac{\omega_1^2(d,\tau)}{\tau^2}\,d\tau < \infty. \tag{1.10}$$

where

$$\omega_1(g,\epsilon) = \sup_{0<\delta\le\epsilon} \|g(\cdot+\delta)-g(\cdot)\|_{L_1[0,1-\delta]}, \quad \epsilon\in(0,1),\ g\in L_1[0,1],$$

is the integral modulus of continuity (see, e.g., [1, Ch. 2, § 7]), we prove the Riesz basis property of the root vectors in H.

These results are generalization of Theorems 6.1 and 6.3 from [5]. There we considered the problem (1.1)–(1.2) with $p=1$, $d=0$, $q=0$. In this article we develop the approach introduced in [5].

The Riesz basis property for problem (1.1)–(1.2) was investigated with the use of many different methods in the series of papers of M. Shubov (see [9] and [10], for instance). It is assumed in [11] that ρ, $p\in W_1^2[0,1]$ and are strictly positive, $d\in W_1^1[0,1]$, $q\in L_\infty[0,1]$ and $\sqrt{\frac{\rho(1)}{p(1)}}\neq|h|$. Our assumptions are weaker. Indeed, for $f\in W_1^1[0,1]$ we have

$$\int_0^1 \frac{\omega_1^2(f,\tau)}{\tau^2}\,d\tau < \infty,$$

since $\omega_1(f,\epsilon)\le\epsilon\|f'\|_{L_1}$. On the other hand one can take a function $\rho(x)=1+x^\alpha$, $1/2<\alpha<1$ which belongs to $W_2^1[0,1]$ but not to $W_1^2[0,1]$. Obviously, the first condition in (1.10) is satisfied.

Furthermore, as far as we are concerned, the explicit rate of convergence in (1.9) in terms of the integral moduli of continuity in this case has not been published yet.

2. ASYMPTOTIC OF EIGENVALUES AND EIGENFUNCTIONS

The resolvent of A_h is determined by the equation

$$\big(A_h-\mu I\big)(u,v)=(f,g),$$

where $(u,v)\in\mathcal{D}(A_h)$ and $(f,g)\in H$. This equation leads to

$$\big(p(x)y'(x)\big)'+\Big(\mu^2-2i\mu d(x)-q(x)\Big)\rho(x)y(x)=F(x,\mu), \tag{2.1}$$

$$u(0)=0, \quad U[u](\mu):=u'(1)+i\mu hu(1)=-ihf(1), \tag{2.2}$$

where

$$F(x,\mu):=\rho(x)\Big[i\big(g(x)+2d(x)f(x)\big)-\mu f(x)\Big]\in L_1[0,1]$$

and

$$v(x) := i\big(f(x) + \mu u(x)\big). \tag{2.3}$$

We are going to use knowledge about the asymptotical behavior of fundamental system of solutions for (2.1). First, we need to introduce the notation

$$\varrho(x) := p(x)\rho(x) \in W_2^1[0,1], \quad b(x) := \sqrt{\frac{\rho(x)}{p(x)}} = \frac{\sqrt{\varrho(x)}}{p(x)} \in W_2^1[0,1].$$

Recall that p and ρ are positive and bounded, hence ϱ and b satisfy

$$0 < m_\varrho \le \varrho(x) \le M_\varrho, \quad 0 < m_b \le b(x) \le M_b, \quad x \in [0,1],$$

where m_ϱ, m_b are positive constants independent of x. We will also use

$$\kappa(f,s) = \omega_1(f, s^{-1}) + s^{-1}\|f\|_{L_1}, \ s > 1$$

and

$$Q_0(x,\mu) := i\mu b(x) + d(x)b(x),$$

$$q_0(x,\mu) := \int_0^x Q_0(\tau,\mu)d\tau, \quad q_1(x,\mu) := \int_x^1 Q_0(\tau,\mu)d\tau, \tag{2.4}$$

$$q(s,t,\mu) := \int_s^t Q_0(\tau,\mu)d\tau, \quad \xi_0(\mu) := \int_0^1 Q_0(\tau,\mu)d\tau.$$

Let r_0, $r_1 \ge 0$. We define

$$\mathbb{C}_\pm(r_0,r_1) := \{\mu \in \mathbb{C};\ |\mu| > r_0,\ \pm \operatorname{Im}\mu > -r_1\},$$

and

$$\mathbb{C}_\pm(r_0) := \mathbb{C}_\pm(r_0, 0).$$

We assume that (1.3)–(1.6) and (1.8) are satisfied. According to [4, Theorem 1] for any $r_1 \ge 0$ there exists $r_0 > 0$ and the fundamental system of solutions $u_1(x,\mu)$, $u_2(x,\mu)$ of equation (1.7), which for any $x \in [0,1]$ is analytical on $\mu \in \mathbb{C}_+(r_0,r_1)$ and admits for $|\mu| \to \infty$ the asymptotical expressions

$$u_1(x,\mu) = \varrho^{-1/4}(x)e^{q_0(x,\mu)}\Big[1 + O\big(\delta(|\mu|)\big)\Big], \tag{2.5}$$

$$p(x)u_1'(x,\mu) = i\mu\varrho^{1/4}(x)e^{q_0(x,\mu)}\Big[1 + O\big(\delta(|\mu|)\big)\Big], \tag{2.6}$$

$$u_2(x,\mu) = \varrho^{-1/4}(x)e^{q_1(x,\mu)}\Big[1 + O\big(\delta(|\mu|)\big)\Big], \tag{2.7}$$

$$p(x)u_2'(x,\mu) = -i\mu\varrho^{1/4}(x)e^{q_1(x,\mu)}\Big[1 + O\big(\delta(|\mu|)\big)\Big], \tag{2.8}$$

where

$$\delta(|\mu|) := \kappa(\varrho', |\mu|) + \kappa(d, |\mu|) + |\mu|^{-1}\|q\|_{L_1}. \tag{2.9}$$

Remark 2.1. The fundamental system of solutions $\widetilde{u}_j(x,\mu)$, $j=1,2$, in $\mathbb{C}_-(r_0,r_1)$ is given by $\widetilde{u}_j(x,\mu) := s_j(x,-\mu)$, $j=1,2$, where $s_j(x,\mu)$ is the solution of

$$\big(p(x)s'(x)\big)' + \Big(\mu^2 + 2i\mu d(x) - q(x)\Big)\rho(x)s(x) = 0, \quad x \in [0,1]. \tag{2.10}$$

We obtain explicit formulas for $\mu \in \mathbb{C}_-(r_0,r_1)$ from (2.5)–(2.8) exchanging μ and d with $-\mu$ and $-d$. Obviously, this system is analytical with respect to $\mu \in \mathbb{C}_-(r_0,r_1)$.

Now we look for solutions of (2.1)–(2.2) in the form

$$u(x,\mu) = C_1u_1(x,\mu) + C_2u_2(x,\mu) + u_0(x,\mu),$$

where

$$u_0(x,\mu) := \frac{u_2(x,\mu)}{w(\mu)}\int_0^x u_1(s,\mu)F(s,\mu)ds + \frac{u_1(x,\mu)}{w(\mu)}\int_x^1 u_2(s,\mu)F(s,\mu)ds,$$

is the particular solution of (2.1). We obtain that

$$\begin{aligned} u(x,\mu) = {} & \frac{u_1(x,\mu)}{\Delta(\mu)}\Big(u_2(0,\mu)\big(ihf(1) + U[u_0](\mu)\big) - u_0(0,\mu)U[u_2](\mu)\Big) \\ & + \frac{u_2(x,\mu)}{\Delta(\mu)}\Big(u_0(0,\mu)U[u_1](\mu) - u_1(0,\mu)\big(ihf(1) + U[u_0](\mu)\big)\Big) + u_0(x,\mu), \end{aligned}$$

where

$$\Delta(\mu) := u_1(0,\mu)U[u_2](\mu) - u_2(0,\mu)U[u_1](\mu).$$

We see that the resolvent exists for μ which is not a zero of Δ. That is why we analyze asymptotical behavior of Δ to investigate spectral properties of operator A_h. Using (2.5)–(2.8) we get

$$\Delta(\mu) = \frac{i\mu}{\sqrt[4]{\varrho(0)\varrho(1)}}\Big(\big(h - b(1)\big)\big[1 + O\big(\delta(|\mu|)\big)\big] - \big(h + b(1)\big)e^{2\xi_0(\mu)}\big[1 + O\big(\delta(|\mu|)\big)\big]\Big). \tag{2.11}$$

Therefore Δ is not identically zero for $\mu \in \mathbb{C}_+(r_0,r_1)$. This implies that the resolvent set of A_h is non-empty, thus A_h is closed.

