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OPTIMAL CONSUMPTION PROBLEM IN THE VASICEK MODEL

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Abstract. We consider the problem of an optimal consumption strategy on the infinite time horizon based on the hyperbolic absolute risk aversion utility when the interest rate is an Ornstein-Uhlenbeck process. Using the method of subsolution and supersolution we obtain the existence of solutions of the dynamic programming equation. We illustrate the paper with a numerical example of the optimal consumption strategy and the value function.

Keywords: stochastic control, interest rate model, optimal consumption, HJB equation.

Mathematics Subject Classification: 93E20, 60H30.

1. INTRODUCTION

An investor allocates his capital in a bank and consumes an arbitrary part of his wealth in every time moment. The purpose of this paper is to describe an optimal consumption strategy which the investor can follow. In our real world, even for the money in a bank, the interest rate may fluctuate from time to time. Therefore, we consider the Vasicek model. Namely, we assume that the interest rate is an Ornstein-Uhlenbeck process. This enables us to use some results from papers [2] and [12].

In [2], Fleming and Pang introduced the method of subsolution and supersolution to solve a problem of optimal investment and consumption when the interest rate is given by a diffusion process. In contrast to their paper we cannot use a constant function as a subsolution (for more details see Remark 6.2).

Using some sophisticated methods, Synowiec solved in [12] some problems of capital consumption when the interest rate may fluctuate from time to time. Nevertheless, he could not directly use the method of subsolution and supersolution from [2]. In his work the subsolution is of the form $\underline{K}(r) = 0$, so after the logarithmic transformation $Z(r) = \ln K(r)$, which is necessary for the mentioned method, he got $\underline{Z}(r) = -\infty$. In this paper we find a new subsolution for which the method introduced in [2] works.

Similar problems have been considered by many authors (for example, see [1,6,11]). In contrast to this paper in these articles the problem has been studied for a finite horizon. A lot of information on this and related topics can be found in [14] and the references therein. In [9], the problem of optimal investment and consumption, when the interest rate is given by a diffusion process, is solved on the infinite horizon for the logarithmic utility function. It is also worth mentioning that the multidimensional version of the method of subsolution and supersolution, introduced in [2], is discussed in [4] and [5].

Our solution of the consumption problem is based on stochastic control theory. We use the Hamilton-Jacobi-Bellman verification theorem to find the optimal consumption and reduce the problem to an ordinary differential equation. Then we find a subsolution and a supersolution of a related equation. Employing some results of Fleming and Pang given in [2], we prove the existence of its solutions, which gives us the existence of solutions of the ordinary differential equation. Finally, in the last section we give a numerical example of the optimal consumption strategy and the value function.

Our paper is mainly based on [12] which is written in Polish. Nevertheless, there is an English version [13] of this work available, so we refer the reader to it for some proofs.

2. FORMULATION OF THE PROBLEM

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)$ generated by a one-dimensional Brownian motion (W_t). Define r_t as the interest rate (i.e. the rate offered by a bank) at a time $t \geq 0$. Assume that (r_t) is the Ornstein-Uhlenbeck process, which means that it satisfies the following stochastic differential equation

$$\begin{cases} dr_t = (a - br_t)dt + \sigma dW_t, \quad t > 0, \\ r_0 = r \in \mathbb{R}, \end{cases} \tag{2.1}$$

where a, b, $\sigma > 0$ are constants. Suppose that an investor allocates his capital in a bank and consumes an arbitrary part of his wealth in every time moment. Let us denote by (V_t) the wealth process of the investor. Without any loss of generality we assume that the consumption rate is of the form $C_t = c_t V_t$, where the $(\mathcal{F}_t)$-adapted process (c_t) is called the consumption. Then V_t changes according to the differential equation with random coefficients

$$\begin{cases} dV_t = (r_t V_t - C_t)\, dt, \quad t > 0, \\ V_0 = v > 0, \end{cases}$$

so

$$V_t = v \exp\left\{ \int_0^t (r_s - c_s) ds \right\}, \quad t \geq 0.$$

We will use $V_t^{(c,r,v)}$ if we want to emphasize the dependence of V_t on the consumption, the interest rate and the initial value of the investor's wealth. The set of all admissible consumption processes are defined as

$$\mathcal{U} := \{c_t \colon c_t \text{ is } (\mathcal{F}_t)\text{-adapted and } c_t \geq 0\}.$$

