10.1515/acsc-2017-0027

Archives of Control Sciences
Volume 27(LXIII), 2017
No. 3, *pages 441–451*

Relationship between the observability of standard and fractional linear systems

TADEUSZ KACZOREK

The relationship between the observability of standard and fractional discrete-time and continuous-time linear systems are addressed. It is shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

Key words: fractional, standard, linear, discrete-time, continuous-time, system, observability.

1. Introduction

The notion of controllability and observability of linear systems have been introduced by Kalman [14, 15]. Those notions are the basic concepts of the modern control theory [1, 6, 13, 16, 21, 24, 25]. They have been extended to positive and fractional linear and nonlinear systems [2, 4, 5, 7-11, 22, 23]. The mathematical fundamentals of fractional calculus are given in the monographs [18-20]. The positive fractional linear systems have been introduced in [8, 11].

In the paper [17] it has been shown that the fractional discrete-time and continuous-time linear systems are controllable if and only if the standard discrete-time and continuous-time systems are controllable.

In this paper it will be shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning standard and fractional discrete-time and continuous-time linear systems are recalled. The relationship between the observability of the standard and fractional discrete-time linear systems is considered in section 3 and of continuous-time linear systems in section 4. Concluding remarks are given in section 5.

The authors is with Bialystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Bialystok, Poland. E-mail: kaczorek@ee.pw.edu.pl

The studies have been carried out in the framework of work No. S/WE/1/2016 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

Received 01.03.2017.

The following notation will be used: $\Re^{n\times m}$ is the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$, Z_+ is the set of nonnegative integers, I_n is the $n\times n$ identity matrix.

2. Preliminaries

Consider the standard discrete-time linear system

$$x_{i+1}=Ax_i+Bu_i,\quad i\in Z_+=\{0,1,...\},\tag{1a}$$

$$y_i=Cx_i,\tag{1b}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

The solution to the equation (1a) is given by

$$x_i=A^ix_0+\sum_{j=0}^{i-1}A^{i-j-1}Bu_j.\tag{2}$$

Substituting (2) into (1b) we obtain

$$y_i=CA^ix_0+\sum_{j=0}^{i-1}CA^{i-j-1}Bu_j.\tag{3}$$

Now let us consider the fractional discrete-time linear system

$$\Delta^{\alpha}x_{i+1}=Ax_i+Bu_i,\quad 0<\alpha<2,\tag{4a}$$

$$y_i=Cx_i,\tag{4b}$$

where

$$\Delta^{\alpha}x_i=\sum_{j=0}^{i}(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix}x_{i-j},\tag{4c}$$

$$\begin{pmatrix}\alpha\\ j\end{pmatrix}=\begin{cases}1 & \text{for}\quad j=0\\ \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for}\quad j=1,2,...\end{cases}\tag{4d}$$

is the fractional α-order difference of x_i and $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

Substitution of (4c) into (4a) yields

$$x_{i+1}=(A+I_n\alpha)x_i+\sum_{j=2}^{i+1}c_jx_{i-j+1}+Bu_i,\quad i\in Z_+,\tag{5a}$$

where

$$c_j = c_j(\alpha) = (-1)^{j+1}\begin{pmatrix}\alpha\\ j\end{pmatrix}, \quad j = 2,3,\dots \tag{5b}$$

The solution to the equation (5a) has the form [11]

$$x_{i+1} = (A + I_n\alpha)x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, \quad i \in Z_+, \tag{6a}$$

where

$$\Phi_{j+1} = \Phi_j(A + I_n\alpha) + \sum_{k=2}^{j+1} c_k \Phi_{j-k+1}, \quad \Phi_0 = I_n \tag{6b}$$

and c_k is defined by (5b).