If $b(1) \neq |h|$ (for $h \in \mathbb{R}$), then the zeroes of Δ in $C_+(r_0,r_1)$ are

$$\mu_{\pm n} = \frac{1}{\mathcal{T}}\left(\pm\pi(n+l) - i\frac{\log|\alpha|}{2} + i\mathcal{N}\right) + \mathrm{O}(\delta(n)), \quad \alpha = \frac{h - b(1)}{h + b(1)}, \quad n \to \infty, \tag{2.12}$$

where

$$\mathcal{T} := \int_0^1 b(\tau)d\tau, \quad \mathcal{N} := \int_0^1 b(\tau)d(\tau)d\tau$$

and $\delta(n) = \delta(|n|)$.

What is more, formula (2.12) shows that the eigenvalues of A_h in $C_+(r_0, r_1)$ lie in a stripe $|\operatorname{Im}\mu| < r$, $r > 0$. Analogously, we can derive the asymptotical behavior of Δ in $C_-(r_0, r_1)$. This leads to the conclusion that the zeroes of Δ located in $C_-(r_0, r_1)$ lie in a stripe too. Therefore to find the asymptotical behavior of eigenvalues of A_h it is sufficient to take appropriate r_1' and find zeroes of (2.11) in $C_+(r_0, r_1')$. Obviously, this leads again to (2.12).

Analogously as in [6], we can prove that A_h is densely defined and due to embedding theorems for Sobolev spaces it has a compact resolvent. Furthermore, the following lemma is true.

Lemma 2.2. *Suppose that conditions* (1.3)–(1.6) *are satisfied and* $b(1) \neq |h|$. *Then the following facts are true:*

(a) *There exists a sequence of positive numbers*

$$R_n := \frac{\pi n}{\mathcal{T}} + O(1), \quad n \in \{0\} \cup \mathbb{N}, \quad n \to \infty,$$

such that on the contours

$$\gamma_n = \{\mu \in \mathbb{C} : \; |\mu| = R_n\}$$

the resolvent of A_h *exists. Furthermore*

$$|\Delta(\mu)| \geq c|\mu|, \quad \mu \in \gamma_n,$$

with a certain constant $c > 0$.

(b) *The spectrum of operator* A_h *is given by*

$$\sigma(A_h) = \{\mu_{0,j}\}_{j=1}^{n_0} \cup \{\mu_n\}_{n=-\infty,\, n\neq 0}^{\infty},$$

for some $n_0 \geq 0$, *such that*

$$|\mu_{0,j}| < |\mu_{\pm n}| < |\mu_{\pm(n+1)}|, \quad j = 1, \ldots, n_0, \quad n \in \mathbb{N}.$$

If $n_0 = 0$, *then the first part of* $\sigma(A_h)$ *is empty. Furthermore, all eigenvalues* $\{\mu_n\}_{n=-\infty,\, n\neq 0}^{\infty}$ *are simple and for an appropriate number* $l \in \mathbb{Z}$ *their asymptotical behavior is described by* (2.12).

Routine calculations reveal that the adjoint of A_h is given by

$$A_h^* = -i \begin{bmatrix} 0 & I \\ 1/\rho \frac{d}{dx}\left(p \frac{d}{dx} \cdot\right) - q & 2d \end{bmatrix}$$

and its domain is

$$\mathcal{D}(A_h^*) = \left\{(u, v) \in W_2^2[0,1] \times \widehat{W}_2^1[0,1] : \; u(0) = 0, \; u'(1) - \overline{h}v(1) = 0\right\}.$$

Using Lemma 2.2 we can describe the behavior of eigenfunctions of A_h and A_h^*. Let y_1 be the solution of (1.7), such that $y_1(0, \mu) = 0$ and $y_1'(0, \mu) = 1$ and $\widetilde{y}_1$ be the solution of (2.10), such that $\widetilde{y}_1(0, \mu) = 0$ and $\widetilde{y}_1'(0, \mu) = 1$.

Corollary 2.3. *The eigenfunctions*

$$Y_n = (y_1(\cdot, \mu_n), i\mu_n y_1(\cdot, \mu_n))$$

of operator A_h *associated with eigenvalues* μ_n *are described by*

$$y_1(x, \mu_n) = \frac{p(0)}{2i\mu_n \sqrt[4]{\varrho(0)\varrho(x)}} \Big(e^{q_0(x,\mu_n)} \big[1 + O\big(\delta_n\big)\big] - e^{-q_0(x,\mu_n)} \big[1 + O\big(\delta_n\big)\big] \Big) \tag{2.13}$$

and

$$p(x)y_1'(x, \mu_n) = \frac{p(0)}{2} \sqrt[4]{\frac{\varrho(x)}{\varrho(0)}} \Big(e^{q_0(x,\mu_n)} \big[1 + O\big(\delta_n\big)\big] + e^{-q_0(x,\mu_n))} \big[1 + O\big(\delta_n\big)\big] \Big), \tag{2.14}$$

where $n \to \pm\infty$ *and* $\delta_n := \delta(|n|)$, $n \in \mathbb{Z}$.
The eigenfunctions $\widetilde{Y}_n$ *of operator* A_h^* *associated with* $\overline{\mu}_n$ *are given by* $\widetilde{Y}_n := (\widetilde{y}_1, i\overline{\mu}_n\widetilde{y}_1)$, *where the behavior of* $\overline{\mu}_n$ *is described by* (2.12) *and*

$$\widetilde{y}_1(x, \overline{\mu}_n) = \frac{p(0)}{2i\overline{\mu}_n \sqrt[4]{\varrho(0)\varrho(x)}} \Big(e^{q_0^*(x,\overline{\mu}_n)} \big[1 + O\big(\delta_n\big)\big] - e^{-q_0^*(x,\overline{\mu}_n)} \big[1 + O\big(\delta_n\big)\big] \Big), \tag{2.15}$$

$$p(x)\widetilde{y}_1'(x, \overline{\mu}_n) = \frac{p(0)}{2} \sqrt[4]{\frac{\varrho(x)}{\varrho(0)}} \Big(e^{q_0^*(x,\overline{\mu}_n)} \big[1 + O\big(\delta_n\big)\big] + e^{-q_0^*(x,\overline{\mu}_n))} \big[1 + O\big(\delta_n\big)\big] \Big) \tag{2.16}$$

and

$$q_0^*(x, \mu) := i\mu \int\limits_0^x b(\tau)d\tau - \int\limits_0^x d(\tau)b(\tau)d\tau.$$

What is more, using this corollary we can obtain that Y_n and $\widetilde{Y}_n$ are almost normalized and asymptotically biorthogonal. This fact will be useful in the proof of the Riesz basis property.

Lemma 2.4. *For* $Y_n := (y_1, i\mu_n y_1)$ *and* $\widetilde{Y}_n := (\widetilde{y}_1, i\overline{\mu}_n\widetilde{y}_1)$ *we have*

$$\langle Y_n, \widetilde{Y}_n \rangle_H = a\Big(1 + O\big(\delta_n\big)\Big), \quad n \to \infty,$$

where $a > 0$.