Our objective is to find a consumption $\hat{c} \in \mathcal{U}$ such that

$$J^{(\hat{c})}(r,v) := \sup_{c \in \mathcal{U}} J^{(c)}(r,v),$$

where

$$J^{(c)}(r,v) := \mathbb{E}^{(r,v)}\left[\int_0^{+\infty} e^{-\gamma t} U\left(c_t V_t^{(c,r,v)}\right) dt\right]$$

and the utility function is of the form

$$U(C) := \frac{C^\alpha}{\alpha}, \quad \alpha \in (0,1).$$

We call

$$\varphi(r,v) := J^{(\hat{c})}(r,v)$$

the value function and assume that the discount rate γ satisfies the inequality

$$\gamma > \frac{\alpha a}{b} + \frac{\alpha^2 \sigma^2}{2(1-\alpha)b^2}. \tag{2.2}$$

For more information about an appropriate γ selection see Theorem 4.5.

3. SOLUTION BY THE HAMILTON-JACOBI-BELLMAN VERIFICATION THEOREM

Let

$$Qf(r) := \frac{1}{2}\sigma^2 f''(r) + (a - br) f'(r),$$

be the formal generator of the diffusion process given by (2.1). The result below comes from [13, Proposition 1] and is the solution of our problem, if there exists a function $K(r)$ which solves (3.1) and satisfies (3.3). Its proof is based on the Hamilton-Jacobi-Bellman verification theorem (see [8, Theorem 3.1]) and is rather standard, so we give only a sketch of it. More information on stochastic control theory can be found, for example, in [3] and [10].

Theorem 3.1. *Let $K \in C^2(\mathbb{R})$ be such that*

$$QK(r) + (\alpha r - \gamma)K(r) + (1-\alpha)K^{\frac{\alpha}{\alpha-1}}(r) = 0, \quad r \in \mathbb{R}. \tag{3.1}$$

Then

$$\varphi(r,v) = \frac{1}{\alpha} K(r) v^\alpha$$

is the value function and

$$\hat{c} = K^{\frac{1}{\alpha-1}} \tag{3.2}$$

is the optimal consumption whenever for all $r \in \mathbb{R}$ *and* $\hat{c} \in \mathcal{U}$ *we have*

$$\lim_{n\to+\infty} \mathbb{E}\left[e^{-\gamma n} K(r_n) \left(V_n^{(\hat{c},r,v)}\right)^{\alpha}\right] = 0. \tag{3.3}$$

Proof. According to the Hamilton-Jacobi-Bellman verification theorem we look for a function $\varphi \in C^2(\mathbb{R} \times (0, +\infty))$ such that

$$-\gamma\varphi(r,v) + \sup_{c\geq 0}\left\{(r-c)v\varphi_v(r,v) + (a-br)\varphi_r(r,v) + \frac{1}{2}\sigma^2\varphi_{rr}(r,v) + \frac{(cv)^{\alpha}}{\alpha}\right\} = 0. \tag{3.4}$$

We expect $\varphi(r,v)$ to be of the form

$$\varphi(r,v) = \frac{1}{\alpha}K(r)v^{\alpha},$$

for a certain function $K \in C^2(\mathbb{R})$. Then equation (3.4) takes the form

$$-\frac{\gamma}{\alpha}K(r)v^{\alpha} + \sup_{c\geq 0}\left\{\frac{1}{\alpha}QK(r)v^{\alpha} + (r-c)K(r)v^{\alpha} + \frac{(cv)^{\alpha}}{\alpha}\right\} = 0$$

and the supremum is attained at $\hat{c}$ given by (3.2). Hence $K(r)$ satisfies (3.1) and the Hamilton-Jacobi-Bellman verification theorem gives us the desired conclusion. □

Note that the Hamilton-Jacobi-Bellman verification theorem enabled us to find the optimal consumption and reduce the problem to the ordinary differential equation (3.1). In this paper, using the method of subsolution and supersolution introduced in [2], we show that there exists a function $K(r)$ which solves (3.1) and satisfies condition (3.3) of Theorem 3.1.

4. AUXILIARY RESULTS

Let

$$N(r) := \mathbb{E}^{(r)}\left[\int_0^{+\infty} \exp\left\{\frac{1}{1-\alpha}\left(-\gamma t + \alpha\int_0^t r_s ds\right)\right\}dt\right].$$

The following two results come from [13]. The proof of the next theorem with all details can be found in [13, Proposition 4], so we omit it.

