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SOME ALGEBRAIC PROPERTIES OF PREPONDERANTLY CONTINUOUS FUNCTIONS

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Abstract. In the presented paper we study some properties of preponderantly continuous functions and functions satisfying the property A_1. For any family $\mathcal{F}$ of real-valued functions we define $\mathcal{MAX}_{\mathcal{F}} = \{g\colon \max\{f,g\} \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$ and $\mathcal{MIN}_{\mathcal{F}} = \{g\colon \min\{f,g\} \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$. The aim of the paper is to find $\mathcal{MIN}_{\mathcal{F}}$ for two discussed classes of functions.

1. Preliminaries

Let $\mathbb{R}$, $\mathbb{N}$ be the set of real numbers and natural numbers, respectively. Next, let I denote a closed interval, U any open subset of $\mathbb{R}$ and $\operatorname{Int}(A)$ is the interior of a set $A \subset \mathbb{R}$ in the natural metric. Let λ stand for Lebesgue measure in $\mathbb{R}$. For each measurable set $E \subset \mathbb{R}$ we define the lower and upper density of E at $x_0 \in \mathbb{R}$ by:

$$\underline{d}\,(E, x_0) = \liminf_{\lambda(I)\to 0, x_0 \in I} \frac{\lambda(I\cap E)}{\lambda(I)} \quad\text{and}\quad \overline{d}\,(E, x_0) = \limsup_{\lambda(I)\to 0, x_0 \in I} \frac{\lambda(I\cap E)}{\lambda(I)}.$$

If $\underline{d}\,(E, x_0) = \overline{d}\,(E, x_0)$, we denote this common value by $d(E, x_0)$ and call it the density of E at x_0. In a similar way, we also define the one-sided lower and upper density of the set E at the point x_0: $\underline{d}^+\,(E, x_0)$, $\underline{d}^-\,(E, x_0)$, $\overline{d}^+\,(E, x_0)$ and $\overline{d}^-\,(E, x_0)$. It is easy to check that $\underline{d}\,(E, x_0) = \min\{\,\underline{d}^+\,(E, x_0), \underline{d}^-\,(E, x_0)\}$ and $\overline{d}\,(E, x_0) = \max\{\,\overline{d}^+\,(E, x_0), \overline{d}^-\,(E, x_0)\}$. If $\underline{d}^+\,(E, x_0) = \overline{d}^+\,(E, x_0)$ ($\underline{d}^-\,(E, x_0) = \overline{d}^-\,(E, x_0)$), then we denote this common value by $d^+(E, x_0)$ ($d^-(E, x_0)$) and call it the right (the left) density of E at x_0.

There are a few nonequivalent definitions of preponderant density and preponderant continuity [3]. We will use the following.

Definition 1. [1, 3] *A point* $x_0 \in \mathbb{R}$ *is said to be the point of preponderant density in Denjoy sense of a measurable set* $E \subset \mathbb{R}$ *if* $\underline{d}(E, x_0) > \frac{1}{2}$.

Similarly, we can define the preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ at the right and at the left. Moreover, a point $x_0 \in \mathbb{R}$ is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in Denjoy sense of the measurable set E at the right and at the left.

Definition 2. [1, 3] *A function* $f\colon U \to \mathbb{R}$ *is said to be preponderantly continuous in Denjoy sense at* $x_0 \in U$ *if there exists a measurable set* $E \subset U$ *containing* x_0 *such that* $\underline{d}(E, x_0) > \frac{1}{2}$ *and* $f_{|E}$ *is continuous at* x_0. *A function* $f\colon U \to \mathbb{R}$ *is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous in Denjoy sense at each point* $x_0 \in U$. *The class of all functions which are preponderantly continuous in Denjoy sense will be denoted by* $\mathcal{PD}$.

Grande [2] defined a property of real functions called the property A_1. Based on this, we may define a similar property, which extends the notion of preponderant continuity.

