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GLOBAL STABILITY OF DISCRETE–TIME FEEDBACK NONLINEAR SYSTEMS WITH DESCRIPTOR POSITIVE LINEAR PARTS AND INTERVAL STATE MATRICES

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The global stability of discrete-time nonlinear systems with descriptor positive linear parts, positive scalar feedbacks and interval state matrices is addressed. Sufficient conditions for the global stability of this class of nonlinear systems are established. The effectiveness of these conditions is illustrated using numerical examples.

Keywords: global stability, positive systems, nonlinear systems, discrete-time systems, feedback.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology, medicine, etc. An overview of the state of the art in positive systems theory is given in the monographs of Berman and Plemmous (1994), Farina and Rinaldi (2000) or Kaczorek (2011).

Descriptor positive systems were analyzed by Borawski (2017) and Sajewski (2017). The positivity and stability of linear electrical circuits were investigated by Kaczorek and Rogowski (2015), and Borawski (2017). The stability analysis of positive descriptor linear systems were presented by Rami and Napp (2012) as well as Virnik (2008). The stability of linear and nonlinear positive systems was addressed by Kaczorek (2019a). The exponential stability for positive linear discrete-time systems in ordered Banach spaces was studied by Gluck and Mironchenko (2020). The global stability of nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts was investigated by Kaczorek (2019b), who also analysed the global stability of nonlinear continuous-time standard and fractional positive systems (Kaczorek, 2020). The constrained regulation problem for fractional-order nonlinear continuous-time systems was investigated by Si *et al.* (2021).

Realistic dynamic systems are nonlinear and usually only interval parameters of linear part are known. The problem of global stability of nonlinear systems with interval matrices of their linear parts is very important and topical. The stability problem of interval linear systems was analyzed by Kharitonov (1978).

In this paper the global stability of discrete-time nonlinear systems with positive descriptor linear parts, positive scalar feedbacks and interval state matrices will be addressed. Electrical circuits with positive linear parts and nonlinear elements are examples of such systems. Two examples of positive nonlinear electrical circuits were presented by Kaczorek (2021).

The paper is organized as follows. In Section 2 basic definitions and theorems concerning positive discrete-time descriptor linear systems are recalled. The stability of positive interval linear systems is considered in Section 3. New sufficient conditions for the global stability of nonlinear feedback systems with positive linear descriptor discrete-time and interval state matrices

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are established in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n\times m$ real matrices; $\mathbb{R}_+^{n\times m}$, the set of $n\times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$; I_n, the $n\times n$ identity matrix.

2. Positive discrete-time descriptor linear systems

Consider the descriptor discrete-time linear system

$$Ex_{i+1}=Ax_i+Bu_i,\quad i=0,1,\dots, \tag{1}$$

$$y_i=Cx_i, \tag{2}$$

where $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$, $y_i\in\mathbb{R}^p$ are the state, input and output vectors and $E,A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$. It is assumed that the pencil (E,A) of (1) is regular, i.e.,

$$\det[Ez-A]\neq 0\quad\text{for some } z\in\mathbb{C}, \tag{3}$$

where $\mathbb{C}$ is the field of complex numbers.

Definition 1. The descriptor system (1), (2) is called (internally) *positive* if $x_i\in\mathbb{R}_+^n$, $y_i\in\mathbb{R}_+^p$, $i=0,1,\dots$ for every consistent nonnegative initial conditions $x_0\in\mathbb{R}_+^n$ and all inputs $u_i\in\mathbb{R}_+^m$.

