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On the theory of λ-matrices based MIMO control system design*

by

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Abstract: In this paper we have described a new design algorithm for the whole set of latent-structure assignments via the approaches of block structure of λ-matrices placement. The procedure that has been developed is based on decoupling of the interactions between control loops in a multivariable plant. The procedure is performed using matrix polynomial solvent reconstruction for the decoupling purposes. However, for the design of the trajectory tracking controller, each input-output pair is treated respectively by designing SISO controllers. A second procedure is the MIMO PID compensator design via the model-matching method. This latter algorithm has been developed in order to avoid the internal or the hidden instability, which may occur in the first method, due to the block zeros - block poles cancellation.

Keywords: block roots, λ-matrices, decoupling, MIMO PID compensator, model-matching.

1. Introduction

A large-scale MIMO linear system, described by a state space equation, is often decomposed into smaller subsystems that can be more easily analyzed and designed. The dynamic properties of the MIMO system depend on the block-poles of its characteristic matrix polynomial. These block poles are nothing more than the solvents of the closed-loop denominator matrix polynomial of the considered MIMO system, see Yaici, Hariche (2014a,b), Dahimene (2009), Shieh, Tsang and Yates (1983), and Bekhiti et al. (2015). The solvents play an important role in the spectral decomposition of the respective matrices. The relationship between the solvents and the latent roots of the matrix polynomial will be briefly presented and explained here. For further information, see

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Shieh, Tsang and Coleman (1981), Dennis, Traub and Weber (1978), and Gohberg, Lancaster and Rodman (1982). One of the most important features of a multivariable system is the possible cross-coupling or interactions between its variables, i.e., one input variable may affect all the output variables. These interrelations make it impossible for the control engineer to design each loop independently. In this case, adjusting controller parameters of one loop affects the performance of another, Wang (2003). The decoupling problem can be stated and solved on the basis of the frequency-domain representation of state feedback control, Kucera (1979). Therefore, in this work, the decoupling problem is only considered in the frequency domain. By using the relations connecting the parametric approach in the time and in the frequency domains, Chen (1984), the equivalent time-domain results can also be derived.

Unfortunately, the proposed decoupling controller does not assure robust tracking in the presence of modeling errors. Hence, in order to partially avoid this problem, a new model-matching controller is stated and elaborated. In the model-matching problem, a controller is designed to generate an input to the system, so that the output tracks exactly the output of a given reference model. In this work, we are looking for the new MIMO control algorithms. The paper is organized as follow: the present first section is an introduction to the study here considered. Then, the second section will include theoretical preliminaries and a brief review on matrix polynomials. It is followed with the section, which deals with the decoupling controller based on the block structure assignment. In the fourth section we investigate the elaboration of the model-matching controller.

2. Survey on matrix polynomials and state space description

The preliminary theory, concerning matrix polynomials, considered here, can be found in Yaici and Hariche (2014a,b), Dennis, Traub and Weber (1976, 1978), Gohberg, Kaashoek and Rodman (1978), and Gohberg, Lancaster and Rodman (1982). At this juncture, we are going to present formal theorems and definitions of the characteristic λ-matrices and the canonical MFDs for MIMO systems, which are the counterparts of the characteristic polynomials and the transfer functions for SISO systems, respectively, see Hippe and O'Reilly (1987).

2.1. Matrix polynomials and solvents

In this subsection, we attempt to present some of the important results obtained in the theory of matrix polynomials.

Definition 1 *Given the set of $m \times m$ complex matrices $A_0, A_1, ..., A_l$, the following matrix valued function of the complex variable λ is called a matrix polynomial of degree l and order m:*

$$A(\lambda) = A_0\lambda^l + A_1\lambda^{l-1} + ... + A_{l-1}\lambda + A_l. \tag{1}$$

DEFINITION 2 *The complex number λ_i is called a latent root of the matrix polynomial $A(\lambda)$ if it is a solution of the scalar polynomial equation $det(A(\lambda)) = 0$. The nontrivial vector p, solution of $A(\lambda_i)p = 0_m$, is called a primary right latent vector associated with λ_i. Similarly, the nontrivial vector q, solution of $q^T A(\lambda_i) = 0_m$, is called a primary left latent vector associated with the latent value λ_i.*

THEOREM 1 *(Gohberg, Lancaster and Rodman, 1982) The number of latent roots of the regular matrix polynomial $A(\lambda)$ in the domain $\mathcal{D}$ enclosed by a contour Γ is given by:*

$$n = \frac{1}{2\pi j} \oint_{\Gamma} trace \left[A^{-1}(\lambda) \frac{dA(\lambda)}{d\lambda} \right] d\lambda$$

each latent root being counted according to its multiplicity.

DEFINITION 3 *A right block root, also called a solvent of λ-matrix $A(\lambda)$, is an $m \times m$ real matrix R such that:*

$$A_0 R^l + A_1 R^{l-1} + ... + A_{l-1} R + A_l = O_m \Leftrightarrow A_R(R) = \sum_{i=0}^{l} A_i R^{l-i} = O_m, \quad (2)$$

while a left solvent is an $m \times m$ real matrix L such that:

$$L^l A_0 + L^{l-1} A_1 + ... + L A_{l-1} + A_l = O_m \Leftrightarrow A_L(L) = \sum_{i=0}^{l} L^{l-i} A_i = O_m. \quad (3)$$

THEOREM 2 *(Shieh ans Tsay, 1981) If $A(\lambda)$ has n linearly independent right latent vectors $(p_1, ..., p_n)$ (or left latent vectors $(q_1, q_2, ..., q_n)$) corresponding to latent roots $(\lambda_1, \lambda_2, ..., \lambda_n)$, then $P \Lambda P^{-1}$ $(Q \Lambda Q^{-1})$ is a right (left) solvent, where: $P = [p_1, p_2, ..., p_n], (Q = [q_1, q_2, ..., q_n]^T)$ and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_n)$.*

PROOF See Shieh and Tsay (1981).