Definition 3. [2, 3] *A function* $f\colon U \to \mathbb{R}$ *is said to have the property* A_1 *in Denjoy sense at* $x_0 \in U$ *if there exist measurable sets* $E_1 \subset U$ *and* $E_2 \subset U$ *containing* x_0 *such that* x_0 *is the point of preponderant density in Denjoy sense of both sets* E_1 *and* E_2, $f_{|E_1}$ *is upper semi-continuous at* x_0 *and* $f_{|E_2}$ *is lower semi-continuous at* x_0. *A function* $f\colon U \to \mathbb{R}$ *has the property* A_1 *in Denjoy sense if it has the property* A_1 *in Denjoy sense at each* $x_0 \in U$. *The class of all functions which have the property* A_1 *in Denjoy sense will be denoted by* $\mathcal{GPD}$.

Corollary. $\mathcal{PD} \subset \mathcal{GPD}$.

2. Auxiliary lemmas

We will present some known facts and the useful lemma.

Theorem 1. [3, *Corollary 9*] $\mathcal{GPD} \subset \mathcal{B}_1$ *and* $\mathcal{PD} \subset \mathcal{B}_1$, *where* $\mathcal{B}_1$ *is the set of Baire class 1 functions.*

Theorem 2. [3, *Theorem 2*]

(i) *A measurable function* $f\colon U \to \mathbb{R}$ *is preponderantly continuous in Denjoy sense at* $x_0 \in U$ *iff* $\lim_{n\to\infty} \underline{d}\Big(\{x \in U\colon |f(x) - f(x_0)| < \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$,

(ii) A measurable function $f\colon U \to \mathbb{R}$ *has the property* A_1 *in Denjoy sense at* $x_0 \in U$ *iff* $\lim_{n\to\infty} \underline{d}\Big(\{x \in U\colon f(x) < f(x_0) + \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$ *and* $\lim_{n\to\infty} \underline{d}\Big(\{x \in U\colon f(x) > f(x_0) - \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$.

Theorem 3. [3, *Corollary 6*] *Let* $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$*, where* $b_{n+1} < a_n$ *for every* n *and* $x_0 = \lim_{n\to\infty} a_n$*. Then*

1. $\underline{d}^+ (E, x_0) = \liminf_{n\to\infty} \dfrac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}$

2. $\overline{d}^+ (E, x_0) = \limsup_{n\to\infty} \dfrac{\lambda([x_0, b_n] \cap E)}{\lambda([x_0, b_n])}$.

Lemma 1. *Let* $\frac{1}{2} < \gamma < 1$, $x \in \mathbb{R}$ *and let* E *be a measurable subset of* $\mathbb{R}$ *such that* $\overline{d}^+(E, x) = c > 0$*. Then there exists a sequence of closed intervals* $\{I_n = [a_n, b_n]\colon n \geq 1\}$ *for which* $x < \ldots < b_{n+1} < a_n < \ldots$, $d^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) = \gamma$, *and* $\overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \geq \frac{1}{2}c$.

Proof. Let $c_n = x + \frac{1}{n}$ for $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \dfrac{\lambda([c_{n+1}, c_n])}{\lambda([x, c_{n+1}])} = \lim_{n\to\infty} \dfrac{\frac{1}{n(n+1)}}{\frac{1}{n+1}} = 0.$

Put $U_n^1 = [c_{n+1}, c_{n+1} + \gamma(c_n - c_{n+1})]$ and $U_n^2 = [c_n - \gamma(c_n - c_{n+1}), c_n]$ for $n \geq 1$. Then $\lambda(U_n^1) = \lambda(U_n^2) = \gamma\lambda([c_{n+1}, c_n])$, $[c_{n+1}, c_n] = U_n^1 \cup U_n^2$ and $\lambda(E \cap U_n^1) + \lambda(E \cap U_n^2) \geq \lambda(E \cap [c_{n+1}, c_n])$. It follows that for each $n \geq 1$ we can find a closed interval $J_n \subset [c_{n+1}, c_n]$ such that $\lambda(J_n) = \gamma\lambda([c_{n+1}, c_n])$ and $\lambda(E \cap J_n) \geq \frac{1}{2}\lambda(E \cap [c_{n+1}, c_n])$. Hence $\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, c_k]\Big) = \gamma\lambda([x, c_k])$ for $k \geq 1$.