3. MAIN RESULT

Recall that the compact resolvent $R(A_h, \mu)$ of the operator A_h exists on the contours γ_n. This is why we can define finite dimensional Riesz projectors

$$\mathcal{P}_n := -\frac{1}{2\pi i} \int\limits_{\gamma_n} R(A_h, \mu) d\mu, \quad n \in \{0\} \cup \mathbb{N}.$$

In particular, we will use

$$\widetilde{\mathcal{P}}_0 := \mathcal{P}_0, \ \ \widetilde{\mathcal{P}}_n := \mathcal{P}_n - \mathcal{P}_{n-1}, \quad n \in \mathbb{N}.$$

According to Lemma 2.2 we can choose R_n, such that the subspaces $\widetilde{\mathcal{P}}_n H$, $n = 1, 2, \ldots$ are spanned by two eigenfunctions associated with μ_n and μ_{-n} and located in the ring $R_{n-1} < |\mu| < R_n$. Then $\widetilde{\mathcal{P}}_0 H$ is finite-dimensional and spanned by a finite number of eigen- and associated functions.

Our main aim is to prove that for every $w \in H$ the unique decomposition

$$w = \sum_{j=0}^{\infty} \widetilde{\mathcal{P}}_j w,$$

is true, hence the root vectors of A_h form a basis of subspaces $H_n := \widetilde{\mathcal{P}}_n H$, where $n = 0, 1, 2, \ldots$. Note that $\mathcal{P}_n = \sum_{j=0}^{n} \widetilde{\mathcal{P}}_j$, thus it is sufficient to show that for every $w \in H$ there holds

$$\lim_{n\to\infty} \|\mathcal{P}_n w - w\|_H = 0.$$

What is more, we want to investigate the rate of convergence for a decomposition $w = \sum_{j=0}^{\infty} \widetilde{\mathcal{P}}_j w$. We introduce the modulus of continuity

$$\widetilde{\omega}_2(f, \epsilon) := \sup_{|\delta|\le\epsilon} \left\{ \int_0^1 |\widetilde{f}(t+\delta) - \widetilde{f}(t)|^2 dt \right\}^{1/2}, \quad \epsilon > 0, f \in L_2[0,1],$$

where $\widetilde{f} \in L_2(\mathbb{R})$ is an extension of $f \in L_2[0,1]$ by zero for $x \in \mathbb{R} \setminus [0,1]$. We denote

$$\mathcal{E}(f', g, s) := \widetilde{\omega}_2(f', s^{-1/2}) + \widetilde{\omega}_2(g, s^{-1/2}) + \delta(s)\big(\|f'\|_{L_2} + \|g\|_{L_2}\big).$$

Note that $\widetilde{\omega}_2(f, \epsilon) \to 0$, when $\epsilon \to 0$ and recall that $\delta(s) \to 0$, if $s \to \infty$. Consequently, $\mathcal{E}(f', g, s) \to 0$, if $s \to \infty$. Summarizing, the first main result of this paper is the following theorem.

Theorem 3.1. *Suppose that* (1.3)–(1.6) *and* (1.8) *are satisfied. Then the system of root vectors of the operator A_h forms a basis of subspaces in the space H. Furthermore, there exists $c > 0$, such that for every $w = (f, g) \in H$ and $n = 1, 2, \ldots$ there holds*

$$\|\mathcal{P}_n w - w\|_H \le c\,\mathcal{E}(f', g, n).$$

The main idea of the proof of Theorem 3.1 is the following. We will use projectors in H denoted by $P_1 \colon H \to \widehat{W}_2^1$ and $P_2 \colon H \to \widehat{L}_2$. Going back to identity (2.3) we derive that for $w = (f, g) \in H$

$$P_2 R(A_h, \mu) w = if + i\mu P_1 R(A_h, \mu) w.$$

This leads to

$$\begin{aligned}\|\mathcal{P}_n w - w\|_H &= \left\| -\frac{1}{2\pi i}\int\limits_{\gamma_n} R(A_h,\mu) w d\mu - w \right\|_H \\ &\le c \left\| -\frac{1}{2\pi i}\int\limits_{\gamma_n} \big(P_1 R(A_h,\mu)w\big)' d\mu - f' \right\|_{L_2} \\ &\quad + \left\| -\frac{1}{2\pi}\int\limits_{\gamma_n} \mu P_1 R(A_h,\mu) w d\mu - g \right\|_{L_2([0,1];\rho)}.\end{aligned}$$

Let

$$\tilde{q}(s,x) := \int\limits_s^x b(\tau)d\tau, \quad v(s,x) := \int\limits_s^x b(\tau)d(\tau)d\tau, \quad s,\ x \in [0,1]$$

and

$$f_1(x) := \varrho^{1/4}(x) f'(x) \in L_2[0,1], \quad g_1(x) := \rho(x)\varrho^{-1/4}(x) g(x) \in L_2[0,1].$$

The first step is to transform the resolvent to obtain

$$\begin{aligned}\left\| -\frac{1}{2\pi i}\int\limits_{\gamma_n} \big(P_1 R(A_h,\mu)w\big)' d\mu - f' \right\|_{L_2} &\le c_1 \Big(\big\| \mathcal{W}_{R_n} f_1 - f' \big\|_{L_2} \\ &\quad + \big\| \mathcal{M}_{R_n} g_1 \big\|_{L_2} + \mathcal{E}(f', g, n) \Big)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\left\| -\frac{1}{2\pi}\int\limits_{\gamma_n} \mu P_1 R(A_h,\mu) w d\mu - g \right\|_{L_2([0,1];\rho)} &\le c_2 \left(\left\| \frac{p}{\varrho^{1/2}} \big(\mathcal{W}_{R_n} g_1\big) - g \right\|_{L_2([0,1];\rho)} \right. \\ &\quad \left. + \left\| \frac{\varrho^{1/2}}{p} \big(\mathcal{M}_{R_n} f_1\big) \right\|_{L_2([0,1];\rho)} + \mathcal{E}(f', g, n) \right),\end{aligned} \tag{3.2}$$

where c_1, $c_2 > 0$ and

$$\big(\mathcal{W}_R y\big)(x) = \frac{\varrho^{1/4}(x)}{\pi p(x)} \int\limits_0^1 \frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)} \cosh\big(v(s,x)\big) y(s) ds,$$

$$\big(\mathcal{M}_R y\big)(x) = \frac{1}{\pi \varrho^{1/4}(x)} \int\limits_0^1 \frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)} \sinh\big(v(s,x)\big) y(s) ds.$$

Now we are going to show how these two inequalities imply the thesis of Theorem 3.1. Consider now the second one. Note that

$$\begin{aligned}\Big\|\frac{p}{\varrho^{1/2}}\big(\mathcal{W}_{R_n}g_1\big)-g\Big\|_{L_2([0,1];\rho)}&+\Big\|\frac{\varrho^{1/2}}{p}\big(\mathcal{M}_{R_n}f_1\big)\Big\|_{L_2([0,1];\rho)}\\&\le\Big\|p^{1/2}\big(\mathcal{W}_{R_n}g_1\big)-\rho^{1/2}g\Big\|_{L_2}+\Big\|\frac{\varrho^{1/2}\rho^{1/2}}{p}\big(\mathcal{M}_{R_n}f_1\big)\Big\|_{L_2}\\&\le c\Big(\big\|\big(\mathcal{W}_{R_n}g_1\big)-bg\big\|_{L_2}+\big\|\mathcal{M}_{R_n}f_1\big\|_{L_2}\Big),\end{aligned}$$

thus it is sufficient to prove the following statement.