THEOREM 3 *(Gohberh, Lancaster and Rodman, 1982) Let Γ be a contour that contains m latent roots and let the $m \times m$ matrix $M = \dfrac{1}{2\pi j} \oint_{\Gamma} A^{-1}(\lambda) d\lambda$ be nonsingular, then we have as right and left solvents, respectively:*

$$R = \frac{1}{2\pi j} \oint_{\Gamma} \lambda A^{-1}(\lambda) d\lambda . M^{-1} \qquad L = M^{-1} . \frac{1}{2\pi j} \oint_{\Gamma} \lambda A^{-1}(\lambda) d\lambda$$

and we see that M is a similarity transformation between R and L.

2.2. Characteristic λ-matrices of MIMO systems

Consider a linear time-invariant system, described by a state equation in general coordinates:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(t) \\ Y(t) = CX(t) + Du(t) \end{cases} \tag{4}$$

where: $X \in R^n$, $Y \in R^p$, $u \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$.

The system (4) is block controllable of index l if the matrix

i. $\Phi = [B, AB, A^2B, ..., A^{l-1}B]$ has full rank
ii. $l = \dfrac{n}{m}$ is an integer.

Theorem 4 *(Shieh, Chang and McInnis, 1986) The multivariable control system described in (4) can be transformed into a block controller form if two conditions are satisfied:*

i. $l = \dfrac{n}{m}$ *is an integer.*
ii. The system is block controllable of index l.

If both conditions are satisfied, then the change of coordinates $X_c(t) = T_cX(t)$ transforms the system into the following block controller form

$$\begin{cases} \dot{X}_c(t) = A_cX_c(t) + B_cu(t) \\ Y(t) = C_cX_c(t) + D_cu(t) \end{cases} \tag{5}$$

$$\begin{cases} T_c = \begin{pmatrix} T_{c1} \\ T_{c1}A \\ \vdots \\ T_{c1}A^{l-1} \end{pmatrix} \\ T_{c1} = [O_m, ..., I_m]\,\Phi^{-1} \\ A_c = T_cAT_c{}^{-1} = \begin{pmatrix} O_m & I_m & \cdots & O_m \\ O_m & O_m & \cdots & O_m \\ \vdots & \vdots & \cdots & O_m \\ O_m & O_m & \cdots & I_m \\ -A_l & -A_{l-1} & \cdots & -A_1 \end{pmatrix} \\ B_c = T_cB = \begin{pmatrix} O_m & O_m & \cdots & I_m \end{pmatrix}^T \\ C_c = CT_c{}^{-1} = \begin{pmatrix} C_l & C_{l-1} & \cdots & C_1 \end{pmatrix} \end{cases}$$

with: $X_c \in R^n, A_i \in R^{m\times m}, C_i \in R^{p\times m}, i = 1, ..., l$, I_m and O_m being $m \times m$ identity and null matrices, respectively, and the superscript T denoting the transpose. For proof, see Shieh, Chang and McInnis (1986).

The characteristic polynomial in a SISO system is directly obtained from the nonzero elements in the last row of the system matrix, when transformed into the controllable canonical form, and the characteristic polynomial is a scalar polynomial. For multivariable control systems, the characteristic polynomial is a matrix polynomial. The right matrix fraction description (RMFD) of the system can be formulated directly from (5) as:

$$H(\lambda) = N_R(\lambda){D_R}^{-1}(\lambda) \tag{6}$$

where the matrix $D_R(\lambda)$ is the right denominator, given by

$$D_R(\lambda) = A_0\lambda^l + A_1\lambda^{l-1} + ... + A_{l-1}\lambda + A_l \tag{7}$$

and the right numerator $N_R(\lambda)$ is given by

$$N_R(\lambda) = C_1\lambda^{l-1} + C_2\lambda^{l-2} + ... + C_{l-1}\lambda + C_l. \tag{8}$$

Note that the matrix coefficients of $D_R(\lambda)$ and $N_R(\lambda)$ can be directly obtained from those nontrivial block entries of the block controllable canonical form in (5). $D_R(\lambda)$ is referred to as the right characteristic λ-matrix of the system (4). In fact, $D_R(\lambda)$ can be directly determined as

$${D_R}^{-1}(\lambda) = {(E_1^l)}^T(\lambda I_n - A_c)^{-1} = (E_1^l)T_c(\lambda I_n - A)^{-1}B \tag{9}$$

where ${(E_1^l)}^T = \begin{bmatrix} I_m & O_m & \cdots & O_m \end{bmatrix} \in R^{m\times m}$.

Upon examining T_c, we have the following new result:

$$T_c = P(A_c, B_c)P^{-1}(A, B) \tag{10}$$

$$P(A, B) = \begin{bmatrix} B, AB, ..., A^{l-1}B \end{bmatrix} \tag{11}$$

$$P(A_c, B_c) = \begin{bmatrix} B_c, A_cB_c, ..., {A_c}^{l-1}B \end{bmatrix}. \tag{12}$$

Substituting (10) into (9) yields the right characteristic λ-matrix of the system in (4),

$${D_R}^{-1}(\lambda) = {(E_1^l)}^T P^{-1}(A, B)(\lambda I_n - A)^{-1}B. \tag{13}$$

Basing on the definition of the characteristic λ-matrix, we can introduce the block poles of an MFD from the solvents of a λ-matrix.

2.3. Block spectral decomposition:

Assume we are given an l^{th} degree m^{th} order monic λ-matrix $D_R(\lambda)$

$$D_{RL}(X) = X^lA_0 + X^{l-1}A_1 + ... + XA_{l-1} + A_l \tag{14}$$

where $X \in C^{m\times m}$. If there is an $L_i \in C^{m\times m}$ such that $D_{RL}(L_i) = O_m$, then L_i is referred to as a left solvent of $D_R(\lambda)$. If there exists a set of left solvents $\{L_i, i = 1, ..., l\}$ such that

$$\bigcup_{i=1}^{l} \sigma(L_i) = \sigma(D_R(\lambda)) \tag{15}$$

then $D_R(\lambda)$ has a complete set of left solvents (Shieh and Tsay, 1981). When $D_R(\lambda)$ has a complete set of left solvents, the RMFD of (4) has a block partial fraction expansion.