Let $z \in (x, c_1)$. There is $k \geq 1$ such that $z \in [c_{k+1}, c_k]$. Then

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\Big) = \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n\Big) + \lambda(J_k \cap [c_{k+1}, z]) \leq \gamma\lambda([x, z]) + \lambda([c_{k+1}, c_k]),$$

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\Big) = \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n\Big) + \lambda(J_k \cap [c_{k+1}, z]) \geq \gamma\lambda([x, z]) - \lambda([c_{k+1}, c_k])$$

and

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap E \cap [x, z]\Big) \geq \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n \cap E\Big) \geq \frac{1}{2}\lambda([x, z] \cap E) - \lambda([c_{n+1}, c_n]).$$

Therefore

$$\gamma-\tfrac{1}{n}=\gamma-\frac{\frac{1}{n}-\frac{1}{n+1}}{\frac{1}{n+1}}\le\frac{\lambda\Big(\bigcup\limits_{n=1}^{\infty}J_n\cap[x,z]\Big)}{\lambda([x,z])}\le\gamma+\frac{\frac{1}{n}-\frac{1}{n+1}}{\frac{1}{n}}=\gamma+\tfrac{1}{n+1}$$

and

$$\frac{\lambda\Big(\bigcup\limits_{n=1}^{\infty}J_n\cap E\cap[x,z]\Big)}{\lambda([x,z])}\ge\frac{1}{2}\frac{\lambda(E\cap[x,z])}{\lambda([x,z])}-\frac{\frac{1}{n}-\frac{1}{n+1}}{\frac{1}{n}}=\frac{1}{2}\frac{\lambda(E\cap[x,z])}{\lambda([x,z])}-\tfrac{1}{n}.$$

It follows that $d^+\Big(\bigcup\limits_{n=1}^{\infty}J_n,x\Big)=\gamma$ and $\overline{d}^+\Big(\bigcup\limits_{n=1}^{\infty}J_n\cap E,x\Big)\ge\frac{1}{2}\overline{d}^+(E,x)$.

We have proven that $d^+\left(\bigcup\limits_{n=1}^{\infty}J_n,x\right)=\gamma$ and $\overline{d}^+\left(E\cap\bigcup\limits_{n=1}^{\infty}J_n,x\right)\ge\frac{1}{2}\overline{d}^+(E,x)$, but the elements of the sequence need not be disjoint.

Let $\{I_n\colon n\ge 1\}$ be a sequence of closed intervals such that $I_n\subset\operatorname{Int}J_n$ for all $n\in\mathbb{N}$ and $\overline{d}^+\left(\bigcup\limits_{n=1}^{\infty}(J_n\setminus I_n),x\right)=0$. Then the sequence $\{I_n\colon n\ge 1\}$ possesses all the required properties. □

3. $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for $\mathcal{PD}$ and $\mathcal{GPD}$

Definition 4. *For any family $\mathcal{F}$ of functions from U to $\mathbb{R}$ we define*

$$\mathcal{MIN}_{\mathcal{F}}=\{g\colon U\to\mathbb{R}\colon\forall_{f\in\mathcal{F}}\ \min\{f,g\}\in\mathcal{F}\}\,.$$

and

$$\mathcal{MAX}_{\mathcal{F}}=\{g\colon U\to\mathbb{R}\colon\forall_{f\in\mathcal{F}}\ \max\{f,g\}\in\mathcal{F}\}\,.$$

Remark 1. *Observe that* $\max\{f,g\}=-\min\{-f,-g\}$ *and if $\mathcal{F}$ has the property $f\in\mathcal{F}\Rightarrow -f\in\mathcal{F}$, then*

$$\mathcal{MAX}_{\mathcal{F}}=\{g\colon U\to\mathbb{R}\colon\ -g\in\mathcal{MIN}_{\mathcal{F}}\}.$$

We will find $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for $\mathcal{PD}$ and $\mathcal{GPD}$.