Lemma 3.2. *There exist positive constants c_1 and c_2, such that for $R>1$ there holds*

$$\big\|\big(\mathcal{W}_{R_n}f_1\big)-f'\big\|_{L_2}+\|\big(\mathcal{M}_{R_n}f_1\big)\|_{L_2}\le c_1\widetilde{\omega}_2(f',R^{-1/2}),\quad f\in\widehat{W}_2^1[0,1],$$

$$\big\|\big(\mathcal{W}_{R_n}g_1\big)-bg\big\|_{L_2}+\|\big(\mathcal{M}_{R_n}g_1\big)\|_{L_2}\le c_2\widetilde{\omega}_2(g,R^{-1/2}),\quad g\in L_2[0,1].$$

Adding and subtracting one in $\mathcal{W}_{R_n}$, we obtain

$$\big\|\big(\mathcal{W}_{R_n}f_1\big)-f'\big\|_{L_2}\le c\Big\{\big\|f'-V_{R_n}f'\big\|_{L_2}+\big\|Z_R(f_1b^{-1})\big\|_{L_2}\Big\},\tag{3.3}$$

where the operator $V_R\colon L_2[0,1]\to L_2[0,1]$ is given by

$$(V_Rf)(x):=\frac{\varrho^{1/4}(x)}{\pi p(x)}\int_0^1\frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}\varrho^{1/4}(s)f(s)ds$$

and

$$(Z_Ry)(x):=\int_0^1\frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}\big(\cosh(v(s,x))-1\big)b(s)y(s)ds.$$

Analogously as we did in [5], one can prove the modified version of Corollary 7.6.

Corollary 3.3. *There exists a constant $c>0$, independent of $f\in L_2[0,1]$, such that*

$$\|f-V_Rf\|_{L_2}\le c\,\widetilde{\omega}_2(f,R^{-1}),\quad f\in L_2[0,1],\ R>1.$$

It left to estimate the expression Z_R.

Lemma 3.4. *There exists $c>0$, such that the following inequality holds*

$$\|Z_Rf\|_{L_2}\le c\widetilde{\omega}_2(f,R^{-1/2}),\quad f\in L_2[0,1],\ R>1.$$

Proof. We want to use integration by parts and [5, Prop. 7.2]. Note that for $s=x$ we have $\tilde{q}(s,x)=0$ and there is a singularity in Z_R. That is why we split Z_R into

$$(D_Ry)(x)=\int_0^x\frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}\big(\cosh(v(s,x))-1\big)b(s)y(s)ds,$$

$$(\widetilde{D}_Ry)(x)=\int_x^1\frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}\big(\cosh(v(s,x))-1\big)b(s)y(s)ds.$$

We will prove the thesis only for D_R, since the reasoning for $\widetilde{D}_R$ is analogous. Note that if M_b is an upper bound for the function b, then

$$M_b(x-s) \geq \tilde{q}(s,x) \geq m_b(x-s), \quad 0 \leq s \leq x \leq 1,$$

and

$$v(s,x) = \int\limits_s^x b(\tau)d(\tau)d\tau \leq M_b\|d\|_{L_1}.$$

The second inequality implies

$$\int\limits_s^x b(\tau)d(\tau)\,d\tau \leq M_b\|d\|_{L_2}(x-s)^{1/2} \leq \frac{M_b}{m_b^{1/2}}\|d\|_{L_2}\tilde{q}^{1/2}(s,x), \quad 0 \leq s \leq x \leq 1. \tag{3.4}$$

Due to this inequality we obtain

$$\cosh\big(v(s,x)\big) - 1 \leq C\int\limits_s^x b(\tau)d(\tau)\,d\tau \leq c_0\tilde{q}^{1/2}(s,x),$$

$$\sinh\big(v(s,x)\big) \leq c_0\tilde{q}^{1/2}(s,x),$$

with $c_0 := CM_bm_b^{-1/2}\|d\|_{L_2}$ and $C := e^{M_b\|d\|_{L_1}}$. Consequently, we derive

$$\begin{aligned}|\big(D_Rf\big)(x)| &\leq \int\limits_0^x \frac{\cosh\big(v(s,x)\big)-1}{\tilde{q}(s,x)}b(s)|y(s)|\,ds \leq \frac{c_0M_b}{m_b^{1/2}}\int\limits_0^x \frac{|y(s)|}{(x-s)^{1/2}}\,ds \\ &\leq \frac{c_0M_b}{m_b^{1/2}}\int\limits_0^1 \frac{|y(s)|}{|x-s|^{1/2}}\,ds.\end{aligned}$$

The kernel $|x-s|^{-1/2}$ is a weak singularity (see [12, Ch. 2, § 6]), hence

$$\|D_Rf\|_{L_2} \leq c\|f\|_{L_2}, \quad f \in L_2[0,1],\ R > 1.$$

Assume now that $f \in \widetilde{W}_2^1[0,1]$, where

$$\widetilde{W}_p^1[0,1] := \{y \in W_p^1[0,1] :\ y(0) = 0 = y(1)\}.$$

Define

$$P(s,x) := \frac{\cosh\big(v(s,x)\big)-1}{\tilde{q}(s,x)}.$$

Integrating by parts we get

$$\begin{aligned}R\big(D_Rf\big)(x) &= -\int\limits_0^x P(s,x)\frac{\partial}{\partial s}\big(1-\cos(R\tilde{q}(s,x))\big)f(s)ds \\ &= (D_{1,R}f)(x) + (D_{2,R}f)(x),\end{aligned} \tag{3.5}$$

where

$$\big(D_{1,R}f\big)(x) := \int_0^x \frac{\partial}{\partial s}P(s,x)\big(1-\cos(R\tilde{q}(s,x))\big)f(s)ds,$$

$$\big(D_{2,R}f\big)(x) := \int_0^x P(s,x)\,\big(1-\cos(R\tilde{q}(s,x))\big)\,f'(s)\,ds.$$

The expression $D_{2,R}$ can be estimated in the similar way to this in (3.4), namely

$$|\big(D_{2,R}f\big)(x)| \le 2\int_0^x P(s,x)|f'(s)|\,ds \le \frac{2c_0}{m_b^{1/2}}\int_0^x \frac{|f'(s)|}{(x-s)^{1/2}}\,ds.$$

Once more due to the weak singularity of $|x-s|^{-1/2}$ we obtain

$$\|\big(D_{2,R}f\big)\|_{L_2} \le \frac{2c_0}{m_b^{1/2}}\left\|\int_0^1 \frac{|f'(s)|}{|x-s|^{1/2}}\,ds\right\|_{L_2} \le c\|f'\|_{L_2}.$$

We go back to $D_{1,R}$. First of all we see that

$$1-\cos(R\tilde{q}(s,x)) = 2\sin^2(R\tilde{q}(s,x)/2),$$

whence

$$|\big(D_{1,R}f\big)(x)| \le 2\|f\|_C\int_0^x \left|\frac{\partial}{\partial s}P(s,x)\right|\sin^2(R\tilde{q}(s,x)/2)\,ds.$$

After differentiation we have

$$\frac{\partial}{\partial s}P(s,x) = \frac{1}{\tilde{q}^2(s,x)}\Big\{-b(s)d(s)\tilde{q}(s,x)\sinh\big(v(s,x)\big) + b(s)\big(\cosh\big(v(s,x)\big)-1\big)\Big\},$$

and this implies

$$\left|\frac{\partial}{\partial s}P(s,x)\right| \le c_0M_b\frac{d(s)\tilde{q}^{3/2}(s,x)+\tilde{q}^{1/2}(s,x)}{\tilde{q}^2(s,x)}.$$

Then we can write

$$\begin{aligned}\frac{1}{c_0M_b}&\int_0^x \left|\frac{\partial}{\partial s}P(s,x)\right|\sin^2(R\tilde{q}(s,x)/2)\,ds\\ &\le \int_0^x \frac{d(s)\tilde{q}^{3/2}(s,x)+\tilde{q}^{1/2}(s,x)}{\tilde{q}^2(s,x)}\sin^2(R\tilde{q}(s,x)/2)\,ds\\ &\le \int_0^x d(s)\,\frac{\sin^2(R\tilde{q}(s,x)/2)}{\tilde{q}^{1/2}(s,x)}\,ds + \int_0^x \frac{\sin^2(R\tilde{q}(s,x)/2)}{\tilde{q}^{3/2}(s,x)}\,ds.\end{aligned}$$

We are going to estimate the norms of these integrals in $L_2[0,1]$. Using $\sin^2(s) \le \sqrt{|s|}$, $s \in \mathbb{R}$ we obtain

$$\left\| \int\limits_0^x d(s)\, \frac{\sin^2(R\tilde{q}(s,x)/2)}{\tilde{q}^{1/2}(s,x)} ds \right\|_C \le \sqrt{\frac{R}{2}} \|d\|_{L_1}.$$