Lemma 1 *(Shieh and Tsay, 1982) Let $\{L_i, i = 1, ..., l\}$ be a complete set of left solvents of $D_R(\lambda)$, then*

$$H(\lambda) = N_R(\lambda) D_R{}^{-1}(\lambda) = \sum_{i=1}^{l} H_i(\lambda I_m - L_i)^{-1} \tag{16}$$

$$\textit{with} \qquad H_i = \sum_{i=1}^{l} C_j Z_i L_i^{l-j}, i = 1, ..., l, \tag{17}$$

where: $Z_i \in C^{m\times m}, i = 1, ..., l$ can be determined from the following matrix equation:

$$[Z_1, Z_2, ..., Z_l] = [O_m, O_m, ..., I_m] V^{-B}(L_1, L_2, ..., L_l),$$

and $V^{-B}(L_1, L_2, ..., L_l)$ is the inverse of the block transpose of the left block Vandermonde matrix and is defined in Yaici and Hariche (2014a).

Lemma 1 indicates that the system of (4) is decomposed into l parallel subsystems, whose RMFD can be expressed as $H_i(\lambda I_m - L_i)^{-1}$. The solvents $\{L_i, i = 1, ..., l\}$ in (17) are called the right block poles of the RMFD in (16) and H_i are the associated block residues of the block partial fraction of the RMFD. If an open-loop system does not have a complete set of right block poles, then it cannot be decomposed into (16).

The state matrix described in (4) can be transformed into a block diagonal canonical form using the so called right block Vandermonde matrix as follows:

$$A_c = V_R \Lambda V_R{}^{-1} \tag{18}$$

where
V_R : is the right block Vandermonde matrix
Λ : is the block diagonal form matrix

$$V_R = \begin{pmatrix} X_{c1}{}^T \\ \vdots \\ X_{cl}{}^T \end{pmatrix}^T, \; V_R{}^{-1} = \begin{pmatrix} Y_{c1} \\ \vdots \\ Y_{cl} \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} R_1 & \dots & O_m \\ \vdots & \ddots & \vdots \\ O_m & \dots & R_l \end{pmatrix}, \; X_{ci} = \begin{pmatrix} I_m \\ R_i \\ \vdots \\ {R_i}^{l-1} \end{pmatrix}.$$

If (18) is expanded, then the block spectral decomposition of the matrix A_c can be written as:

$$A_c = \sum_{i=1}^{l} X_{ci} R_i Y_{ci} \tag{19}$$

$$\text{Eq (19)} \Leftrightarrow A = \sum_{i=1}^{l} ({T_c}^{-1} X_{ci}) R_i (Y_{ci} T_c) = \sum_{i=1}^{l} X_i R_i Y_i \tag{20}$$

where
$X_i = {T_c}^{-1} X_{ci}$: is the right block vector corresponding to R_i
$Y_i = Y_{ci} T_c$: is the left block vector corresponding to R_i .

Equation (18) leads to the following matrix eigenstructure

$$\text{Eq(18)} \Leftrightarrow \quad A X_i = X_i R_i \;\text{ for }\; i = 1, ..., l \tag{21}$$

where R_i is called a right block eigenvalue of the matrix A, and X_i, of full rank, is the corresponding right block eigenvector of the matrix A. Further, Y_i, of full rank, is the corresponding left block eigenvector of the matrix A.

Properties of projectors:

The matrices $P_i = X_i Y_i$ are called projectors and satisfy the following properties:

1. $\sum\limits_{i=1}^{l} P_i = \sum\limits_{i=1}^{l} X_i Y_i = I_n$

2. $Y_j X_i = \begin{cases} O_m & i \neq j \\ I_m & i = j \end{cases}$

3. $P_i = X_i Y_i$ and ${P_i}^2 = P_i$

4. $P_i P_j = O_m$ iff $i \neq j$.

2.4. Block partial fraction expansion and response

Multi-input, multi-output systems generally lead to matrix fraction descriptions (MFD) of rational matrices and/or block partial fraction expansions, expressed in terms of projectors contribution that we consider now:

$$\text{Eq. (20)} \Leftrightarrow \quad (\lambda I_n - A)^{-1} = \sum_{i=1}^{l} X_i (\lambda I_m - R_i)^{-1} Y_i \tag{22}$$

$$\text{Eq. (20)} \Leftrightarrow \quad e^{At} = \sum_{i=1}^{l} X_i e^{R_i t} Y_i. \tag{23}$$

Remark 1 The pair (X_i, R_i) is called right block root block vector pair or, in other words, we call it block structure, which characterizes completely the system and alters both stability and the shape of the response. A transfer-function matrix of a linear time-invariant multivariable system can be formulated in terms of block structure as follows:

$$H(\lambda) = N_R(\lambda)D_R(\lambda)^{-1} = C(\lambda I_n - A)^{-1}B \Rightarrow Y(\lambda) = C(\lambda I_n - A)^{-1}Bu(\lambda)$$

$$Y(\lambda) = \left(\sum_{i=1}^{l} CX_i(\lambda I_m - R_i)^{-1}Y_iB \right) u(\lambda). \tag{24}$$

Note that this decoupled structure is in the canonical block diagonal form, and is schematically illustrated in Fig. 1:

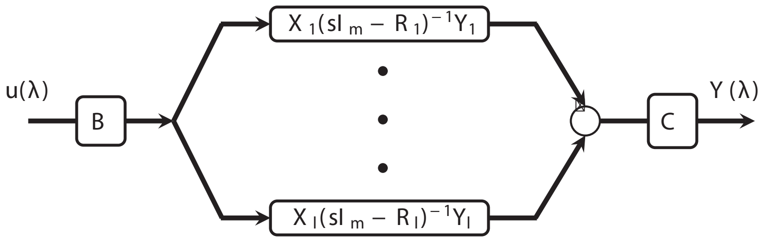

Figure 1. The MIMO solvent decoupling structure

Latent structure assignment, which is the (block-roots, block-vectors) placement, provides for large degree of freedom in the design of feedback gain matrix, because the latent structure is more general and alters both the stability and the transient responses.

2.5. Latent structure from the eigenstructure

Assume that V_i is an eigenvector of A, corresponding to λ_i; the corresponding latent vector is obtained directly using the following similarity transformation (see Yaici and Hariche, 2014b; and Yaici, Hariche and Clark, 2014):

$$AV_i = \lambda_i V_i \Leftrightarrow \nu_i = T_{c1}V_i. \tag{25}$$

Let an i^{th} set $(i = 1...l)$ of m linearly independent latent vectors be $\{\nu_{i1}, ..., \nu_{im}\}$ and its corresponding latent values $\{\lambda_{i1}, ..., \lambda_{im}\}$. Then we can determine a right block root using the following equation:

$$R_i = [\nu_{i1}, ..., \nu_{im}]diag([\lambda_{i1}, ..., \lambda_{im}])[\nu_{i1}, ..., \nu_{im}]^{-1}. \tag{26}$$

3. Parametric block roots assignment

3.1. Objectives

In order to place spectral factors, we can use directly a block structure assignment procedure, which will decouple the system via the assignment of latent structure of the numerator at the denominator and set the indefectible roots to zeros.