Lemma 2. *$\mathcal{MIN}_{\mathcal{PD}}\subset\mathcal{PD}$ and $\mathcal{MIN}_{\mathcal{GPD}}\subset\mathcal{GPD}$.*

Proof. To prove it, it suffices to take any $f\in\mathcal{MIN}_{\mathcal{PD}}$ ($f\in\mathcal{MIN}_{\mathcal{GPD}}$) and for each $x_0\in U$ define a constant function $g(x)=f(x_0)+1$. Then $g\in\mathcal{PD}\cap\mathcal{GPD}$ and, since $\min\{f,g\}\in\mathcal{PD}$ ($\min\{f,g\}\in\mathcal{GPD}$), it is easy to verify, applying Theorem 2, that f is preponderantly continuous in Denjoy sense at x_0 (g has the property A_1 in Denjoy sense at x_0). □

Lemma 3. *If $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$) and g is approximately continuous, then* $\max\{f,g\}, \min\{f,g\} \in \mathcal{PD}$ *(*$\max\{f,g\}, \min\{f,g\} \in \mathcal{GPD}$*).*

Proof. Fix any $x_0 \in U$. Since $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$), there exists a measurable set E (there exist two measurable sets E_1 and E_2) such that $x_0 \in E$ ($x_0 \in E_1 \cap E_2$), x_0 is a point of Denjoy preponderant density of E (of both sets E_1 and E_2) and $f_{|E}$ is continuous at x_0 ($f_{|E_1}$ is upper semi-continuous at x_0 and $f_{|E_2}$ is lower semi-continuous at x_0). Similarly, since g is approximately continuous at x_0, there exists a measurable set F such that $x_0 \in F$, $\underline{d}(F, x_0) = 1$ and $g_{|F}$ is continuous at x_0. Then $\min\{f,g\}_{|E\cap F}$ and $\max\{f,g\}_{|E\cap F}$ are continuous at x_0 ($\min\{f,g\}_{|E_1\cap F}$, $\max\{f,g\}_{|E_1\cap F}$ are upper semi-continuous at x_0 and $\min\{f,g\}_{|E_2\cap F}$, $\max\{f,g\}_{|E_2\cap F}$ are lower semi-continuous at x_0). Moreover, $\underline{d}(E\cap F, x_0) \geq \underline{d}(E, x_0) - \overline{d}(\mathbb{R}\setminus F, x_0) > \frac{1}{2}$ ($\underline{d}(E_1 \cap F, x_0) > \frac{1}{2}$ and $\overline{d}(E_2 \cap F, x_0) > \frac{1}{2}$). It follows that $\min\{f,g\}$ and $\max\{f,g\}$ are preponderantly continuous in Denjoy sense at x_0 ($\min\{f,g\}$ and $\max\{f,g\}$ satisfy the property A_1 in Denjoy sense at x_0). Since x_0 was an arbitrary point, $\min\{f,g\}, \max\{f,g\} \in \mathcal{PD}$ ($\min\{f,g\}, \max\{f,g\} \in \mathcal{GPD}$). □

Lemma 4. *If $g \in \mathcal{PD}$ is not approximately lower semi-continuous at $x_0 \in U$, then there exists $f \in \mathcal{PD}$ such that* $\min\{f,g\} \notin \mathcal{GPD}$*.*

Proof. We may assume that g is not approximately lower semi-continuous at x_0 at the right. Then there exists $\varepsilon > 0$ such that $\overline{d}^{+}(\{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0) = c > 0$. Applying Lemma 1, we can find a sequence of closed intervals $\{I_n = [a_n, b_n] \colon n \geq 1\}$ such that $x_0 < \ldots < b_{n+1} < a_n < \ldots$, $d^{+}\big(\bigcup_{n=1}^{\infty} I_n, x_0\big) = \frac{1}{2} + \frac{1}{4}c$ and $\overline{d}\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) > \frac{1}{2}c$. Pick a sequence of pairwise disjoint closed intervals $\{J_n = [c_n, d_n] \colon n \geq 1\}$ such that $I_n \subset \operatorname{Int}(J_n)$ and $\overline{d}\Big(\bigcup_{n=1}^{\infty}(J_n \setminus I_n), x_0\Big) = 0$. Define a function $f \colon U \to \mathbb{R}$ letting

$$f(y) = \begin{cases} g(x_0) & \text{if} \quad y \in \big(U \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x_0) - 2\varepsilon & \text{if} \quad y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on each interval } [c_n, a_n] \text{ and } [b_n, d_n], & n = 1, 2, \ldots . \end{cases}$$