Substituting (6a) into (4b) we obtain

$$y_i = C\Phi_i x_0 + \sum_{j=0}^{i-1} C\Phi_{i-j-1}Bu_j. \tag{7}$$

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{8a}$$

$$y(t) = Cx(t), \tag{8b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

The solution to the equation (8a) has the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{9}$$

and

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau. \tag{10}$$

Now let us consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 2 \tag{11a}$$

$$y(t) = Cx(t), \tag{11b}$$

where

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n} \tag{12}$$

is the Caputo fractional derivative of order $n-1<\alpha<n$ ($n \in N$) of $x(t)$, $\Gamma(x)$ is the Euler gamma function, $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

The solution of the equation (11a) is given by [11]

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x_0 = x(0), \tag{13a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{13b}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{13c}$$

and

$$y(t) = C\Phi_0(t)x_0 + \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau. \tag{14}$$

Theorem 4 (Cayley-Hamilton) *Let* $A \in \Re^{n\times n}$ *and*

$$\det[I_n\lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0. \tag{15}$$

Then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n = 0. \tag{16}$$

Proof Proof is given in [3, 12].

Theorem 5 (Kronecker-Capelli) *The linear matrix equation*

$$Ax = b, \quad A \in \Re^{n\times n}, \quad b \in \Re^n \tag{17}$$

has a solution $x \in \Re^n$ *if and only if*

$$\operatorname{rank}[A, b] = \operatorname{rank} A. \tag{18}$$

Proof Proof is given in [12].

3. Observability of standard and fractional discrete-time linear systems

It is well-known [1, 2, 7] that the observability of the standard and fractional linear systems depends only of the pair (A,C) and it is independent of the matrix B.

Definition 13 *The standard linear discrete-time linear system* (1) *is called observable in the interval* $[0,q]$ *if knowing the output* y_i *for* $i = 0, 1, ..., q-1$, $q \leqslant n$, *it is possible to find the unique* x_0 *of the system.*

Theorem 6 *The standard linear discrete-time linear system* (1) *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{19}$$

Proof Proof is given in [1, 6, 13].

Definition 14 *The fractional discrete-time linear system* (4) *is called observable in the interval* $[0,q]$ *if knowing the output* y_i *for* $i = 0, 1, ..., q-1$, $q < n$, *it is possible to find the unique* x_0 *of the system.*

We shall show that the fractional discrete-time linear system (4) is observable in the interval $[0,q]$ if and only if the standard linear discrete-time system (1) is observable in the same interval.

From (7) for $B = 0$ and (6b) for $i = 0, 1, ..., q-1$ we have

$$y_{0q} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ \vdots \\ C\Phi_{q-1} \end{bmatrix} x_0 = O_{0q}x_0, \tag{20a}$$

where

$$O_{0q} = \begin{bmatrix} C \\ C(A + I_n\alpha) \\ C[(A + I_n\alpha)^2 + c_2 I_n] \\ \vdots \\ C[(A + I_n\alpha)^{q-1} + ... + (\alpha^{q-1} + ... + c_{q-1})I_n] \end{bmatrix}. \tag{20b}$$

By Kronecker-Capelli theorem the equation (20a) has a unique solution x_0 for any given y_{0q} if and only if

$$\operatorname{rank} O_{0q} = n. \tag{20c}$$

Therefore, the following theorem has been proved.

Theorem 7 *The fractional discrete-time linear system* (4) *or equivalently* (5a), (4b), *is observable in the interval* $[0,q]$ *if and only if the condition* (20c) *is satisfied.*

It will be shown that the condition (20c) is equivalent to the condition (19). Note that

$$O_{0q} = \begin{bmatrix} C \\ C(A+I_n\alpha) \\ C[(A+I_n\alpha)^2 + c_2 I_n] \\ \vdots \\ C[(A+I_n\alpha)^{q-1} + \ldots + (\alpha^{q-1} + \ldots + c_{q-1})I_n] \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ \alpha I_n & I_n & 0 & \cdots & 0 \\ (c_2+\alpha^2)I_n & 2\alpha I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (c_{q-1}+\ldots+\alpha^{q-1})I_n & \cdots & \cdots & \cdots & I_n \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \tag{21}$$

since

$$(A+I_n\alpha)^k = A^k + k\alpha A^{k-1} + \ldots + \alpha^k I_n \text{ for } k = 2,3,\ldots,q-1. \tag{22}$$

From (21) it follows that

$$\operatorname{rank} O_{0q} = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \tag{23}$$

since the matrix

$$\begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ \alpha I_n & I_n & 0 & \cdots & 0 \\ (c_2+\alpha^2)I_n & 2\alpha I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (c_{q-1}+\ldots+\alpha^{q-1})I_n & \cdots & \cdots & \cdots & I_n \end{bmatrix} \tag{24}$$

is nonsingular for all values of α and c_k, $k = 1,2,\ldots,q-1$. Therefore, the following theorem has been proved.