Let us define the RMFD transfer function $H(\lambda) = N(\lambda)D(\lambda)^{-1}$ with:

$$D(\lambda) = (D_l\lambda^l + ... + D_1\lambda + D_0) \quad \text{and} \quad D_l = I_m$$

$$N(\lambda) = (N_k\lambda^k + ... + N_1\lambda + N_0).$$

The desired characteristic matrix polynomial is of the form:

$$D_d(\lambda) = (D_{dl}\lambda^l + ... + D_{d1}\lambda + D_{d0}). \tag{27}$$

If we know that the desired solvents (R_i with $i = 1, ...l$) are block roots with respect to the desired matrix polynomial, then:

$$\begin{aligned} D_d(R_i) = O_m \quad &\Leftrightarrow \quad D_{d0} + D_{d1}R_i + ... + D_{dl}{R_i}^l = O_m \\ &\Leftrightarrow \quad [D_{d0}, D_{d1}, ..., D_{d(l-1)}]X_{ci} + {R_i}^l = O_m \\ &\Leftrightarrow \quad [D_{d0}, D_{d1}, ..., D_{d(l-1)}]T_cX_i = -{R_i}^l. \end{aligned}$$

Hence, in order to obtain the desired block roots relocation, we employ the matrix shift property: $D_{di} = D_i - K_{ci}$, and we get:

$$D_d(R_i) = O_m \Leftrightarrow [D_0, ..., D_{l-1}]T_cX_i - [K_{c0}, ..., K_{c(l-1)}]T_cX_i = -{R_i}^l,$$

or, in a more compact form,

$$D_d(R_i) = O_m \Leftrightarrow [D_0, ..., D_{l-1}]T_cX_i + {R_i}^l = K_cT_cX_i = KX_i$$

$$D_d(R_i) = O_m \Leftrightarrow D(R_i) = KX_i \quad \text{for } i = 1, 2, ..., l.$$

The feedback gain matrix is easily obtained from the block structure data by the newt formula:

$$K = [D(R_1), ..., D(R_l)][X_1, ..., X_l]^{-1}. \tag{28}$$

3.2. Construction of the decoupling block roots

In the present section we have introduced an alternative method for constructing the linear state-feedback control law from the desired block structure (R_i, X_i) using some algebraic approaches and λ-matrix theory. This approach is the counterpart of eigenstructure assignment for MIMO system design, which has been discussed by many authors. But a natural question to be asked here at this stage is the following: what are the desired block roots that will perfectly decouple the system? In order to answer this question we propose an extension of the scalar poles-zeros cancellation method to the more general case (block poles-block zeros cancellation method).

First, we should determine the block zeros of the corresponding numerator matrix polynomial $N(\lambda)$, and then we will force them to be the block roots of the denominator via the state feedback gain matrix. Without prior knowledge of the eigenvalues and eigenvectors of the matrix, the Newton-Raphson method (Shieh, Tsay and Coleman, 1981) has been successfully utilized for finding the solvents. Also, the block-power method has been developed by Tsai, Shieh and Shen (1988) for finding the solvents and spectral factors of a general nonsingular polynomial matrix. Moreover, there are quite a few numerical methods for computing the block roots of matrix polynomials without any prior knowledge of the eigenvalues and eigenvectors of the matrix polynomial. In this paper we will use one of the very well-known methods for constructing the complete set of solvents.

Let now $Z_1, \cdots, Z_k \in R^{m\times m}$ be the block zeros of $N(\lambda)$, then the desired block poles are given directly by: $R_i = Z_i, \quad R_j = O_m$. Knowing that R_i and R_j for $i = 1, \cdots, k$, and $j = k+1, \cdots, l$ are the block roots of $D_d(\lambda)$ means that $[D_{d0}, \cdots, D_{d(l-1)}]V_Z = -[Z_1^l, \cdots, Z_k^l, O_{m\times m(l-k)}]$, where

$$V_Z = \begin{pmatrix} I_m & \cdots & I_m & O_m & \cdots & O_m \\ Z_1 & \cdots & Z_k & O_m & \cdots & O_m \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ Z_1^{l-1} & \cdots & Z_k^{l-1} & O_m & \cdots & O_m \end{pmatrix}.$$

Some algebraic manipulations yield

$$K = \left([D_0, \cdots, D_{l-1}]V_Z + [Z_1^l, \cdots, Z_k^l, O_{m\times m(l-k)}]\right)\left(T_c^{-1}V_Z\right)^{-1}.$$

The procedure:

- Assume that all states are available and measurable.
- Check the Block Observability and Block Controllability of a given state space model of the square dynamic system.
- Construct the right numerator and right denominator matrix polynomials.
- Find a complete set of block roots for the numerator matrix polynomial.
- Choose the k solvents of the numerator as block roots to denominator and force the remaining ones to zeros.
- Design the state feedback gain matrix which will assign the whole set of block structure. Here, at this point, we are ready to design the SISO tracking regulators for each of the input-output pairs, because the system is perfectly decoupled.