Obviously, $\min\{f(x_0), g(x_0)\} = g(x_0)$ and $f \in \mathcal{PD}$, because f is continuous at each point except at x_0 and x_0 is a point of preponderant density in Denjoy sense of $\big(E \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n$. Let $E = \{y \colon \min\{f(y), g(y)\} > g(x_0) - \varepsilon\}$.

Then $E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\underline{d}(E, x_0) \le \underline{d}^+(E, x_0) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\Big) + \overline{d}^+\Big(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\Big) =$$

$$= d^+\Big(\bigcup_{n=1}^{\infty} I_n, x_0\Big) - \overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) \le$$

$$\le \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}.$$

This implies that $\min\{f, g\}$ does not have the property A_1 in Denjoy sense at x_0 and $\min\{f, g\} \notin \mathcal{PGD}$, which completes the proof. □

Theorem 4. *$\mathcal{MIN}_{\mathcal{PD}} = \mathbf{A}$, where $\mathbf{A}$ is the set of approximately continuous functions.*

Proof. By Lemma 3, we have inclusion $\mathbf{A} \subset \mathcal{MIN}_{\mathcal{PD}}$.

Suppose that g is not approximately continuous at x_0. If g is not approximately lower semi-continuous at x_0, then applying Lemma 4, we obtain that $g \notin \mathcal{MIN}_{\mathcal{PD}}$. Assume that g is not approximately upper semi-continuous at $x_0 \in U$. Without loss of generality we may assume that g is not approximately upper semi-continuous at x_0 at the right. Then we can find $\varepsilon > 0$ such that $\overline{d}^+\big(\{x > x_0 \colon f(x) > f(x_0) + \varepsilon\}, x_0\big) = c > 0$.

As it was shown earlier, we can find $\varepsilon > 0$ and two sequences $\{I_n = [a_n, b_n] \colon n \ge 1\}$, $\{J_n = [c_n, d_n] \colon n \ge 1\}$ of closed intervals such that $x_0 < \ldots < d_{n+1} < c_n < \ldots$, $I_n \subset \mathrm{Int}(J_n)$ for $n \in \mathbb{N}$, $d^+\Big(\bigcup_{n=1}^{\infty} I_n, x_0\Big) = \frac{1}{2} + \frac{1}{4}c$, $\overline{d}^+\Big(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\Big) = 0$ and $\overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) > f(x_0) + \varepsilon\}, x_0\Big) > \frac{1}{2}c$. Define $f \colon U \to \mathbb{R}$ letting:

$$f(y) = \begin{cases} g(x_0) + 2 \cdot \varepsilon & \text{if} \quad y \in \big(U \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x_0) - 2 \cdot \varepsilon & \text{if} \quad y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on the intervals } [c_n, a_n] \text{ and } [b_n, d_n], & n = 1, 2, \ldots . \end{cases}$$

It is clear that $f \in \mathcal{PD}$, since it is discontinuous only at x_0 and x_0 is a point of preponderant density in Denjoy sense of $\big(U \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n$. Moreover, $\min\{f(x_0), g(x_0)\} = g(x_0)$. Let $E = \{x \in U \colon |\min\{f(y), g(y)\} - g(x_0)| < \varepsilon\}$.