Theorem 4.1. *If*

$$N(r) < +\infty \quad \textit{and} \quad \mathbb{E}^{(r)}\left[\int_0^{+\infty} \exp\left\{\frac{2}{1-\alpha}\left(-\gamma t + \alpha\int_0^t r_s ds\right)\right\}dt\right] < +\infty \tag{4.1}$$

for each $r \in \mathbb{R}$, then $N \in C^2(\mathbb{R})$ and

$$QN(r) + \frac{\alpha r - \gamma}{1-\alpha} N(r) = -1. \tag{4.2}$$

Theorem 4.2. *If condition* (4.1) *is fulfilled for each $r \in \mathbb{R}$, then the function*

$$\overline{K}(r) := N^{1-\alpha}(r) \tag{4.3}$$

satisfies

$$Q\overline{K}(r) + (\alpha r - \gamma)\overline{K}(r) + (1-\alpha)\overline{K}^{\frac{\alpha}{\alpha-1}}(r) \le 0, \quad r \in \mathbb{R}. \tag{4.4}$$

Proof. Using (4.2) we get

$$Q\overline{K}(r) + (\alpha r - \gamma)\overline{K}(r) + (1-\alpha)\overline{K}^{\frac{\alpha}{\alpha-1}}(r) = -\frac{\alpha\sigma^2\left(\overline{K}'(r)\right)^2}{2(1-\alpha)\overline{K}(r)} \le 0, \quad r \in \mathbb{R}.$$

□

Theorem 4.3. *The function*

$$\underline{K}(r) := \mathbb{E}^{(r)}\left[\int_0^{+\infty} c^\alpha \exp\left\{-\gamma t + \alpha\int_0^t (r_s - c)ds\right\}dt\right], \tag{4.5}$$

where $c \in (0,1)$, satisfies

$$Q\underline{K}(r) + (\alpha r - \gamma)\underline{K}(r) + (1-\alpha)\underline{K}^{\frac{\alpha}{\alpha-1}}(r) \ge 0, \quad r \in \mathbb{R}. \tag{4.6}$$

Proof. From the Feynman-Kac theorem we deduce that the function $\underline{K}(r)$ satisfies the equation

$$Q\underline{K}(r) + (\alpha r - \gamma - \alpha c)\underline{K}(r) = -c^\alpha,$$

so

$$Q\underline{K}(r) + (\alpha r - \gamma)\underline{K}(r) + (1-\alpha)\underline{K}^{\frac{\alpha}{\alpha-1}}(r) = \alpha c\underline{K}(r) - c^\alpha + (1-\alpha)\underline{K}^{\frac{\alpha}{\alpha-1}}(r).$$

Define

$$f(k) := \alpha ck - c^\alpha + (1-\alpha)k^{\frac{\alpha}{\alpha-1}}.$$

The function $f(k)$ is differentiable for each $k > 0$ and has a minimum at the point $k^* = c^{\alpha-1}$. Note that $f(k^*) = 0$ and

$$\lim_{k\to 0^+} f(k) = \lim_{k\to+\infty} f(k) = +\infty.$$

This means that

$$f(k) \ge 0 \quad \text{for each} \quad k > 0.$$

Since $\underline{K}(r) > 0$ for each $r \in \mathbb{R}$, it follows that

$$\alpha c\underline{K}(r) - c^\alpha + (1-\alpha)\underline{K}^{\frac{\alpha}{\alpha-1}}(r) \ge 0.$$

Thus $\underline{K}(r)$ satisfies (4.6). □

Theorem 4.4. *Suppose that condition* (4.1) *is satisfied and*

$$\lim_{n\to+\infty}\mathbb{E}\left[e^{-\gamma n}\overline{K}(r_n)\left(V_n^{(c,r,v)}\right)^{\alpha}\right]=0 \tag{4.7}$$

for all $r\in\mathbb{R}$ *and* $c\in\mathcal{U}$. *Then*

$$\underline{K}(r)\leq\overline{K}(r),\quad r\in\mathbb{R}.$$

Proof. Since $\overline{K}(r)$ satisfies (4.4) and condition (4.7), using the Hamilton-Jacobi-Bellman verification theorem, we get

$$\varphi(r,v)\leq\frac{1}{\alpha}v^{\alpha}\overline{K}(r),\quad r\in\mathbb{R},\ v>0. \tag{4.8}$$