Example 1 A basic element in power generation is the turbogenerator, the dynamic model of which has six states, two inputs and two outputs ($n = 6, m = 2$ and $p = 2$) see Fredriksson and Egardt (2001), Friedrich, Liu and Oehlerking (2009), Haiyan (2006), and Jung, Glover and Christen (2005). Using the appropriate data, see Magdi and Xia (2012), the system matrices are given by:

$$A = \begin{pmatrix} 50.9686 & 183.4988 & 185.0527 & 377.4427 & 397.5263 & -895.7863 \\ 49.3505 & 174.8767 & 181.8536 & 362.9740 & 383.8789 & -863.0960 \\ 22.3703 & 78.4855 & 67.7505 & 154.7968 & 158.4009 & -365.6538 \\ 16.3520 & 61.2286 & 53.8233 & 118.8125 & 125.0745 & -284.0468 \\ 27.5073 & 100.9073 & 112.8257 & 213.2357 & 228.3972 & -507.7180 \\ 37.3706 & 135.7728 & 137.5260 & 278.2199 & 294.2728 & -661.8056 \end{pmatrix},$$

$$B = \begin{pmatrix} 6.9516 & 8.5300 \\ 4.9912 & 8.7393 \\ 5.3580 & 2.7029 \\ 4.4518 & 2.0846 \\ 1.2393 & 5.6498 \\ 4.9036 & 6.4031 \end{pmatrix}$$

$$C = \begin{pmatrix} 1.5201 & -8.4744 & 8.8746 & 9.4849 & 22.1231 & -15.6123 \\ 2.1220 & -7.8463 & 10.3791 & 11.5325 & 24.5930 & -21.0454 \end{pmatrix},$$

$$D = \begin{pmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{pmatrix}.$$

Finding the decoupling gain matrix via block structure assignment for a turbo generator system $rank(M) = rank([B, AB, ..., A^2B]) = 6$ and $\dfrac{n}{m} = 3 \Rightarrow$ block controllable system. The block controllability transformation is given by:

$$T_c = \begin{pmatrix} 0.1370 & 0.0088 & -0.0708 & 0.2064 & 0.1293 & -0.3459 \\ 0.1169 & -0.2028 & -0.1018 & -0.2104 & -0.1754 & 0.3874 \\ -0.1619 & -0.1583 & 0.2815 & -0.2001 & 0.1801 & 0.2192 \\ -0.1193 & -0.0017 & 0.0049 & 0.1065 & 0.0884 & 0.0464 \\ 0.1052 & 0.3790 & 0.0126 & 0.6104 & 0.0594 & -0.9139 \\ -0.1440 & -0.0497 & 0.0474 & -0.4465 & -0.1065 & 0.6352 \end{pmatrix}.$$

Then, the numerator and denominator matrices are:

$D(\lambda) = D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0$
$N(\lambda) = N_3\lambda^3 + N_2\lambda^2 + N_1\lambda + N_0,$

such that:

$$D_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 4.5688 & 3.7077 \\ -8.2327 & 16.4313 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} -6.1967 & 24.4111 \\ -59.0804 & 75.6239 \end{pmatrix}, \qquad D_0 = \begin{pmatrix} -32.7837 & 39.7278 \\ -104.1149 & 104.2085 \end{pmatrix}$$

$$N_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 8.9054 & 7.6895 \\ 9.8203 & 5.8135 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 58.1433 & 28.5030 \\ 63.8164 & 21.9189 \end{pmatrix}, \qquad N_0 = \begin{pmatrix} 93.1159 & 14.9399 \\ 100.0705 & 9.9390 \end{pmatrix}.$$

By applying the generalized Newton method or block power method we can find the solvents of $N(\lambda)$:

$$N(R_{Ni}) = O_2 \Rightarrow \quad R_{N1} = \begin{pmatrix} -2.1727 & 0.1564 \\ -1.2948 & -0.8273 \end{pmatrix}, \; R_{N2} = \begin{pmatrix} -4.1727 & -1.3671 \\ 0.1481 & -2.8273 \end{pmatrix}.$$

Using now our proposition we set the following:

$$\begin{cases} R_1 = R_{N1} \\ R_2 = R_{N2} \\ R_3 = O_2 \end{cases} \quad \text{and} \quad X_i = T_c{}^{-1} \begin{pmatrix} I_m \\ R_i \\ R_i{}^2 \end{pmatrix} \quad i = 1, 2, 3.$$

The block structure pairs (R_i, X_i) are given below:

$$R_1 = \begin{pmatrix} -2.1727 & 0.1564 \\ -1.2948 & -0.8273 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -4.1727 & -1.3671 \\ 0.1481 & -2.8273 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 64.0353 & 8.6571 \\ 58.7618 & 2.4826 \\ 23.6798 & 4.7298 \\ 17.0027 & 0.2638 \\ 29.8439 & 0.3218 \\ 40.4194 & 2.8015 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 105.1777 & 132.3271 \\ 80.8622 & 111.9225 \\ 69.2170 & 69.9568 \\ 68.5704 & 44.5486 \\ 19.3865 & 46.5400 \\ 74.8234 & 84.9200 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 9.2638 & 8.4644 \\ 5.7850 & 2.1083 \\ 0.1608 & 5.5341 \\ 1.1992 & 6.3069 \\ 8.6177 & 0.3233 \\ 4.8291 & 6.1582 \end{pmatrix}.$$

The desired right evaluation of the original denominator is as follows:

$$W_1 = R_1{}^3 + D_2 R_1{}^2 + D_1 R_1 + D_0 = \begin{pmatrix} -25.0920 & 19.3012 \\ -56.1029 & 44.3949 \end{pmatrix}$$

$$W_2 = R_2{}^3 + D_2 R_2{}^2 + D_1 R_2 + D_0 = \begin{pmatrix} 1.0790 & 1.2084 \\ 0.3792 & -0.2103 \end{pmatrix}$$

$$W_3 = R_3{}^3 + D_2 R_3{}^2 + D_1 R_3 + D_0 = \begin{pmatrix} -32.7837 & 39.7278 \\ -104.1149 & 104.2085 \end{pmatrix}.$$

The state feedback gain matrix is designed, following our method, as:

$$K = -[W_1, W_2, W_3][X_1, X_2, X_3]^{-1}$$

$$K = \begin{pmatrix} 0.9660 & 6.7688 & 5.9075 & 12.0559 & 12.3487 & -28.4003 \\ 3.8210 & 16.4655 & 19.2893 & 34.4455 & 37.9502 & -82.4143 \end{pmatrix}$$

$$F = N_2{}^{-1} = \begin{pmatrix} -0.2449 & 0.3239 \\ 0.4136 & -0.3751 \end{pmatrix}$$

$$A_d = (A - BK), \qquad B_d = BF, \qquad C_d = C.$$

Construction of the desired matrix polynomial coefficients from those block spectral data, see Yaici and Haniche (2014b) and Bekhiti et al. (2016):

$$[D_{d3}, D_{d2}, D_{d1}, D_{d0}] = -[R_1^3, R_2^3, R_3^3] \begin{pmatrix} I & I & I \\ R_1 & R_2 & R_3 \\ R_1^2 & R_2^2 & R_3^2 \end{pmatrix}^{-1}$$

results in the following desired λ-matrix:

$$D_d(\lambda) = D_{d3}\lambda^2 + D_{d2}\lambda^2 + D_{d1}\lambda + D_{d0}.$$

Then, the closed loop transfer matrix is completely decoupled as follows:

$$H_{closed}(\lambda) = C(\lambda I - A + BK)^{-1}BF = N(\lambda){D_d}^{-1}(\lambda)F = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}.$$

We obtain a trajectory tracking with a sufficiently small error when we use a SISO PID regulator for each of the input-output pairs. The result for arbitrary input signals is shown in the subsequent Figs. 2 and 3.