Then $E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\underline{d}(E, x_0) \le \underline{d}^+(E, x_0) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\Big) + \overline{d}^+\Big(\bigcup_{n=1}^{\infty}(J_n \setminus I_n), x_0\Big) =$$

$$= d^+\Big(\bigcup_{n=1}^{\infty} I_n, x_0\Big) - \overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) \le$$

$$\le \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}.$$

Therefore $\min\{f, g\}$ is not Denjoy preponderantly continuous at x_0. It follows that $\min\{f, g\} \notin \mathcal{PD}$. We have proven that if $g \notin \mathbf{A}$, then $g \notin \mathcal{MIN}_{\mathcal{PD}}$. Hence $\mathcal{MIN}_{\mathcal{PD}} \subset \mathbf{A}$, which completes the proof. □

Applying Remark 1, we have:

Corollary.

$$\mathcal{MAX}_{\mathcal{PD}} = \mathbf{A}.$$

Theorem 5. *$\mathcal{MIN}_{\mathcal{GPD}} = \mathcal{GPD} \cap \{f \colon f$ is approximately lower semi-continuous$\}$.*

Proof. Let $g \in \mathcal{MIN}_{\mathcal{GPD}}$. Remark 1 and Lemma 4 imply that $g \in \mathcal{GPD}$ and g is lower semi-continuous.

Let $f, g \colon U \to \mathbb{R}$, $f, g \in \mathcal{GPD}$, $x_0 \in I$ and g be approximately lower semi-continuous at x_0. If $\min\{f(x_0), g(x_0)\} = g(x_0)$, then

$$\{y \in U \colon g(y) < g(x_0) + \varepsilon\} \subset \big\{y \in U \colon \min\{f(y), g(y)\} < g(x_0) + \varepsilon\big\}$$

and if $\min\{f(x_0), g(x_0)\} = f(x_0)$, then

$$\{y \in U \colon f(y) < f(x_0) + \varepsilon\} \subset \big\{y \in U \colon \min\{f(y), g(y)\} < f(x_0) + \varepsilon\big\}$$

for each $\varepsilon > 0$. In both cases x_0 is a point of preponderant density in Denjoy sense of

$$\big\{y \in U \colon \min\{f(y), g(y)\} < \min\{f(x_0), g(x_0)\} + \varepsilon\big\}$$

for each $x_0 \in I$ and each $\varepsilon > 0$, because $f, g \in \mathcal{GPD}$.

On the other hand, the set $\{y \in U \colon g(y) > g(x_0) - \varepsilon\} \cap \{y \colon f(y) > f(x_0) - \varepsilon\}$ is contained in $\big\{y \in U \colon \min\{f(y), g(y)\} > \min\{f(x_0), g(x_0)\} - \varepsilon\big\}$. Since $f \in \mathcal{GPD}$ and g is approximately lower semi-continuous at x_0, we have

$$\underline{d}\big(\{y \colon f(y) > f(x_0) - \varepsilon\}, x_0\big) > \frac{1}{2} \quad \text{and} \quad \underline{d}\big(\{y \colon g(y) > g(x_0) - \varepsilon\}, x_0\big) = 1.$$

Therefore

$$\begin{aligned}&\underline{d}\big(\{y\in U\colon\ \min\{f(y),g(y)\}>\min\{f(x_0),g(x_0)\}-\varepsilon\big\},x_0\big)\geq\\ &\geq \underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big)-\overline{d}\big(\mathbb{R}\setminus\{y\in U\colon g(y)>g(x_0)-\varepsilon\},x_0\big)\geq\\ &\qquad\geq \underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big).\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\underline{d}\big(\big\{y\in U\colon\ \min\{f(y),g(y)\}>\min\{f(x_0),g(x_0)\}-\varepsilon\big\},x_0\big)\geq\\ &\qquad\geq\lim_{\varepsilon\to 0^+}\underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big)>\frac{1}{2}.\end{aligned}$$

It follows that $\min\{f,g\}$ has property A_1 in Denjoy sense at x_0. Since x_0 was an arbitrary point of U, we have $\min\{f,g\}\in\mathcal{GPD}$. Therefore $g\in\mathcal{MIN}_{\mathcal{GPD}}$. This completes the proof. □

Corollary.

$$\mathcal{MAX}_{\mathcal{GPD}}=\mathcal{GPD}\cap\{f\colon f\ \textit{is approximately upper semi-continuous}\}.$$

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