It is assumed that the singular matrix E has only $n_1<n$ linearly independent columns and the pencil (E,A) is regular. In this case, by the Weierstrass–Kronecker theorem (Dai, 1989; Kaczorek and Rogowski, 2015; Virnik, 2008), there exist nonsingular monomial matrices $P\in\mathbb{R}^{n\times n}$ and $Q\in\mathbb{R}^{n\times n}$ (in each row and in each column only one entry is positive and the remaining entries are zero) such that

$$PEQ=\begin{bmatrix} I_{n_1} & 0\\ 0 & N\end{bmatrix},\quad PAQ=\begin{bmatrix} A_1 & 0\\ 0 & I_{n_2}\end{bmatrix}, \tag{4}$$

$n=n_1+n_2$, where $N\in\mathbb{R}^{n_2\times n_2}$ is the nilpotent matrix such that $N^\mu=0$, $N^{\mu-1}\neq 0$, μ is the nilpotency index, $A_1\in\mathbb{R}^{n_1\times n_1}$ and $n_1=\deg\det[Es-A]=\operatorname{rank}E$.

Premultiplying (1) by the matrix $P\in\mathbb{R}^{n\times n}$ and defining the new state vector

$$\begin{bmatrix} x_{1i}\\ x_{2i}\end{bmatrix}=Q^{-1}x_i,\quad x_{1i}\in\mathbb{R}^{n_1},\quad x_{2i}\in\mathbb{R}^{n_2},\quad i=0,1,\dots \tag{5}$$

we obtain

$$x_{1,i+1}=A_1x_{1i}+B_1u_i, \tag{6}$$

$$Nx_{2,i+1}=x_{2i}+B_2u_i, \tag{7}$$

where $A_1\in\mathbb{R}^{n_1\times n_1}$, $B_1\in\mathbb{R}^{n_1\times m}$, $B_2\in\mathbb{R}^{n_2\times m}$ and

$$\begin{bmatrix} B_1\\ B_2\end{bmatrix}=PB.$$

Note that if $Q\in\mathbb{R}_+^{n\times n}$ is monomial then $Q^{-1}\in\mathbb{R}_+^{n\times n}$ and $x_{1i}\in\mathbb{R}_+^{n_1}$ and $x_{2i}\in\mathbb{R}_+^{n_2}$ for $i=0,1,\dots$ if $x_i\in\mathbb{R}_+^n$, $i=0,1,\dots$. Defining $CQ=[\,C_1\quad C_2\,]$, $C_1\in\mathbb{R}_+^{p\times n_1}$, $C_2\in\mathbb{R}^{p\times n_2}$ for any $C\in\mathbb{R}_+^{p\times n}$ from (2) we have

$$y_i=C_1x_{1i}+C_2x_{2i}. \tag{8}$$

The transfer matrix of the system (1), (2) is given by

$$T(z)=C[Ez-A]^{-1}B\in\mathbb{R}^{p\times m}(z), \tag{9}$$

where $\mathbb{R}^{p\times m}(z)$ is the set of $p\times m$ rational matrices in z. It is easy to verify that

$$\begin{aligned} T(z)&=C[Ez-A]^{-1}B\\ &=CQ[P(Ez-A)Q]^{-1}PB\\ &=\begin{bmatrix} C_1 & C_2\end{bmatrix}\begin{bmatrix} I_{n_1}z-A_1 & 0\\ 0 & Nz-I_{n_2}\end{bmatrix}^{-1}\\ &\quad\cdot\begin{bmatrix} B_1\\ B_2\end{bmatrix}\\ &=C_1[I_{n_1}z-A_1]^{-1}B_1-C_2[I_{n_2}+Nz\\ &\quad+\dots+N^{\mu-1}z^{\mu-1}]B_2. \end{aligned} \tag{10}$$

From the above we have the following result.

Theorem 1. (Kaczorek and Rogowski, 2015) *The descriptor discrete-time system (1), (2) is positive if and only if*

$$A_1\in\mathbb{R}_+^{n_1\times n_1},\quad B_1\in\mathbb{R}_+^{n_1\times m},\quad -B_2\in\mathbb{R}_+^{n_2\times m}, \tag{11}$$

$$C_1\in\mathbb{R}_+^{p\times n_1},\quad C_2\in\mathbb{R}_+^{p\times n_2}.$$

Theorem 2. (Kaczorek, 2011) *The positive linear discrete-time system (6) is asymptotically stable (the matrix A_1 is Schur) if and only if one of the following equivalent conditions are satisfied:*

1. *All coefficients of the characteristic polynomial*

$$\begin{aligned} p_{n_1}(z)&=\det[I_{n_1}(z+1)-A_1]\\ &=z^{n_1}+a_{n-1}z^{n_1-1}+\dots\\ &\quad+a_1z+a_0 \end{aligned} \tag{12}$$

are positive, i.e., $a_i>0$ for $i=0,1,\dots,n-1$.