Theorem 8 *The fractional discrete-time linear system* (4) *is observable in the interval* $[0,q]$*,* $q \leqslant n$*, if and only if the standard discrete-time linear system* (1) *is observable in the same interval* $[0,q]$.

Example 1 Consider the standard system (1) and the fractional system (4) for $\alpha = 0.5$ with the same matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad C = [\ 1 \quad 1\ ]. \tag{25}$$

Using (19) and (25) for $q = 2$ we obtain

$$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = 2 \tag{26}$$

and by Theorem 6 the standard system is observable in the interval $[0,2]$.

For the fractional system with (25) using (20b) we obtain

$$\operatorname{rank}\begin{bmatrix} C \\ C(A+\alpha I_2) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ -0.5 & -1.5 \end{bmatrix} = 2. \tag{27}$$

By Theorem 7 the fractional system with (25) is also observable in the interval $[0,2]$.

4. Observability of standard and fractional continuous-time linear systems

Definition 15 *The standard continuous-time linear system* (8) *is called observable in the interval* $[0,t_f]$ *if knowing the output* $y(t)$ *for* $t \in [0,t_f]$ *it is possible to find the unique* x_0 *of the system.*

Theorem 9 *The standard continuous-time linear system* (8) *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{28}$$

Proof Proof is given in [1, 6, 13].

Definition 16 *The fractional continuous-time linear system* (11) *is called observable in the interval* $[0,t_f]$ *if knowing the output* $y(t)$ *for* $t \in [0,t_f]$ *it is possible to find the unique* x_0 *of the system.*

We shall show that the fractional continuous-time linear system (11) is observable in the interval $[0,t_f]$ if and only if the standard continuous-time linear system (8) is observable in the same interval.

Using the Cayley-Hamilton theorem (the equality (10)) it is possible to eliminate the powers $k = n, n+1, \dots$ of the matrix A^k in (13b) and we obtain

$$\Phi_0(t) = \sum_{k=0}^{n-1} c_k(t)A^k. \tag{29}$$

The coefficients c_k in (29) can be computed as follows.

To simplify the calculations it is assumed the eigenvalues λ_k of the matrix A are distinct, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case using (29) we obtain

$$\begin{bmatrix} \Phi_0(\lambda_1) \\ \Phi_0(\lambda_2) \\ \vdots \\ \Phi_0(\lambda_n) \end{bmatrix} = H \begin{bmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix}, \tag{30}$$

where

$$H = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}. \tag{31}$$

If the eigenvalues are distinct, then the matrix (31) is nonsingular and from (30) we have

$$\begin{bmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix} = H^{-1} \begin{bmatrix} \Phi_0(\lambda_1) \\ \Phi_0(\lambda_2) \\ \vdots \\ \Phi_0(\lambda_n) \end{bmatrix}. \tag{32}$$

The coefficients $c_k(t)$, $k = 0, 1, ..., n-1$ can be also found using the well-known Lagrange-Sylvester formula [3, 12].

Substitution of (29) into (14) for $B = 0$ yields

$$y(t) = C\Phi_0(t)x_0 = \sum_{k=0}^{n-1} c_k(t)CA^k = \begin{bmatrix} c_0(t) & c_1(t) & \cdots & c_{n-1}(t) \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0. \tag{33}$$

From (33) it follows that it is possible to find $y(t)$ for given $t \in [0, t_f]$, if and only if

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{34}$$

since $c_k(t) \neq 0$ for $t \in [0, t_f]$. Therefore, the following theorem has been proved.

Theorem 10 *The fractional continuous-time linear system* (11) *is observable in the interval* $[0, t_f]$ *if and only if the standard continuous-time linear system* (8) *is observable in the same interval.*

Example 2 Consider the standard system (8) and the fractional system (11) with the same matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{35}$$

Using (28) and (35) we obtain

$$\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \tag{36}$$

and by Theorem 10 the standard system is observable. In this case for the fractional system (11) with (35) we obtain

$$\Phi_0(t) = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)} = I_2 + \frac{At^\alpha}{\alpha} = \begin{bmatrix} 1 & \frac{t^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix} = c_0(t)I_2 + c_1(t)A, \tag{37}$$

where

$$c_0(t) = 1, \ c_1(t) = \frac{t^\alpha}{\alpha}. \tag{38}$$

By Theorem 10 the fractional system is also observable.