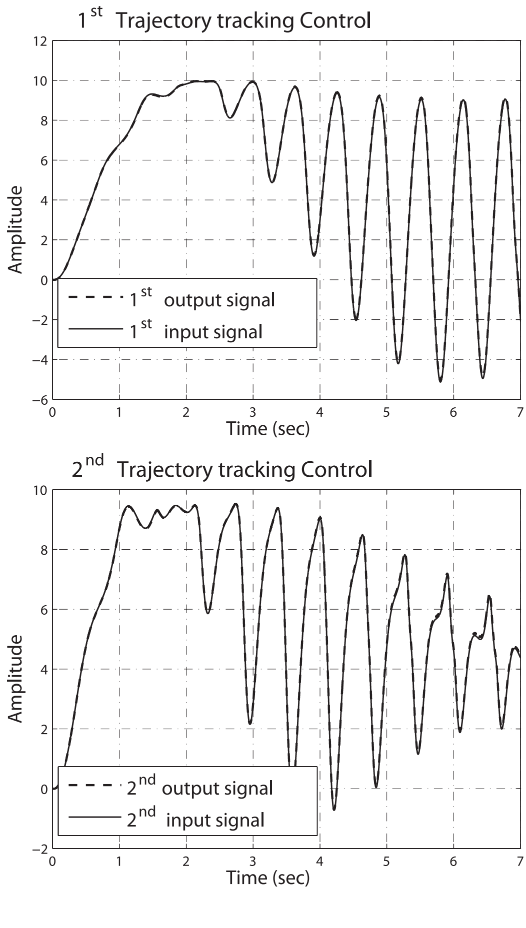

Figure 2. Trajectory tracking of the MIMO decoupled system

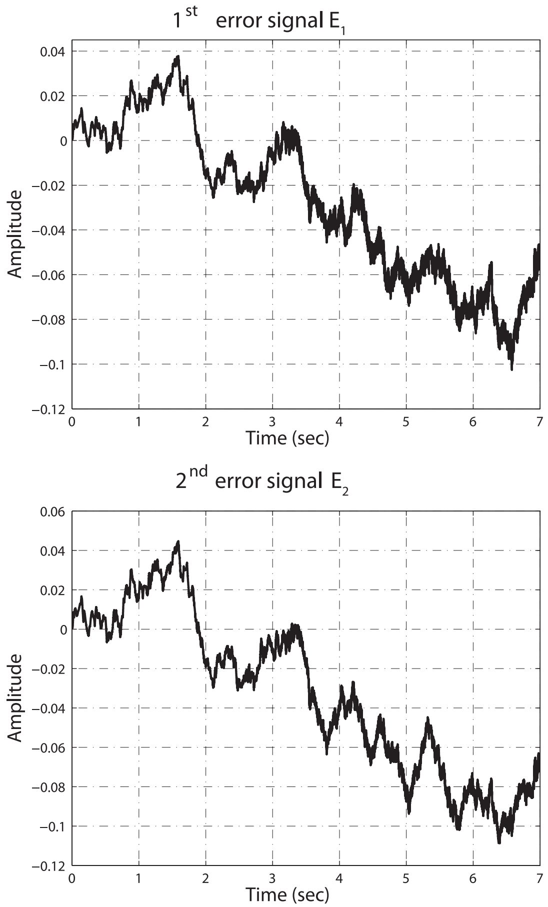

Figure 3. The error of the MIMO decoupled system

3.3. Internal dynamics and zero dynamics

The dynamics of the non-observable states are called the internal dynamics. The stability of these dynamics is required for the development of the control law. For a MIMO linear system, the internal dynamics are stable if the block zeros of the matrix transfer function have latent roots lying in the left half-plane of the complex plane. We introduce the notion of zero dynamics to study the stability of the internal dynamics of a MIMO linear system. Unfortunately, if at least one block zero of the numerator is unstable, then we have a hidden instability, when we do a decoupling state feedback. We have to find how to move or to relocate block zeros to the desired stable locations. But this can be done only in systems with input-output matrix.

4. Model matching based MIMO PID controller design

Objectives: our objective in this section is the design of multivariable PID controller based matrix fraction description so as to achieve a set of performance characteristics or matching a desired system with a given model. Via the help of partial fraction expansion and minimal realization we will parameterize the matrix coefficients of PID controller in terms of state space description. Both tracking and regulation problems are treated in our work.

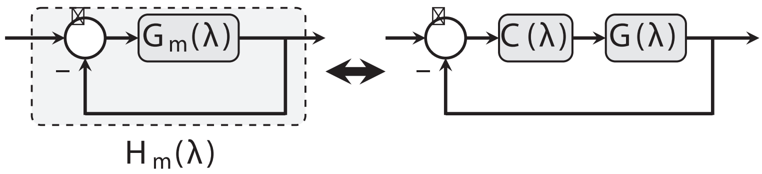

Figure 4. The matching MIMO PID controller

Design procedure:

- First, design the transfer matrix of the closed loop model to be achieved, $H_m(\lambda)$ with DC-gain$(H_m) = H_m(0) = I$.
  Reaching some desired objectives means that the needed performance characteristics are secured via the design (i.e. the system response characteristics, robustness, stability performance, sensitivity regulation, tracking with sufficiently small errors etc.).

- Second, reconstruct the forward transfer matrix $G_m(\lambda)$

$$G_m(\lambda) = H_m(\lambda)(I - H_m(\lambda))^{-1} = \frac{1}{(\Delta_m(\lambda))} N_m(\lambda), \tag{29}$$

  where $\Delta(\lambda)$ is the common denominator or the characteristic polynomial.