On the other hand, if we assume that $c_t=c$, where $c\in(0,1)$, we have

$$V_t=v\exp\left\{\int_0^t(r_s-c)ds\right\},\quad t\geq 0,$$

and

$$\begin{aligned}J^{(c)}(r,v)&=\mathbb{E}^{(r,v)}\left[\int_0^{+\infty}e^{-\gamma t}\frac{1}{\alpha}c^{\alpha}v^{\alpha}\exp\left\{\alpha\int_0^t(r_s-c)\,ds\right\}dt\right]\\&=\frac{1}{\alpha}v^{\alpha}\mathbb{E}^{(r)}\left[\int_0^{+\infty}c^{\alpha}\exp\left\{-\gamma t+\alpha\int_0^t(r_s-c)\,ds\right\}dt\right]=\frac{1}{\alpha}v^{\alpha}\underline{K}(r).\end{aligned}$$

Obviously,

$$\frac{1}{\alpha}v^{\alpha}\underline{K}(r)\leq\sup_{c\in\mathcal{U}}J^{(c)}(r,v)=\varphi(r,v),\quad r\in\mathbb{R},\ v>0. \tag{4.9}$$

Finally, from (4.8) and (4.9) we infer that

$$\underline{K}(r)\leq\overline{K}(r),\quad r\in\mathbb{R}.$$

□

The theorem below comes from [13]. For its detailed proof we refer the reader to Steps 1 and 2 in the proof of [13, Theorem 5].

Theorem 4.5. *For a* γ *given by* (2.2), *conditions* (4.1) *and* (4.7) *are satisfied.*

5. METHOD OF SUBSOLUTION AND SUPERSOLUTION

In this section we briefly describe the method of subsolution and supersolution introduced in [2].

Consider a second order differential equation

$$Z''(r)=H(r,Z(r),Z'(r)),\quad r\in\mathbb{R}. \tag{5.1}$$

Definition 5.1. A function $u \in C^2(\mathbb{R})$ is said to be a subsolution of (5.1) if

$$u''(r) \geq H(r, u(r), u'(r)), \quad r \in \mathbb{R},$$

and a supersolution of (5.1) if

$$u''(r) \leq H(r, u(r), u'(r)), \quad r \in \mathbb{R}.$$

Definition 5.2. Let $\underline{Z}(r)$ and $\overline{Z}(r)$ be a subsolution and a supersolution of (5.1), respectively. If

$$\underline{Z}(r) \leq \overline{Z}(r), \quad r \in \mathbb{R},$$

then $(\underline{Z}(r), \overline{Z}(r))$ is said to be a pair of ordered subsolution and supersolution of (5.1).

The method of subsolution and supersolution, which can be found in [2], is summarized in the next theorem.

Theorem 5.3. *Assume that:*

(H1) *function $H(r, z, p)$ is continuous,*
(H2) *function $H(r, z, p)$ is strictly increasing in z,*
(H3) *pair $(\underline{Z}(r), \overline{Z}(r))$ is an ordered subsolution and supersolution of* (5.1)*,*
(H4) *for each $J = [r_1, r_2] \subset \mathbb{R}$, for all $r \in J$ and $|z| \leq 3M$, where*

$$M = \max\left\{\sup_{r \in J} |\underline{Z}(r)|, \sup_{r \in J} |\overline{Z}(r)|\right\},$$

there exist $C_1 > 0$ and $C_2 \geq 0$ such that

$$|H(r, z, p)| \leq C_1(p^2 + C_2).$$

Then there exists a solution $Z(r)$ of equation (5.1) *such that*

$$\underline{Z}(r) \leq Z(r) \leq \overline{Z}(r), \quad r \in \mathbb{R}.$$

6. THE EXISTENCE OF SOLUTION

Consider a second order differential equation

$$Z''(r) = H(r, Z(r), Z'(r)), \quad r \in \mathbb{R}, \tag{6.1}$$

where

$$H(r, z, p) := \frac{2}{\sigma^2 e^z} \hat{H}(r, z, p)$$

and

$$\hat{H}(r, z, p) := -\frac{1}{2}\sigma^2 p^2 e^z - (a - br)pe^z - (\alpha r - \gamma)\, e^z - (1 - \alpha)e^{-\frac{\alpha z}{1-\alpha}}.$$