We use two changes of variables to estimate the second integral. The first one is $t = \tilde{q}(s,x)$ and the second is $\tau = Rt/2$. This leads to

$$\begin{aligned} \int\limits_0^x \frac{\sin^2(R\tilde{q}(s,x)/2)}{\tilde{q}^{3/2}(s,x)}\, ds &= \int\limits_0^{q(0,x)} \frac{\sin^2(Rt/2)}{t^{3/2}}\, \frac{dt}{b(s(t))} \le \frac{1}{m_b} \int\limits_0^{\infty} \frac{\sin^2(Rt/2)}{t^{3/2}}\, dt \\ &= \frac{\sqrt{2}R^{1/2}}{m_b} \int\limits_0^{\infty} \frac{\sin^2 \tau}{\tau^{3/2}}\, d\tau = cR^{1/2}, \end{aligned}$$

with some $c > 0$. Merging two previous inequalities and this for $D_{2,R}$ in (3.5) we obtain

$$\|\big(D_R f\big)(x)\|_{L_2} \le cR^{-1/2} \|f'\|_{L_2}, \quad f \in \widetilde{W}_2^1[0,1],$$

with some $c > 0$. We checked all the assumptions of [5, Prop. 7.2] and this completes the proof. ☐

Remark 3.5. In the formula (3.3) we need estimations in situation when D_R acts on $f_1 b^{-1} \in L_2[0,1]$ instead of f. However, using this lemma for $f_1 b^{-1}$ and then properties of $\widetilde{\omega}_2$ (see [5, Prop. 7.5]), we obtain $\widetilde{\omega}_2(f', R^{-1/2})$ in the thesis. The last part missing in the proof of Lemma 3.2 is the estimation of $\mathcal{M}_R$. The proof goes along the same lines as in Lemma 3.4 for D_R and $\widetilde{D}_R$. It is sufficient to split $\mathcal{M}_R$ into two integrals and repeat whole reasoning taking instead of $\cosh\big(v(s,x)\big) - 1$ the expression $\sinh\big(v(s,x)\big)$.

4. TRANSFORMATION OF THE RESOLVENT

Now we want to find the representation for the resolvent which allows the estimation (3.1)–(3.2). First we will find the formulas for the resolvent in $\mathbb{C}_+(r_0)$. Analogously as we did in [5], we obtain

$$\begin{aligned} -\frac{1}{2\pi}\mu P_1 R(A_h,\mu) w = \frac{1}{2\pi}\Big\{ & f(x) - \mu(L_0 f')(x,\mu) + i\mu(N_0 g)(x,\mu) - (N_0 q f)(x,\mu) \\ & - \mu(M_1 f')(x,\mu) - i\mu(M_2 g)(x,\mu) + (M_2 q f)(x,\mu)\Big\} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} -\frac{1}{2\pi i}\big(P_1 R(A_h,\mu) w\big)' = \frac{1}{2\pi}\Big\{ & i(L_1 f')(x,\mu) + (N_1 g)(x,\mu) + i\mu^{-1}(N_1 q f)(x,\mu) \\ & + i(\widetilde{M}_1 f')(x,\mu) - (\widetilde{M}_2 g)(x,\mu) - i\mu^{-1}(\widetilde{M}_2 q f)(x,\mu)\Big\}, \end{aligned} \tag{4.2}$$

where

$$(L_0 f')(x,\mu) := \frac{u_1(x,\mu)}{\mu w(\mu)} \int_0^x p(s) u_2'(s,\mu) f'(s)ds + \frac{u_2(x,\mu)}{\mu w(\mu)} \int_x^1 p(s) u_1'(s,\mu) f'(s)ds, \tag{4.3}$$

$$(N_0 v)(x,\mu) := \frac{u_1(x,\mu)}{w(\mu)} \int_0^x u_2(s,\mu)\rho(s) v(s)ds + \frac{u_2(x,\mu)}{w(\mu)} \int_x^1 u_1(s,\mu)\rho(s) v(s)ds, \tag{4.4}$$

$$(L_1 v)(x,\mu) := \frac{u_1'(x,\mu)}{\mu w(\mu)} \int_0^x p(s) u_2'(s,\mu) v(s)ds + \frac{u_2'(x,\mu)}{\mu w(\mu)} \int_x^1 p(s) u_1'(s,\mu) v(s)ds, \tag{4.5}$$

$$(N_1 g)(x,\mu) := \frac{u_1'(x,\mu)}{w(\mu)} \int_0^x u_2(s,\mu)\rho(s) g(s)ds + \frac{u_2'(x,\mu)}{w(\mu)} \int_x^1 u_1(s,\mu)\rho(s) g(s)ds, \tag{4.6}$$

$$\begin{aligned}(M_1 v)(x,\mu) := {} & \frac{u_1(x,\mu)u_2(0,\mu)}{\mu w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_2'(s,\mu)U[u_1] - u_1'(s,\mu)U[u_2]\Bigr) v(s)ds \\ & + \frac{u_2(x,\mu)U[u_1]}{\mu w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_1'(s,\mu)u_2(0,\mu) - u_2'(s,\mu)u_1(0,\mu)\Bigr) v(s)ds,\end{aligned}$$

$$\begin{aligned}(M_2 v)(\mu) := {} & \frac{u_1(x,\mu)u_2(0,\mu)}{w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_1(s,\mu)U[u_2] - u_2(s,\mu)U[u_1]\Bigr) \rho(s) v(s)ds \\ & + \frac{u_2(x,\mu)U[u_1]}{w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_2(s,\mu)u_1(0,\mu) - u_1(s,\mu)u_2(0,\mu)\Bigr) \rho(s) v(s)ds,\end{aligned}$$

$$\begin{aligned}(\widetilde{M}_1 v)(x,\mu) := {} & \frac{u_1'(x,\mu)u_2(0,\mu)}{\mu w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_2'(s,\mu)U[u_1] - u_1'(s,\mu)U[u_2]\Bigr) v(s)ds \\ & + \frac{u_2'(x,\mu)U[u_1]}{\mu w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_1'(s,\mu)u_2(0,\mu) - u_2'(s,\mu)u_1(0,\mu)\Bigr) v(s)ds,\end{aligned}$$

$$\begin{aligned}(\widetilde{M}_2 v)(x,\mu) := {} & \frac{u_1'(x,\mu)u_2(0,\mu)}{w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_1(s,\mu)U[u_2] - u_2(s,\mu)U[u_1]\Bigr) \rho(s) v(s)ds \\ & + \frac{u_2'(x,\mu)U[u_1]}{w(\mu)\Delta(\mu)} \int_0^1 \Bigl(u_2(s,\mu)u_1(0,\mu) - u_1(s,\mu)u_2(0,\mu)\Bigr) \rho(s) v(s)ds.\end{aligned}$$

Remark 4.1. According to Remark 2.1 we obtain analogous representation for (4.1) and (4.2) in $\mathbb{C}_-(r_0)$ exchanging $u_j(x,\mu)$, $j=1,2$ with $\widetilde{u}_j(x,\mu)$. Instead of the Wronskian w for $u_j(x,\mu)$ we take the Wronskian for $\widetilde{u}_j(x,\mu)$.