2. *There exists a strictly positive vector $\lambda^T=[\,\lambda_1\ \cdots\ \lambda_n\,]$, $\lambda_k>0$, $k=1,\dots,n$ such that*

$$(A_1-I_{n_1})\lambda<0\quad or\quad \lambda^T(A_1-I_{n_1})<0. \tag{13}$$

From (13) we immediately have the necessary condition for asymptotic stability of the system (6).

Theorem 3. *The positive linear discrete-time system (6) is asymptotically stable if the sum of the entries of each column (row) of the matrix A_1 is less than one.*

Proof. The proof follows from condition (13) for $\lambda^T = [\,1 \;\; \cdots \;\; 1\,]$ since $[\,1 \;\; \cdots \;\; 1\,]^T A_1 < [\,1 \;\; \cdots \;\; 1\,]$ if the sum of entries of each column of the matrix A_1 is less than 1. The proof for rows is similar. ∎

3. Stability of positive interval linear systems

Consider the interval positive linear discrete-time system described by the homogeneous state equation

$$x_{i+1} = A x_i, \tag{14}$$

where $x_i \in \mathbb{R}^n_+$ is the state vector and the matrix $A \in \mathbb{R}^{n\times n}_+$ is the interval matrix in which all entries are only known to within a specific closed intervals defined as follows:

$$\begin{aligned} A &= [\underline{A},\ \overline{A}] \\ &= \{A = [a_{ij}],\ a_{ij} \in [\underline{a}_{ij},\ \overline{a}_{ij}], \\ &\qquad i,j = 1,2,\ \dots, n\}, \end{aligned} \tag{15}$$

where $\underline{a}_{ij}$ and $\overline{a}_{ij}$ are entries of matrices $\underline{A}$ and $\overline{A}$. The matrices $\underline{A}$ and $\overline{A}$ are the left and right bounds of the matrix A. The condition $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}$ is the most common case. However, in this paper we consider the general case defined in (15).

A special case of the interval matrix is the matrix of the form

$$A = (1-q)\underline{A} + q\overline{A} \quad \text{for } 0 \le q \le 1. \tag{16}$$

Each entry a_{ij} of the interval matrix (16) is a convex combination of the entries $\underline{a}_{ij}$ and $\overline{a}_{ij}$ of the matrices $\underline{A}$ and $\overline{A}$. The system (14) with matrix (15) or (16) is called the interval system.

Definition 2. The interval positive system (14) is called *asymptotically stable* if it is asymptotically stable for all matrices $A \in \mathbb{R}^{n\times n}_+$ satisfying the condition (15).

By the condition (13) the positive system (14) is asymptotically stable if and only if there exists a strictly positive vector $\lambda > 0$ such that (13) holds.

For two positive linear systems

$$x_{i+1} = \underline{A} x_i, \quad \underline{A} \in \mathbb{R}^{n\times n}_+ \tag{17}$$

and

$$x_{i+1} = \overline{A} x_i, \quad \overline{A} \in \mathbb{R}^{n\times n}_+ \tag{18}$$

there exists a strictly positive vector $\lambda \in \mathbb{R}^n_+$ such that

$$\underline{A}\lambda < \lambda \quad \text{and} \quad \overline{A}\lambda < \lambda \tag{19}$$

if and only if the systems (17) and (18) are asymptotically stable.