- Third, make the MIMO model matching, where the PID controller matrix transfer function is given by its right MFD formula as:

$$C(\lambda) = (K_I + K_P\lambda + K_D\lambda^2)(\lambda I)^{-1}. \tag{30}$$

  By performing model matching we arrive at:

$$G(\lambda)C(\lambda) = G_m(\lambda) \Leftrightarrow \left(\frac{N(\lambda)(K_I + K_P\lambda + K_D\lambda^2)}{(\lambda.\Delta(\lambda))} \right) = \left(\frac{N_m(\lambda)}{\Delta_m(\lambda)} \right)$$

$$(K_I + K_P\lambda + K_D\lambda^2) = \frac{(\lambda.\Delta(s))}{(\Delta_m(\lambda))} N^{-1}(\lambda) N_m = F(\lambda). \tag{31}$$

Remark 2 In the inversion of the matrix $N^{-1}(\lambda)$ we use the procedure proposed by Yaici and Hariche (2014b):

$$K_I = \left(\; F(\lambda) \mid_{\lambda=0} \; \right) = \lim_{\lambda \to 0} F(\lambda) \tag{32}$$

$$K_P = \left(\frac{dF(\lambda)}{d\lambda} |_{\lambda=0} \right) = \lim_{\lambda \to 0} \frac{dF(\lambda)}{d\lambda} \tag{33}$$

$$K_D = \left(\frac{d^2F(\lambda)}{2d\lambda^2} |_{\lambda=0} \right) = \lim_{s \to 0} \frac{d^2F(\lambda)}{2d\lambda^2}. \tag{34}$$

Now, the subject matter we deal with in this part is the determination of a linear time invariant dynamical equation that has a prescribed rational transfer matrix $F(\lambda)$. This dynamical equation is called a realization of $F(\lambda)$, and is the best one if it has the least possible dimension, sometimes it is called a minimal or irreducible realization.

The study of irreducible or minimal realization is important for the following reasons:
1. In order to apply techniques and computational algorithms developed for the state space representation, the transfer function matrices must be realized as dynamical equations.
2. It is always desirable to simulate the system on an analog or digital computer to check its performance before the system is built.
3. The results can establish the link between the state-variable approach and the transfer function approach.

Note: our objective here is to parameterise the MIMO PID coefficients in terms of state space matrices of $F(\lambda)$, therefore, at this stage, we can take the minimal realization of $F(\lambda)$ using the very well-known algorithms (that are available in MATLAB): $F_{space} = minreal(ss(F))$

$$F(\lambda) = \frac{(\lambda.\Delta(\lambda))}{(\Delta_m(\lambda))} N^{-1}(\lambda) N_m = C_F(\lambda E_F - A_F) B_F + D_F$$

$$(31) \Leftrightarrow K_I + K_P\lambda + K_D\lambda^2 = C_F(\lambda E_F - A_F)B_F + D_F,$$

where (using MATLAB command):

$$\begin{aligned} A_F &= F_{space}.A \\ B_F &= F_{space}.B \\ C_F &= F_{space}.C \\ D_F &= F_{space}.D \\ E_F &= F_{space}.E. \end{aligned}$$

A parametrical derivation of the matrix coefficients in terms of state space matrices is easily obtained:

$$K_I = -C_F {A_F}^{-1} B_F + D_F, \tag{35}$$

$$K_P = -C_F E_F {A_F}^{-2} B_F, \tag{36}$$

$$K_D = -C_F {E_F}^2 {A_F}^{-3} B_F. \tag{37}$$

EXAMPLE 2 Suppose we are given a square transfer matrix of a distillation column system (Wang, 2003):

$$G(\lambda) = \frac{N(\lambda)}{\Delta}$$

with:
$$N(\lambda) = \begin{pmatrix} -6.7461 & 26.6440 \\ 59.4810 & -54.9357 \end{pmatrix} + \begin{pmatrix} 2.5806 & 5.9490 \\ 4.0872 & 2.6221 \end{pmatrix}\lambda, \quad \Delta = (\lambda^2 + 4\lambda + 5).$$

More precisely, in the matrix polynomial of rational coefficients form we can write:

$$G(\lambda) = \begin{pmatrix} \left(\dfrac{2.581\lambda - 6.746}{\lambda^2 + 4\lambda + 5}\right) & \left(\dfrac{5.949\lambda + 26.64}{\lambda^2 + 4\lambda + 5}\right) \\ \left(\dfrac{4.087\lambda + 59.48}{\lambda^2 + 4\lambda + 5}\right) & \left(\dfrac{2.622\lambda - 54.94}{\lambda^2 + 4\lambda + 5}\right) \end{pmatrix}.$$

The desired closed loop transfer matrix is given by:

$$H_m = \begin{pmatrix} \left(\dfrac{16.62\lambda + 580}{\lambda^2 + 45\lambda + 580}\right) & \left(\dfrac{2.205\lambda}{\lambda^2 + 45\lambda + 580}\right) \\ \left(\dfrac{1.018\lambda}{\lambda^2 + 45\lambda + 580}\right) & \left(\dfrac{17.3\lambda + 580}{\lambda^2 + 45\lambda + 580}\right) \end{pmatrix}.$$

We now design a MIMO PID compensator $C(\lambda) = (K_I + K_P\lambda + K_D\lambda^2)(\lambda I)^{-1}$ which will force the original plant $G(\lambda)$ to match the desired system $H_m(\lambda)$.