The expressions N_0, N_1, L_0 and L_1 need more transformation. We introduce operators which occur during the process of construction of fundamental system u_j, $j=1,2$ (see [4], for details). Let

$$r_1(x,\mu) := \frac{\varrho'(x)}{4\varrho(x)} + d(x)b(x) - \frac{iq(x)b(x)}{2\mu}, \tag{4.7}$$

$$r_2(x,\mu) := \frac{\varrho'(x)}{4\varrho(x)} - d(x)b(x) + \frac{iq(x)b(x)}{2\mu}, \tag{4.8}$$

$$\mathcal{Q}(x,\mu) := 2i\mu b(x) + 2d(x)b(x) - \frac{iq(x)b(x)}{\mu},$$

$$k(s,t,\mu) := \exp\Big(\int\limits_s^t \mathcal{Q}(\tau,\mu)d\tau\Big)$$

and

$$(S_1z)(x) := \int\limits_x^1 k(x,t,\mu)r_2(t,\mu)z(t)dt, \qquad T_1z(x) := \int\limits_0^x (S_1z)(t)r_1(t,\mu)dt,$$

$$(S_2z)(x) := \int\limits_0^x k(t,x,\mu)r_1(t,\mu)z(t)dt, \qquad T_2z(x) := \int\limits_x^1 (S_2z)(t)r_2(t,\mu)dt.$$

We will use formulas, which come from the proof of [4, Theorem 1],

$$u_1(x,\mu) = \varrho^{-1/4}(x)e^{q_0(x,\mu)}\big[z_1(x) - \big(S_1z_1\big)(x)\big]\Big(1 + \|q\|_{L_1}O\big(|\mu|^{-1}\big)\Big), \tag{4.9}$$

$$u_1'(x,\mu) = i\mu p^{-1}(x)\varrho^{1/4}(x)e^{q_0(x,\mu)}\big[z_1(x) + \big(S_1z_1\big)(x)\big]\Big(1 + \|q\|_{L_1}O\big(\||\mu|^{-1}\big)\Big), \tag{4.10}$$

$$u_2(x,\mu) = \varrho^{-1/4}(x)e^{q_1(x,\mu)}\big[z_2(x) + \big(S_2z_2\big)(x)\big]\Big(1 + \|q\|_{L_1}O\big(|\mu|^{-1}\big)\Big), \tag{4.11}$$

$$u_2'(x,\mu) = -i\mu p^{-1}(x)\varrho^{1/4}(x)e^{q_1(x,\mu)}\big[z_2(x) - \big(S_2z_2\big)(x)\big]\Big(1 + \|q\|_{L_1}O\big(|\mu|^{-1}\big)\Big), \tag{4.12}$$

where $z_j = z_j(x,\mu)$, $j=1,2$ are the unique solutions in $C[0,1]$ of

$$z_j(x) + \big(T_jz_j\big)(x) = 1, \quad x \in [0,1].$$

for $\mu \in \mathbb{C}_+(r_0)$. Extending results from [4] we can prove the following lemma.

Lemma 4.2. *For* $j = 1, 2$ *and* $\mu \in \mathbb{C}_+(r_0)$ *there holds*

$$\|S_j z\|_C \le \frac{c}{\beta^{1/2}} \|z\|_C, \quad \|S_j\|_C = O(\delta(|\mu|)),$$

$$\|T_j z\|_C \le \frac{c}{\beta^{1/2}} \|z\|_C, \quad \|T_j\|_C = O(\delta(|\mu|)),$$

$$\|S_j T_j\|_C + \|T_j^2\|_C \le \frac{c\delta(|\mu|)}{\beta^{1/2}},$$

$$\|z_j - e\|_C = O(\delta(|\mu|)), \quad j = 1, 2,$$

where $z \in C[0,1]$, e *is a function, such that* $e(x) = 1$, $x \in [0,1]$ *and* $\beta = \operatorname{Im} \mu$. *Furthermore, for the Wronskian* w *we derive*

$$2\mu \frac{e^{\xi_0(\mu)}}{iw(\mu)} = [1 + (T_2 e)(0)] + O\left(\frac{\delta(|\mu|)}{\beta^{1/2}}\right) + O\left(\frac{\|q\|_{L_1}}{|\mu|}\right).$$

We also have

$$z_j = e - T_j e + T_j^2 z_j, \quad S_j z_j = S_j e - S_j T_j z_j. \tag{4.13}$$

Denote

$$f_1(x) := \varrho^{1/4}(x) f'(x) \in L_2[0,1], \quad v_1(x) := \rho(x) \varrho^{-1/4}(x) v(x) \in L_2[0,1]$$

for $f \in W_2^1[0,1]$ and $v \in L_2[0,1]$.

The identities (4.9)–(4.12), (4.13) and Lemma 4.2 leads to the formulas

$$\begin{aligned}
(N_0 g)(x,\mu) = {} & \frac{i}{2\mu\varrho^{1/4}(x)} \Big[-(E_0 g_1)(x,\mu) + (E_1 g_1)(x,\mu) - (E_2 g_1)(x,\mu) \\
& - (F_0 g_1)(x,\mu) \Big] \Big(1 + \|q\|_{L_1} O\big(|\mu|^{-1}\big)\Big), \\
(L_1 f')(x,\mu) = {} & \frac{i\varrho^{1/4}(x)}{2p(x)} \Big[-(E_0 f_1)(x,\mu) - (E_1 f_1)(x,\mu) - (E_2 f_1)(x,\mu) \\
& + (F_1 f_1)(x,\mu) \Big] \Big(1 + \|q\|_{L_1} O\big(|\mu|^{-1}\big)\Big),
\end{aligned}$$

where

$$(E_0y)(x,\mu) := -\int\limits_0^x e^{q(s,x,\mu)}y(s)ds - \int\limits_x^1 e^{q(x,s,\mu)}y(s)ds,$$

$$\begin{aligned}(E_1y)(x,\mu) := &-(S_1e)(x)\int\limits_0^x e^{q(s,x,\mu)}y(s)ds\\ &+(S_2e)(x)\int\limits_x^1 e^{q(x,s,\mu)}y(s)ds\\ &+\int\limits_0^x e^{q(s,x,\mu)}(S_2e)(s)y(s)ds - \int\limits_x^1 e^{q(x,s,\mu)}(S_1e)(s)y(s)ds,\end{aligned}$$

$$\begin{aligned}(E_2y)(x,\mu) := &\,(T_1e)(x)\int\limits_0^x e^{q(s,x,\mu)}y(s)ds\\ &+\int\limits_x^1 e^{q(x,s,\mu)}(T_1e)(s)y(s)ds\\ &+[(T_2e)(x)-(T_2e)(0)]\int\limits_x^1 e^{q(x,s,\mu)}y(s)ds\\ &+\int\limits_0^x e^{q(s,x,\mu)}[(T_2e)(s)-(T_2e)(0)]y(s)ds\end{aligned}$$

and

$$\begin{aligned}(N_1g)(x,\mu) = \frac{1}{2\varrho^{1/4}(x)}\Big[&(\widehat{E}_0g_1)(x,\mu)+(\widehat{E}_1g_1)(x,\mu)+(\widehat{E}_2g_1)(x,\mu)\\ &+(\widehat{F}_2g_1)(x,\mu)\Big]\Big(1+\|q\|_{L_1}O\big(|\mu|^{-1}\big)\Big),\end{aligned}$$

$$\begin{aligned}(L_0f')(x,\mu) = \frac{\varrho^{1/4}(x)}{2p(x)\mu}\Big[&-(\widehat{E}_0f_1)(x,\mu)+(\widehat{E}_1f_1)(x,\mu)-(\widehat{E}_2f_1)(x,\mu)\\ &+(\widetilde{F}_3f_1)(x,\mu)\Big]\Big(1+\|q\|_{L_1}O\big(|\mu|^{-1}\big)\Big),\end{aligned}$$

where

$$(\widehat{E}_0 y)(x,\mu) := -\int\limits_0^x e^{q(s,x,\mu)} y(s)ds + \int\limits_x^1 e^{q(x,s,\mu)} y(s)ds,$$

$$\begin{aligned}(\widehat{E}_1 y)(x,\mu) := &-(S_1e)(x)\int\limits_0^x e^{q(s,x,\mu)} y(s)ds - (S_2e)(x)\int\limits_x^1 e^{q(x,s,\mu)} y(s)ds \\ &- \int\limits_0^x e^{q(s,x,\mu)}(S_2e)(s)y(s)ds - \int\limits_x^1 e^{q(x,s,\mu)}(S_1e)(s)y(s)ds,\end{aligned}$$

$$\begin{aligned}(\widehat{E}_2 y)(x,\mu) := &-(T_1e)(x)\int\limits_0^x e^{q(s,x,\mu)} y(s)ds + \int\limits_x^1 e^{q(x,s,\mu)}(T_1e)(s)y(s)ds \\ &+ [(T_2e)(x)-(T_2e)(0)]\int\limits_x^1 e^{q(x,s,\mu)} y(s)ds \\ &- \int\limits_0^x e^{q(s,x,\mu)}[(T_2e)(s)-(T_2e)(0)]y(s)ds.\end{aligned}$$

Here Fj, $j=0,\ldots,3$, have a very complex form but admit estimations

$$|(F_jy)(x,\mu)| \le c\int\limits_0^1 e^{-\beta m_b|x-s|}|y(s)|\Big(\frac{\delta(|\mu|)}{\beta^{1/2}} + \frac{\|q\|_{L_1}}{|\mu|}\Big)ds, \quad \beta = \operatorname{Im}\mu > 0.$$

Modifying the proof of [5, Lemma 5.1] we can obtain the following fact.