Example 1. Consider the positive linear system (14) with the matrices

$$\underline{A} = \begin{bmatrix} 0.6 & 0.1 \\ 0.3 & 0.3 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0.7 & 0.2 \\ 0.4 & 0.5 \end{bmatrix}. \tag{20}$$

Note that for $\lambda^T = [1 \;\; 1]$ we have

$$\underline{A}\lambda = \begin{bmatrix} 0.6 & 0.1 \\ 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{21}$$

$$\overline{A}\lambda = \begin{bmatrix} 0.7 & 0.2 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{22}$$

Therefore, by the condition (13) the positive system (14) with interval state matrix (20) is asymptotically stable. ♦

Theorem 4. *If the matrices $\underline{A} \in \mathbb{R}^{n\times n}_+$ and $\overline{A} \in \mathbb{R}^{n\times n}_+$ of positive systems (17) and (18) are asymptotically stable then their convex linear combination*

$$A = (1-q)\underline{A} + q\overline{A} \quad \textit{for} \quad 0 \le q \le 1 \tag{23}$$

is also asymptotically stable.

Proof. By (13), if the positive linear systems (17) and (18) are asymptotically stable, then there exists a strictly positive vector $\lambda \in \mathbb{R}^n_+$ such that (19) holds. Using (23) and (19), we obtain

$$\begin{aligned} A\lambda &= [(1-q)A_1 + qA_2]\lambda \\ &= (1-q)A_1\lambda + qA_2\lambda \\ &< (1-q)\lambda + q\lambda = \lambda \end{aligned} \tag{24}$$

for $0 \le q \le 1$.

Therefore, if the positive linear systems (17) and (18) are asymptotically stable and (19) holds, then their convex linear combination is also asymptotically stable. ∎

Theorem 5. *The interval positive system (14) is asymptotically stable if and only if the positive systems (17) and (18) are asymptotically stable.*

Proof. By the condition (13) of Theorem 2 the matrices $\underline{A} \in \mathbb{R}^{n\times n}_+$, $\overline{A} \in \mathbb{R}^{n\times n}_+$ are asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}^n_+$, such that (19) holds. The convex linear combination (23) satisfies the condition $A\lambda < \lambda$ if and only if (19) holds. Therefore, the interval positive system (14) is asymptotically stable if and only if the positive systems (17) and (18) are asymptotically stable. ∎

Example 2. Consider the interval positive linear system (14) with the matrices

$$\underline{A} = \begin{bmatrix} 0.7 & 0.1 \\ 0.4 & 0.3 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}. \tag{25}$$

For the matrices (25) we choose $\lambda^T = [1 \ \ 1]$ and obtain

$$\begin{aligned} \underline{A}\lambda &= \begin{bmatrix} 0.7 & 0.1 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.7 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \overline{A}\lambda &= \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{26}$$

Therefore, by Theorem 5 the interval positive system (14) with (25) is asymptotically stable. ♦

4. Global stability of descriptor nonlinear feedback systems with interval state matrices

Consider the nonlinear feedback system shown in Fig. 1 which consists of the descriptor positive linear part, the nonlinear element with characteristic $u = f(e)$, positive scalar gain feedback h and interval state matrices. The descriptor linear part is described by Eqns. (1) and (2) on the assumption that $m = p = 1$, i.e., a single-input single-output system. The state matrix A is an interval matrix (15). The characteristic $f(e)$ of the nonlinear element (Fig. 2) satisfies the condition

$$0 < f(e) < ke, \quad 0 < k < \infty. \tag{27}$$

The linear part of the system in Fig. 1 is discrete-time and the nonlinear part is continuous-time. Therefore, the continuous-time to discrete-time converter C/D and the discrete-time to continuous-time converter D/C have been added.

We make the following assumptions:

A1. the pencil $(E,\ A)$ is regular (the condition (3) is satisfied),

A2. the matrix E has n_1 linearly independent columns,

A3. rank$E$ = deg det$[Ez - A] = n_1$.