Theorem 6.1. *The functions* $\underline{Z}(r) = \ln \underline{K}(r)$ *and* $\overline{Z}(r) = \ln \overline{K}(r)$ *are a subsolution and a supersolution of equation* (6.1)*, respectively. Moreover,*

$$\underline{Z}(r) \leq \overline{Z}(r), \quad r \in \mathbb{R}.$$

Proof. For $\overline{Z}(r) = \ln \overline{K}(r)$, we have

$$\frac{1}{2}\sigma^2 e^{\overline{Z}(r)}\overline{Z}_{rr}(r) - \hat{H}(r, \overline{Z}(r), \overline{Z}_r(r)) = Q\overline{K}(r) + (\alpha r - \gamma)\overline{K}(r) + (1-\alpha)\overline{K}^{\frac{\alpha}{\alpha-1}}(r) \leq 0$$

for each $r \in \mathbb{R}$, where the latter inequality follows from Theorem 4.2. As a consequence,

$$\overline{Z}_{rr}(r) \leq H(r, \overline{Z}(r), \overline{Z}_r(r)), \quad r \in \mathbb{R}.$$

Similarly, for $\underline{Z}(r) = \ln \underline{K}(r)$, in view of Theorem 4.3, we get

$$\frac{1}{2}\sigma^2 e^{\underline{Z}(r)}\underline{Z}_{rr}(r) - \hat{H}(r, \underline{Z}(r), \underline{Z}_r(r)) = Q\underline{K}(r) + (\alpha r - \gamma)\underline{K}(r) + (1-\alpha)\underline{K}^{\frac{\alpha}{\alpha-1}}(r) \geq 0$$

for each $r \in \mathbb{R}$. As a result, we obtain

$$\underline{Z}_{rr}(r) \geq H(r, \underline{Z}(r), \underline{Z}_r(r)), \quad r \in \mathbb{R}.$$

Now let us recall that, by Theorem 4.4, we have

$$\underline{K}(r) \leq \overline{K}(r), \quad r \in \mathbb{R}.$$

This means that

$$\underline{Z}(r) = \ln \underline{K}(r) \leq \ln \overline{K}(r) = \overline{Z}(r), \quad r \in \mathbb{R}.$$

□

Remark 6.2. Note that for $\tilde{Z}(r) \equiv D$, where $D \in \mathbb{R}$, we have

$$\frac{1}{2}\sigma^2 e^{\tilde{Z}(r)}\tilde{Z}_{rr}(r) - \hat{H}(r, \tilde{Z}(r), \tilde{Z}_r(r)) = (\alpha r - \gamma)\, e^D + (1-\alpha)e^{-\frac{\alpha D}{1-\alpha}}.$$

Since $r \in \mathbb{R}$, the function $\tilde{Z}(r)$ cannot be a subsolution of equation (6.1).

In the next theorem, we show that there exists a function $K(r)$ which solves (3.1).

Theorem 6.3. *Let* $\underline{Z}(r) = \ln \underline{K}(r)$ *and* $\overline{Z}(r) = \ln \overline{K}(r)$*. Then there exists a solution* $Z(r)$ *of equation* (6.1) *such that*

$$\underline{Z}(r) \leq Z(r) \leq \overline{Z}(r), \quad r \in \mathbb{R}.$$

Moreover, the function

$$K(r) := e^{Z(r)}$$

is a solution of equation (3.1) *such that*

$$\underline{K}(r) \leq K(r) \leq \overline{K}(r), \quad r \in \mathbb{R}.$$

Proof. First, we check all the assumptions of Theorem 5.3 for the function

$$H(r,z,p) = \frac{2}{\sigma^2 e^z}\hat{H}(r,z,p) = \frac{2}{\sigma^2}\left[-\frac{1}{2}\sigma^2 p^2 - (a-br)p - \alpha r + \gamma - (1-\alpha)e^{-\frac{z}{1-\alpha}}\right].$$