Lemma 4.3. *There exists a constant $c>0$, such that for $R>1$ and $j=0,\ldots,3$ we have*

$$\Big\|\int\limits_{\Gamma_R^+}(F_jy)(x,\mu)|d\mu|\Big\|_{L_2} \le c\Big(\|y\|_{L_2}\delta(R) + \widetilde{\omega}_2(y, R^{-1/2})\Big), \quad y \in L_2[0,1].$$

Now we focus on showing how to derive the main part of the resolvent.

Remark 4.4. The main part of the resolvent comes from the integration of E_0 and $\widehat{E}_0$ over γ_n^+ and their analogues E_0^- and $\widehat{E}_0$ over γ_n^-. We need to go back to formulas (4.3)–(4.6) and use Remark 4.1 to find the explicit formulas for E_0^- and $\widehat{E}_0$. Then we write integrals over γ_n^- and change variables from $-\mu$ to μ to get integrals over γ_n^+. Next we use the same transformations as in $\mathbb{C}_+(r_0)$ but with the aid of the fundamental system of solutions for (1.7) where d is exchanged with $-d$.

Recall that

$$q(s,x,\mu) = i\mu\tilde{q}(s,x) + v(s,x).$$

Proceeding as we described we get

$$-\frac{1}{2\pi}\int\limits_{\gamma_n}\big(P_1R(A_h,\mu)w\big)'d\mu \approx \frac{1}{4\pi}\Bigg\{\frac{\varrho^{1/4}(x)}{p(x)}\int\limits_{\gamma_n^+}\Big[\big(E_0f_1\big)(x,\mu)+\big(E_0^-f_1\big)(x,\mu)\Big]d\mu$$

$$+\frac{1}{\varrho^{1/4}(x)}\int\limits_{\gamma_n^+}\Big[\big(\widehat{E}_0g_1\big)(x,\mu)+\big(\widehat{E}_0^-g_1\big)(x,\mu)\Big]d\mu\Bigg\},$$

and

$$-\frac{1}{2\pi}\int\limits_{\gamma_n}\mu P_1R(A_h,\mu)wd\mu \approx \frac{1}{4\pi}\Bigg\{\frac{1}{\varrho^{1/4}(x)}\int\limits_{\gamma_n^+}\Big[\big(E_0g_1\big)(x,\mu)+\big(E_0^-g_1\big)(x,\mu)\Big]d\mu$$

$$-\frac{\varrho^{1/4}(x)}{p(x)}\int\limits_{\gamma_n^+}\Big[\big(\widehat{E}_0f_1\big)(x,\mu)+\big(\widehat{E}_0^-f_1\big)(x,\mu)\Big]d\mu\Bigg\},$$

where

$$\big(E_0^-y\big)(x,\mu) := -\int\limits_0^x e^{i\mu\tilde{q}(s,x)}e^{-v(s,x)}y(s)ds-\int\limits_x^1 e^{i\mu\tilde{q}(x,s)}e^{-v(x,s)}y(s)ds,$$

$$\big(\widehat{E}_0^-y\big)(x,\mu) := \int\limits_0^x e^{i\mu\tilde{q}(s,x)}e^{-v(s,x)}y(s)ds-\int\limits_x^1 e^{i\mu\tilde{q}(x,s)}e^{-v(x,s)}y(s)ds.$$

Let

$$\Gamma_R := \{\mu\in\mathbb{C}:\ |\mu|=R\},\quad \Gamma_R^+ := \{\mu\in\mathbb{C}:\ |\mu|=R,\ \operatorname{Im}\mu>0\}.$$

Note that integration over a positive oriented contour Γ_R^+ gives

$$\frac{1}{2}\int\limits_{\Gamma_R^+}(E_0y)(x,\mu)d\mu = \int\limits_0^x \frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}e^{v(s,x)}y(s)ds$$

$$+\int\limits_x^1 \frac{\sin\big(R\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}e^{v(x,s)}y(s)ds.$$

Taking this into account we derive

$$\int\limits_{\gamma_n^+}\Big[\big(E_0y\big)(x,\mu)+\big(E_0^-y\big)(x,\mu)\Big]d\mu = 4\int\limits_0^1\frac{\sin\big(R_n\tilde{q}(s,x)\big)}{\tilde{q}(s,x)}\cosh\big(v(s,x)\big)y(s)ds,$$

$$-\int\limits_{\gamma_n^+} \Big[\big(\widehat{E}_0 y\big)(x,\mu) + \big(\widehat{E}_0^- y\big)(x,\mu)\Big] d\mu = 4\int\limits_0^1 \frac{\sin\big(R_n \tilde{q}(s,x)\big)}{\tilde{q}(s,x)} \sinh\big(v(s,x)\big) y(s) ds.$$

These results leads to expressions with operators $\mathcal{W}_{R_n}$ and $\mathcal{M}_{R_n}$ as in (3.1) and (3.2).

Remark 4.5. It is left to estimate the remainder of the resolvent. For the integrals on γ_n^+ we proceed in the same way as we did in [5]. We have to take into consideration that q contains v, whereas in [5] there was $v = 0$. During all the estimates we always need the modulus of $e^{q(s,x,\mu)}$. Note that

$$|e^{q(s,x,\mu)}| \le c|e^{i\mu\tilde{q}(s,x)}| \tag{4.14}$$

and $\tilde{q}$ is equal to q from [5], thus we can proceed in the same way. Furthermore, in the definitions of the operators E_1 and E_2 we now have r_1 and r_2 given by (4.7) and (4.8) instead of one simpler r given by formula (1.19) from [5]. This changes the rate of convergence but not the proof itself.

To complete the proof on γ_n^- we need to find the form of (4.1)–(4.2) in $\mathbb{C}_-(r_0)$. We described this process in Remark 4.4. Therefore we see that the only thing which has changed is the sign of d (according to (4.14) is not important) and a sign in front of some expressions of the remainder, but here we estimate only the modulus.

5. RIESZ BASIS

The second main result of this paper is the following theorem.

Theorem 5.1. *Suppose that assumptions of Theorem 3.1 are satisfied and additionally*

$$\int\limits_0^1 \frac{\omega_1^2(\rho',\tau)}{\tau^2}\, d\tau < \infty, \qquad \int\limits_0^1 \frac{\omega_1^2(d,\tau)}{\tau^2}\, d\tau < \infty. \tag{5.1}$$

Then the root vectors of the operator A_h forms a Riesz basis for H.