If these assumptions are satisfied in Eqns. (4) and (7), the nilpotent matrix $N = 0$. Note that if $Q \in \mathbb{R}_+^{n\times n}$ is monomial then $Q^{-1} \in \mathbb{R}_+^{n\times n}$ and $x_{1i} \in \mathbb{R}_+^{n_1}$ and $x_{2i} = 0$ for $i = 0, 1, \dots, B_2 = 0$ since $N = 0$ (Assumptions A2 and A3). Similarly, since $C_2 x_{2i} = 0$, from (8) we have

$$y_i = C_1 x_{1i}. \tag{28}$$

Definition 3. A nonlinear positive system is called *globally stable* if it is asymptotically stable for all nonnegative initial conditions $x_0 \in \mathbb{R}_+$.

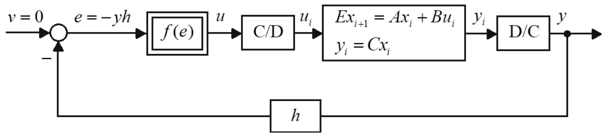

Fig. 1. Nonlinear system.

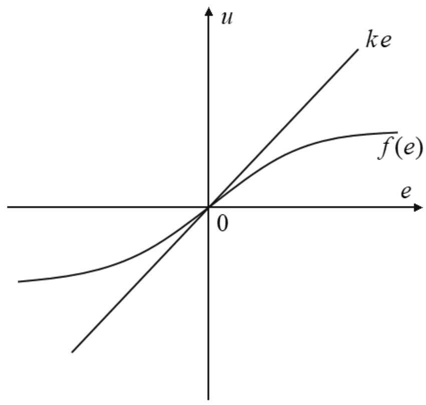

Fig. 2. Characteristic of the nonlinear element.

The following theorem gives sufficient conditions for the global stability of the descriptor positive nonlinear system with the interval matrix.

Theorem 6. *The nonlinear system shown in Fig. 1 consisting of the positive linear part with interval state matrix (15) satisfying Assumptions A1–A 3, the nonlinear element satisfying the condition (27) and the gain feedback h is globally stable if the matrices*

$$\underline{A_1} + khB_1C_1 \in \mathbb{R}_+^{n_1\times n_1} \tag{29}$$

and

$$\overline{A_1} + khB_1C_1 \in \mathbb{R}_+^{n_1\times n_1} \tag{30}$$

are asymptotically stable.

Proof. The proof is based on the Lyapunov method (Lyapunov, 1963; Leipholz, 1970). As the Lyapunov function $V(x_{1i})$ for each system we choose

$$V(x_{1i}) = \lambda^T x_{1i} \geq 0 \quad \text{for}\, x_{1i} \in \mathbb{R}_+^{n_1}, \tag{31}$$

where λ is strictly positive vector, i.e., $\lambda_k > 0$, $k = 1, \dots, n_1$.

Using (31) and (6), we obtain

$$\begin{aligned} \Delta V(x_{1i}) &= V(x_{1,i+1}) - V(x_{1i}) \\ &= \lambda^T (x_{1,i+1} - x_{1i}) \\ &= \lambda^T (\underline{A_1} - I_{n_1}) x_{1i} + B_1 h f(e) \\ &\leq \lambda^T (\underline{A_1} + khB_1C_1) x_{1i} \end{aligned} \tag{32}$$

since $\lambda^T(\underline{A_1} - I_{n_1}) < 0$ and $(I_{n_1} - \underline{A_1}) > B_1 h f(e)$.

From (32) it follows that $\Delta V(x_{1i}) < 0$ if the matrix (29) is asymptotically stable and the nonlinear system is globally stable. A similar result is obtained for (30). ■

To find the maximal value of h for which the nonlinear systems is globally stable the following procedure can be used.

Procedure 1.
Step 1. Find the value of h for which the matrix

$$\underline{A_1} + khB_1C_1 \in \mathbb{R}_+^{n_1 \times n_1} \tag{33}$$

is asymptotically stable. Denote obtained h as $\underline{h}$.

Step 2. Find the value of h for which the matrix

$$\overline{A_1} + khB_1C_1 \in \mathbb{R}_+^{n_1 \times n_1} \tag{34}$$

is asymptotically stable. Denote by $\overline{h}$ the obtained solution.