1. It is plain that the function $H(r,z,p)$ is continuous.
2. Since
$$H_z(r,z,p) = \frac{2}{\sigma^2}e^{-\frac{z}{1-\alpha}} > 0, \quad z \in \mathbb{R},$$
the function $H(r,z,p)$ is strictly increasing in z.
3. In view of Theorem 6.1, $(\underline{Z}(r), \overline{Z}(r))$ is an ordered subsolution and supersolution of (6.1).
4. For each $J = [r_1, r_2] \subset \mathbb{R}$, for all $r \in J$ and $|z| \le 3M$, where
$$M = \max\left\{\sup_{r\in J}|\underline{Z}(r)|, \sup_{r\in J}|\overline{Z}(r)|\right\},$$
there exist $C_1 > 0$ and $C_2 \ge 0$ such that
$$\begin{aligned}|H(r,z,p)| &= \left|-p^2 - \frac{2(a-br)}{\sigma^2}p - \frac{2\alpha}{\sigma^2}r + \frac{2\gamma}{\sigma^2} - \frac{2(1-\alpha)}{\sigma^2}e^{-\frac{z}{1-\alpha}}\right| \\ &\le p^2 + \frac{2a}{\sigma^2}|p| + \frac{2b\hat{r}}{\sigma^2}|p| + \frac{2\alpha}{\sigma^2}\hat{r} + \frac{2\gamma}{\sigma^2} + \frac{2(1-\alpha)}{\sigma^2}e^{\frac{3M}{1-\alpha}} \\ &\le p^2 + \left(\frac{a}{\sigma^2} + \frac{b\hat{r}}{\sigma^2}\right)^2 + p^2 + \frac{2\alpha}{\sigma^2}\hat{r} + \frac{2\gamma}{\sigma^2} + \frac{2(1-\alpha)}{\sigma^2}e^{\frac{3M}{1-\alpha}} \le C_1(p^2 + C_2),\end{aligned}$$
where
$$\hat{r} = \max\{|r_1|, |r_2|\}.$$

Since all the assumptions of Theorem 5.3 are satisfied, we know that there exists a solution $Z(r)$ of equation (6.1) such that

$$\underline{Z}(r) \le Z(r) \le \overline{Z}(r), \quad r \in \mathbb{R}. \tag{6.2}$$

If $K(r) := e^{Z(r)}$, then $K \in C^2(\mathbb{R})$ and for each $r \in \mathbb{R}$

$$0 = \frac{1}{2}\sigma^2 e^{Z(r)} Z_{rr}(r) - \hat{H}(r, Z(r), Z_r(r)) = QK(r) + (\alpha r - \gamma)K(r) + (1-\alpha)K^{\frac{\alpha}{\alpha-1}}(r).$$

Hence $K(r)$ is a solution of equation (3.1). Moreover, from (6.2) we have

$$\underline{K}(r) \le K(r) \le \overline{K}(r), \quad r \in \mathbb{R}.$$ □

Now we verify that the function $K(r)$ is a part of the value function.

Theorem 6.4. *Let $K \in C^2(\mathbb{R})$ be a solution of equation* (3.1) *such that*

$$\underline{K}(r) \le K(r) \le \overline{K}(r), \quad r \in \mathbb{R}, \tag{6.3}$$

where $\overline{K}(r)$ *and* $\underline{K}(r)$ *are given by* (4.3) *and* (4.5), *respectively. Then*

$$\varphi(r,v)=\frac{1}{\alpha}K(r)v^{\alpha}$$

is the value function and

$$\hat{c}=K^{\frac{1}{1-\alpha}}$$

is the optimal consumption.

Proof. It is sufficient to check whether all the assumptions of Theorem 3.1 are fulfilled. To do this note that, since $\overline{K}(r)$ satisfies condition (3.3) (see Theorem 4.5), using (6.3), we can easily show that $K(r)$ satisfies this condition too. □

7. NUMERICAL EXAMPLE

In this section we give a numerical example of calculating a subsolution, a supersolution and a solution of our problem.

Note that equation (2.1) has the following solution

$$r_t = re^{-bt}+\frac{a}{b}\left(1-e^{-bt}\right)+\sigma\int_0^t e^{b(s-t)}dW_s.$$

Thus

$$r_t = re^{-bt}+r_t^0, \tag{7.1}$$

where (r_t^0) is an Ornstein-Uhlenbeck process with the same coefficients as in the case of the process defined in (2.1) but with the initial value $r_0=0$.