Proof. Consider the root vectors of A_h and A_h^*

$$\{Y_n\}_{n=-\infty,\, n\neq 0}^{\infty}, \quad \{\widetilde{Y}_n\}_{n=-\infty,\, n\neq 0}^{\infty}.$$

According to Lemma 2.4 these systems are asymptotically biorthogonal. From Theorem 3.1 we know that root vectors of A_h forms a basis of subspaces in H, thus this system is complete. Recall that all eigenvalues $\mu_{\pm n}$, $n \in \mathbb{N}$ of A_h, save countably many, are simple. Due to [3, Ch. 6, Thm. 2.1] and Lemma 2.4 it is sufficient to show that for every $w \in H$ there hold

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} |\langle Y_n, w\rangle_H|^2 < \infty, \tag{5.2}$$

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \left| \langle \widetilde{Y}_n, w \rangle_H \right|^2 < \infty. \tag{5.3}$$

We are going to use the asymptotical behavior of eigenfunctions

$$Y_n = \big(y_1(x,\mu_n), i\mu_n y_1(x,\mu_n)\big),$$

given by (2.13) oraz (2.14). Let $w := (w_1, w_2) \in H$, then

$$\begin{aligned}
|\langle Y_n, w\rangle_H| \leq{}& \left| \int_0^1 p(x) y_1'(x,\mu_n)\overline{w}_1'(x)\,dx \right| + |\mu_n| \left| \int_0^1 y_1(x,\mu_n)\overline{w}_2(x)\,\rho(x)\,dx \right| \\
&+ \left| \int_0^1 y_1(x,\mu_n)\overline{w}_1(x)\,q(x)\rho(x)\,dx \right| \\
&\leq \frac{1}{2}\sqrt{\frac{p(0)}{b(0)}} \left\{ \left| \int_0^1 e^{q_0(x,\mu_n)}\overline{w}_1'(x)\sqrt{\frac{b(x)}{p(x)}}\,dx \right| + \left| \int_0^1 e^{-q_0(x,\mu_n)}\overline{w}_1'(x)\sqrt{\frac{b(x)}{p(x)}}\,dx \right| \right. \\
&+ \left| \int_0^1 e^{q_0(x,\mu_n)}\overline{w}_2(x)\sqrt{b(x)\rho(x)}\,dx \right| + \left| \int_0^1 e^{-q_0(x,\mu_n)}\overline{w}_2(x)\sqrt{b(x)\rho(x)}\,dx \right| \\
&+ \left| \frac{1}{\mu_n}\int_0^1 e^{q_0(x,\mu_n)}\overline{w}_1(x)\sqrt{b(x)\rho(x)}q(x)\,dx \right| \\
&\left. + \left| \frac{1}{\mu_n}\int_0^1 e^{-q_0(x,\mu_n)}\overline{w}_1(x)\sqrt{b(x)\rho(x)}q(x)\,dx \right| \right\} + c\delta(|n|), \quad n \geq 1,
\end{aligned} \tag{5.4}$$

where q_0 is given by (2.4). Recall that δ is defined by (2.9) and $\omega_1(f,\epsilon)$ converges monotonically to zero, when $\epsilon \to 0$. This fact together with (5.1) imply

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \omega_1^2(\rho', |n|^{-1}) < \infty, \qquad \sum_{n=-\infty,\, n\neq 0}^{\infty} \omega_1^2(d, |n|^{-1}) < \infty.$$

Going back to (2.9) we obtain

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \delta^2(|n|) < \infty.$$

Due to (2.12) we know that μ_n lies in a finite stripe, whence the last two integrals in (5.4) are bounded. What is more (2.12) implies also $|\mu_n|^{-2} < cn^{-2}$, thus an appropriate series will be convergent. Consequently, to prove (5.2) it is sufficient to establish

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \left| \int_0^1 e^{q_0(x,\mu_n)} y(x)\,dx \right|^2 < \infty, \quad y \in L_2[0,1], \tag{5.5}$$

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \left| \int_0^1 e^{-q_0(x,\mu_n)} y(x)\, dx \right|^2 < \infty, \quad y \in L_2[0,1]. \tag{5.6}$$

We start with the proof of (5.5). Note that changing variables

$$2\pi t = \tilde{q}_0(x) = \int_0^x b(\tau) d\tau$$

we can write the integral from (5.5) as

$$F(\mu_n) = \int_0^1 e^{q_0(x,\mu_n)} y(x)\, dx = 2\pi \int_0^a e^{2\pi i t \mu_n} y_1(t) dt, \tag{5.7}$$

where $a := \frac{1}{2\pi}\tilde{q}_0(1)$ and

$$y_1(t) := e^{v(0,x(t))} \frac{y(x(t))}{b(x(t))} \in y_1 \in L_2[0,a].$$

We can treat F like a Fourier transformation for a certian function from $L_2[0,a]$. Let $\mathcal{H}^2$ be a Hardy space of the upper half-plane of $\mathbb{C}$. Recall that its elements are Fourier transforms F of $y \in L_2[0,\infty]$. Due to [8, Thm. 2.1] we have that for every $F \in \mathcal{H}^2$ associated with $y \in L_2[0,\infty]$ the following inequality holds

$$\|F\|^2_{\mathcal{H}^2} = \sup_{\beta>0} \int_{\mathbb{R}} |F(\mu)|^2 d\alpha = \|y\|^2_{L_2[0,\infty]}, \quad \mu = \alpha + i\beta.$$

This inequality and (5.7) imply

$$\left\| \int_0^1 e^{q_0(x,\mu_n)} y(x) dx \right\|^2_{\mathcal{H}^2} = \sup_{\beta>0} \int_{\mathbb{R}} \left| \int_0^1 e^{q_0(x,\mu_n)} y(x) dx \right|^2 d\alpha \le c\|y\|^2_{L_2}. \tag{5.8}$$

Let $\nu(\mu)$ be a measure defined on the half-plane $\operatorname{Im}\mu \ge 0$, which is concentrated in points μ_n, $n = \pm 1, \pm 2, \dots$. Note that

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \left| \int_0^1 e^{q_0(x,\mu_n)} y(x) dx \right|^2 = \int_{\operatorname{Im}\mu>0} \left| \int_0^1 e^{q_0(x,\mu_n)} y(x) dx \right|^2 d\nu(\mu). \tag{5.9}$$

From (2.12) we derive that $\nu(\mu)$ is a Carleson measure (see [2, Ch. 1]). According to the Carleson Theorem [2, Thm. 3.9] we get

$$\int_{\operatorname{Im}\mu>0} |F(\mu)|^2\, d\nu(\mu) \le c\, \|F\|^2_{\mathcal{H}^2}.$$

This fact together with (5.8) and (5.9) give

$$\sum_{n=-\infty,\, n\neq 0}^{\infty} \left| \int_0^1 e^{q_0(x,\mu_n)} y(x)dx \right|^2 \leq c\|y\|_{L_2}^2.$$

The proof of (5.6) is similar. We does not consider $y \in L_2[0,1]$ but $e^{i\eta\tilde{q}_0(\cdot)}y \in L_2[0,1]$, where η is fixed in order to push all the eigenvalues $\eta - \mu_n$ into $\operatorname{Im}\mu > 0$.

Now we go back to (5.3). Using (2.15)–(2.16) we obtain

$$\begin{aligned}
|\langle \widetilde{Y}_n, w\rangle_H| \leq \frac{1}{2}\sqrt{\frac{p(0)}{b(0)}}\Bigg\{ &\left| \int_0^1 e^{q_0^*(x,\overline{\mu}_n)} \overline{w}_1'(x)\sqrt{\frac{b(x)}{p(x)}}\, dx \right| \\
&+ \left| \int_0^1 e^{-q_0^*(x,\overline{\mu}_n)} \overline{w}_1'(x)\sqrt{\frac{b(x)}{p(x)}}\, dx \right| \\
&+ \left| \int_0^1 e^{q_0^*(x,\overline{\mu}_n)} \overline{w}_2(x)\sqrt{b(x)\rho(x)}\, dx \right| \\
&+ \left| \int_0^1 e^{-q_0^*(x,\overline{\mu}_n)} \overline{w}_2(x)\sqrt{b(x)\rho(x)}\, dx \right| \Bigg\} + c\delta(|n|),
\end{aligned}$$

where

$$q_0^*(x,\mu) = i\mu \int_0^x b(\tau)d\tau - \int_0^x d(\tau)b(\tau)d\tau.$$

For expressions with $-q_0^*(x,\overline{\mu}_n)$ we proceed analogously as we did for (5.5), whereas for integrals with $q_0^*(x,\overline{\mu}_n)$ we go along the same lines as for (5.6). ☐

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