Step 3. Select the desired value of h as

$$h = \min(\underline{h}, \overline{h}). \tag{35}$$

From Theorem 6 and Theorem 3 we deduce that the nonlinear positive feedback system shown in Fig. 1 is asymptotically stable only if the sum of all entries in every rows (columns) of the matrices (29) and (30) is less than 1.

Example 3. Consider the nonlinear feedback system with the descriptor linear part with the matrices

$$\underline{A} = \begin{bmatrix} 0.3 & 0 & 0.1 & 0 \\ 0.5 & 0 & 0.5 & 1 \\ 0.5 & 1 & 0.5 & 1 \\ 0.3 & 0 & 0.1 & 1 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0.4 & 0 & 0.2 & 0 \\ 0.65 & 0 & 0.7 & 1 \\ 0.65 & 1 & 0.7 & 1 \\ 0.4 & 0 & 0.2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1.1 \\ 1.1 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 \end{bmatrix} \tag{36}$$

and the nonlinear element satisfying the condition (27) with $k = 0.5$. Find the maximal value of h for which the nonlinear system is globally stable.

In this case the matrices P and Q have the form

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Using (4)–(8) for $N = 0$, we obtain

$$PEQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{38}$$

$$P\underline{A}Q = \begin{bmatrix} \underline{A_1} & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{39}$$

$$P\overline{A}Q = \begin{bmatrix} \overline{A_1} & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{40}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.6 \\ 0 \\ 0 \end{bmatrix}, \tag{41}$$

$$CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 & 0 & 0 \end{bmatrix}. \tag{42}$$

Using Procedure 1 and the matrices $\underline{A_1}$, $\overline{A_1}$, B_1 and C_1, we have the following calculations.

Step 1. Using (29), we obtain

$$\underline{A_1} + khB_1C_1 = \begin{bmatrix} 0.3 + 0.05h & 0.1 + 0.1h \\ 0.2 + 0.06h & 0.4 + 0.12h \end{bmatrix} \tag{43}$$

and the maximal value of h for which the matrix (43) is asymptotically stable is $h < 2.857$, since for this value the coefficients of the polynomial

$$\begin{aligned} &\det[I_2(z+1) - \underline{A_1} - khB_1C_1] \\ &\quad = z^2 + (1.3 - 0.17h)z + 0.4 - 0.14h \end{aligned} \tag{44}$$

are positive. For $h = \underline{h}$ we have $\underline{h} < 2.857$.

Step 2. Using (30) we obtain

$$\overline{A_1} + khB_1C_1 = \begin{bmatrix} 0.4 + 0.05h & 0.2 + 0.1h \\ 0.25 + 0.06h & 0.5 + 0.12h \end{bmatrix} \tag{45}$$

and the maximal value of h for which the matrix is asymptotically stable is $h < 1.866$, since for this value the coefficients of the polynomial

$$\begin{aligned} &\det[I_2(z+1) - \overline{A_1} - khB_1C_1] \\ &\quad = z^2 + (1.1 - 0.17h)z + 0.25 - 0.134h \end{aligned} \tag{46}$$

are positive. For $h = \overline{h}$ we have $\overline{h} < 2.857$.

Step 3. Using (35) and the results of Steps 1 and 2 we obtain

$$h = \min(\underline{h}, \overline{h}) = \min(2.857,\ 1.866) = 1.866. \quad (47)$$

Therefore, the nonlinear system is globally stable for $h < 1.866$. ♦

5. Concluding remarks

The global stability of positive discrete-time nonlinear feedback systems with interval state matrices has been investigated. New sufficient conditions for the global stability of the class of positive nonlinear systems (Theorem 6) have been established. A procedure for computation of the value of the positive scalar feedback coefficient for which the nonlinear feedback system with the interval state matrix is globally stable has been proposed. The effectiveness of these new stability conditions has been demonstrated on the simple example of positive nonlinear systems with interval state matrices. The considerations can be extended to nonlinear feedback fractional systems.

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