Let us recall that our subsolution is of the form

$$\underline{K}(r):=\mathbb{E}^{(r)}\left[\int_0^{+\infty}c^{\alpha}\exp\left\{-\gamma t+\alpha\int_0^t(r_s-c)ds\right\}dt\right].$$

Using Fubini's theorem we can interchange the order of integration. Taking into account (7.1), we have

$$\begin{aligned}\underline{K}(r)&=\int_0^{+\infty}c^{\alpha}\exp\left\{-\gamma t-\alpha ct\right\}\mathbb{E}^{(r)}\left[\exp\left\{\alpha\int_0^t r_s ds\right\}\right]dt\\&=\int_0^{+\infty}c^{\alpha}\exp\left\{-\gamma t-\alpha ct+\alpha r\int_0^t e^{-bs}ds\right\}\mathbb{E}\left[\exp\left\{\alpha\int_0^t r_s^0 ds\right\}\right]dt\\&=\int_0^{+\infty}c^{\alpha}\exp\left\{-\gamma t-\alpha ct+\frac{\alpha r}{b}\left(1-e^{-bt}\right)\right\}\mathbb{E}\left[\exp\left\{\alpha\int_0^t r_s^0 ds\right\}\right]dt.\end{aligned}$$

Note that

$$\int_0^t r_s^0 ds \sim \mathcal{N}\left(\mu_t, \sigma_t^2\right),$$

where

$$\mu_t = \frac{a}{b^2}\left(e^{-bt} + bt - 1\right), \quad \sigma_t^2 = \frac{\sigma^2}{2b^3}\left(2bt - 3 + 4e^{-bt} - e^{-2bt}\right).$$

This means that

$$\mathbb{E}\left[\exp\left\{\alpha \int_0^t r_s^0 ds\right\}\right] = \exp\left\{\alpha\mu_t + \frac{1}{2}\alpha^2\sigma_t^2\right\},$$

so

$$\underline{K}(r) = \int_0^{+\infty} c^\alpha \exp\left\{-\gamma t - \alpha c t + \frac{\alpha r}{b}\left(1 - e^{-bt}\right) + \alpha\mu_t + \frac{1}{2}\alpha^2\sigma_t^2\right\} dt.$$

Let M be a finite time horizon. In order to make numerical approximations we need to discretize the interval $[0, M]$. Namely, we shall consider a partition of $[0, M]$ defined as follows:

$$0 = t_0 < t_1 < \ldots < t_N = M, \quad N \in \mathbb{N},$$

with the convention

$$\Delta t_j := t_{j+1} - t_j, \quad j \in \{0, 1, \ldots, N-1\}.$$

Then

$$\underline{K}(r) \approx \underline{K}^{(M,N)}(r) := \sum_{j=0}^{N-1} \frac{F(t_j, r) + F(t_{j+1}, r)}{2} \Delta t_j,$$

where

$$F(t, r) := c^\alpha \exp\left\{-\gamma t - \alpha c t + \frac{\alpha r}{b}\left(1 - e^{-bt}\right) + \alpha\mu_t + \frac{1}{2}\alpha^2\sigma_t^2\right\}.$$

Acting along the same lines for a supersolution

$$\overline{K}(r) = \left[\mathbb{E}^{(r)}\left[\int_0^{+\infty} \exp\left\{\frac{1}{1-\alpha}\left(-\gamma t + \alpha \int_0^t r_s ds\right)\right\} dt\right]\right]^{1-\alpha},$$

we get

$$\overline{K}(r) \approx \overline{K}^{(M,N)}(r) := \left[\sum_{j=0}^{N-1} \frac{G(t_j, r) + G(t_{j+1}, r)}{2} \Delta t_j\right]^{1-\alpha},$$

where

$$G(t, r) = \exp\left\{-\frac{\gamma t}{1-\alpha} + \frac{\alpha r}{(1-\alpha)b}\left(1 - e^{-bt}\right) + \frac{\alpha}{1-\alpha}\mu_t + \frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^2 \sigma_t^2\right\}.$$

Now, using Mathematica 9.0, we present numerical solutions for parameters

$$a = 0.5, \quad b = 0.2, \quad c = 0.9, \quad \alpha = 0.5, \quad \sigma = 0.02 \quad \text{and} \quad \gamma = 1.5304.$$

Suppose that $M = 30\,000$, $N = 3\,000\,000$ and the partition of $[0, M]$ is of the form

$$0 < 0.01 < 0.02 < \ldots < 30\,000.$$

For $r \in \{-0.48, -0.47, \ldots, 0.48\}$ we calculate the values of $\underline{K}^{(M,N)}(r)$ and $\overline{K}^{(M,N)}(r)$. It appears that

$$\underline{K}^{(M,N)}(-0.48) \approx 0.46, \quad \underline{K}^{(M,N)}(0.48) \approx 0.58,$$

$$\overline{K}^{(M,N)}(-0.48) \approx 0.55, \quad \overline{K}^{(M,N)}(0.48) \approx 0.64.$$

Then we numerically solve equation (3.1) with the boundary conditions

$$K(-0.48) = 0.55, \qquad K(0.48) = 0.64. \tag{7.2}$$

In Figures 1 and 2, we present the graphs of the functions $\underline{K}^{(M,N)}(r)$, $\overline{K}^{(M,N)}(r)$ and the solution of (3.1) with boundary conditions (7.2).