5. Concluding remarks

The relationship between the observability of the standard and fractional discrete-time and continuous-time linear systems has been addressed. It has been shown that: 1) the fractional discrete-time linear systems are observable if and only if the standard discrete-time linear systems are observable (Theorem 8); 2) the fractional continuous-time linear systems are observable if and only if the standard continuous-time linear systems are observable (Theorem 10). The considerations have been illustrated by numerical examples. The considerations can be extended to the standard and fractional time-varying linear systems.

References

[1] P. ANTSAKLIS and A. MICHEL: Linear Systems. Birkhauser, Boston, 2006.

[2] L. FARINA and S. RINALDI: Positive Linear Systems: Theory and Applications. J. Wiley & Sons, New York, 2000.

[3] F.R. GANTMACHER: The Theory of Matrices. Chelsea Pub. Comp., London, 1959.

[4] T. KACZOREK: Constructability and observability of standard and positive electrical circuits. *Electrical Review*, **89**(7) (2013), 132-136.

[5] T. KACZOREK: Controllability and observability of linear electrical circuits *Electrical Review*, **87**(9a) (2011), 248-254.

[6] T. KACZOREK: Linear Control Systems. Vol. 1, J. Wiley, New York, 1999.

[7] T. KACZOREK: Positive 1D and 2D Systems. Springer-Verlag, London, 2002.

[8] T. KACZOREK: Positive linear systems consisting of n subsystems with different fractional orders. *IEEE Trans. Circuits and Systems*, 58(6) (2011), 1203-1210.

[9] T. KACZOREK: Reachability and controllability to zero tests for standard and positive fractional discrete-time systems. *Journal Européen des Systemes Automatisés*, JESA, **42**(6-8) (2008), 769-787.

[10] T. KACZOREK: Reachability and observability of fractional positive electrical circuits. *Computational Problems of Electrical Engineering*, **23**(2) (2013), 28-36.

[11] T. KACZOREK: Selected Problems of Fractional Systems Theory. Springer-Verlag, Berlin, 2011.

[12] T. KACZOREK: Vectors and Matrices in Automation and Electrotechnics. WNT, Warsaw, 1998, (in Polish).

[13] T. KAILATH: Linear Systems. Prentice Hall, Englewood Cliffs, New Yok, 1980.

[14] R. KALMAN: Mathematical description of linear systems. *SIAM J. Control*, **1**(2) (1963), 152-192.

[15] R. KALMAN: On the general theory of control systems. *Prof. First Int. Congress on Automatic Control*, Butterworth, London, (1960), 481-493.

[16] J. KLAMKA: Controllability of Dynamical Systems. Kluwer, Academic Press, Dordrecht, 1991.

[17] J. KLAMKA: Relationship between controllability of standard and fractional linear systems. Submitted to KKA, Krakow, (2017).

[18] K. OLDHAM and J. SPANIER: The Fractional Calculus: Integrations and Differentiations of Arbitrary Order. Academic Press, New York, 1974.

[19] P. OSTALCZYK: Epitome of the Fractional Calculus, Theory and its Applications in Automatics. Technical University of Lodz Press, Lodz, 2008 (in Polish).

[20] I. PODLUBNY: Fractional Differential Equations. Academic Press, San Diego, 1999.

[21] H. ROSENBROCK: State-space and Multivariable Theory. J. Wiley, New York, 1970.

[22] Ł. SAJEWSKI: Reachability of fractional positive continuous-time linear systems with two different fractional orders. *In: Recent Advances in Automation, Robotics and Measuring Techniques, Series in Advances in Intelligent Systems and Computing*, **267** (2014), 239-249.

[23] Ł. SAJEWSKI: Reachability, observability and minimum energy control of fractional positive continuous-time linear systems with two different fractional orders. *Multidimensional Systems and Signal Processing*, **27**(1), (2016), 27-41.

[24] W. WOLOVICH: Linear Multivariable Systems. Springer-Verlag, New York, 1974.

[25] S.H. ŻAK: Systems and Control. Oxford University Press, New York, 2003.