First, we follow the preceding procedure and we compute the inverse of the matrix polynomial using the algorithm proposed by Yaici and Hariche (2014b), and we get:

$$G_m = H_m(\lambda)(I - H_m(\lambda))^{-1} = \begin{pmatrix} {}^{11}G_m(\lambda) & {}^{12}G_m(\lambda) \\ {}^{21}G_m(\lambda) & {}^{22}G_m(\lambda) \end{pmatrix},$$

with:

$${}^{11}G_m(\lambda) = \left(\frac{-16.62\lambda^2 - 1043\lambda - 16070}{\lambda^3 + 56.08\lambda^2 + 783.9\lambda}\right), \; {}^{12}G_m(\lambda) = \left(\frac{-2.205\lambda^2 - 99.21\lambda - 1279}{\lambda^3 + 56.08\lambda^2 + 783.9\lambda}\right)$$

$${}^{21}G_m(\lambda) = \left(\frac{-1.018\lambda^2 - 45.79\lambda - 590.2}{\lambda^3 + 56.08\lambda^2 + 783.9\lambda}\right), \; {}^{22}G_m(\lambda) = \left(\frac{-17.3\lambda^2 - 1073\lambda - 16460}{\lambda^3 + 56.08\lambda^2 + 783.9\lambda}\right)$$

$$F(\lambda) = \lambda G^{-1}(\lambda)G_m(\lambda) = \lambda\frac{\Delta(\lambda)}{(\Delta_m(\lambda))}N^{-1}(\lambda)N_m(\lambda).$$

With the help of the minimal realization and order reduction algorithms that exist in MATLAB, we can use for example *"minreal"* and *"ss"* functions, and we get:

$$\begin{aligned}
K_I &= \lim_{\lambda\to 0} F(\lambda) = \lim_{\lambda\to 0}\left(C_F\left(\lambda E_F - A_F\right)^{-1} B_F\right) \\
&= -C_F {A_F}^{-1} B_F + D_F \\
&= \begin{pmatrix} -4.7191 & -2.6728 \\ -5.0410 & -0.9828 \end{pmatrix}, \\
K_P &= \lim_{\lambda\to 0} \frac{dF(\lambda)}{d\lambda} = -\lim_{\lambda\to 0}\left(C_F E_F\left(\lambda E_F - A_F\right)^{-2} B_F\right) \\
&= -C_F E_F {A_F}^{-2} B_F \\
&= \begin{pmatrix} -159.6226 & 566.7458 \\ 305.9696 & -128.7950 \end{pmatrix}, \\
K_D &= \lim_{\lambda\to 0} \frac{d^2F(\lambda)}{2d\lambda^2} = \lim_{\lambda\to 0}\left(C_F {E_F}^2\left(\lambda E_F - A_F\right)^{-3} B_F\right) \\
&= -C_F {E_F}^2 {A_F}^{-3} B_F \\
&= \begin{pmatrix} -160.7530 & 573.2951 \\ 311.1655 & -130.7223 \end{pmatrix}.
\end{aligned}$$

The simulation results for the arbitrary input signals for MIMO PID controller based on model-matching are shown in the subsequent Figs. 5 and 6.

Discussion: The case study considered illustrates best tracking, regulation and robustness with no oscillations. It shows also the ability of the proposed MIMO PID controller to robustly maintain the best dynamic performance, while matching some desired latent structures and preserving the output feedback compensator behavior. From the obtained results shown in the respective figures, we see that the plant outputs coincide with the reference, no excess is recorded in both transient and permanent regimes, which is well illustrated by the error signals, while both tracking and regulation objectives are verified by the procedure. Finally, the global stability is guaranteed because the system is designed to match a specific stable latent structure. Hence, the proposed systematic procedure has a high degree of design freedom and/or much more flexibility in synthesis.

5. Conclusion

As a concluding statement, we can say that, instead of placing only a set of desired eigenvalues, we are able to assign both latent-vectors and the corresponding latent-values. So, it is more efficient to assign the latent structure via the approach of block pole placement using the state feedback gain matrix, which is a static controller. For block roots of λ-matrices, a decoupling method

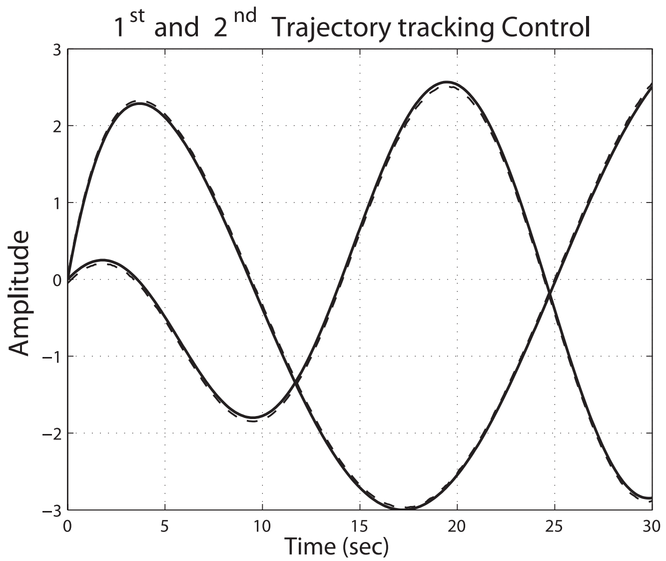

Figure 5. The trajectory tracking control for the 1^{st} and 2^{nd} output

has been proposed to eliminate the interactions between the control loops in MIMO systems. The simulation results show that a high performance is obtained for both regulation and tracking problems with low order controllers. However, there is the problem of internal dynamics due to the nature of the proposed procedure (i.e. block zeros – block poles cancellation). A second algorithm, based on the model-matching method, is presented and shown to be efficient, this dynamic compensator of a special type being called MIMO PID controller. It has a higher degree of freedom in design and avoids the internal dynamics instability.

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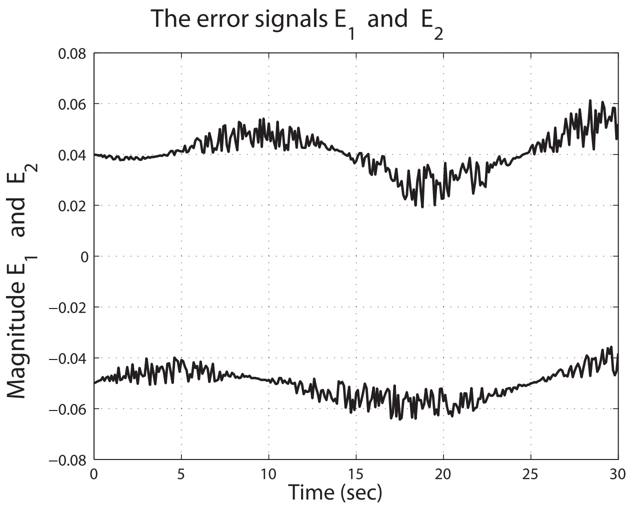

Figure 6. The error signals arising when we use the MIMO PID controller

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