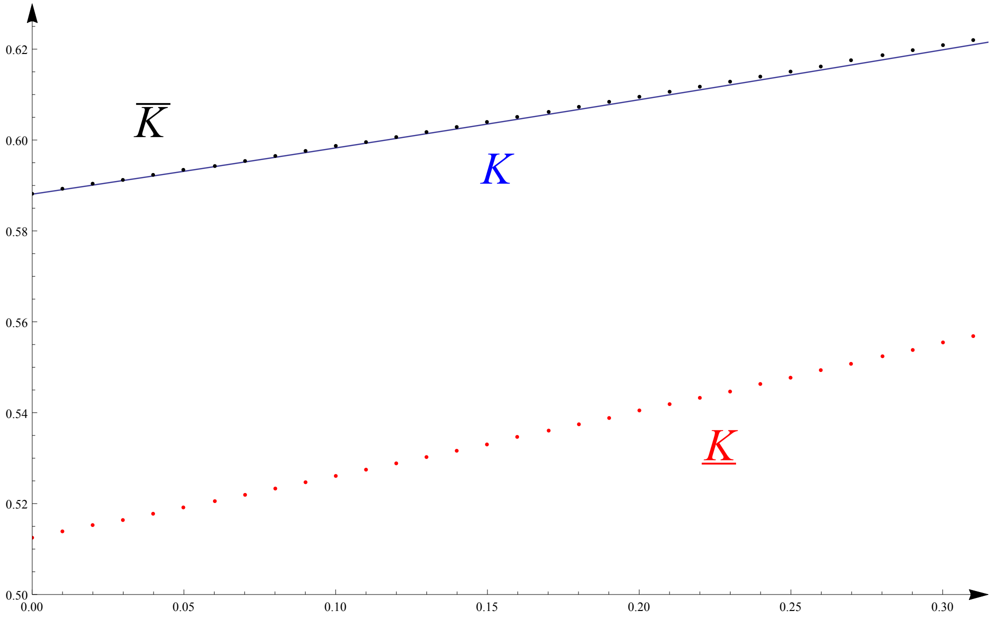

Fig. 1. The continuous blue curve K is the solution of equation (3.1) with boundary conditions (7.2) over the interval $[0, 0.3]$.

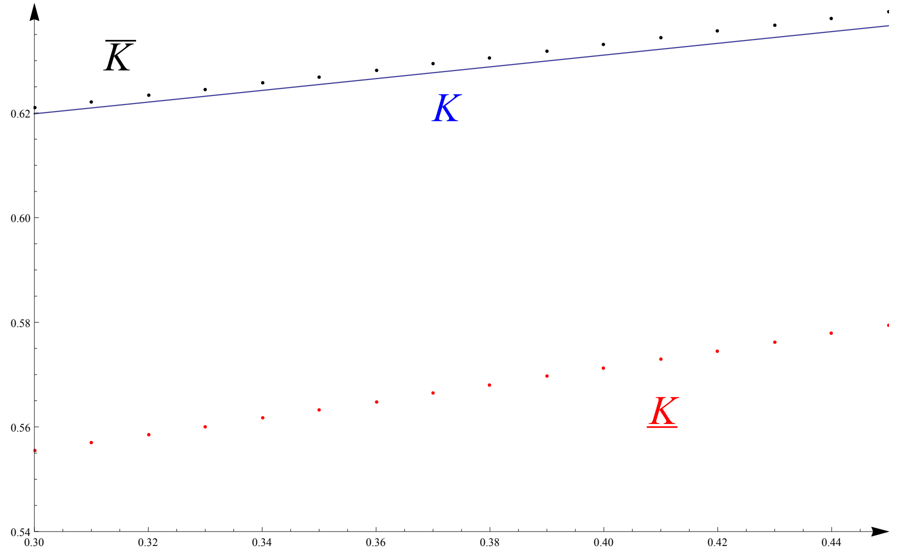

Fig. 2. The continuous blue curve K is the solution of equation (3.1) with boundary conditions (7.2) over the interval $[0.3, 0.45]$.

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