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Rayleigh-Bénard convection in an elastico-viscous Walters' (model B') nanofluid layer

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Abstract. In this study, the onset of convection in an elastico-viscous Walters' (model B') nanofluid horizontal layer heated from below is considered. The Walters' (model B') fluid model is employed to describe the rheological behavior of the nanofluid. By applying the linear stability theory and a normal mode analysis method, the dispersion relation has been derived. For the case of stationary convection, it is observed that the Walters' (model B') elastico-viscous nanofluid behaves like an ordinary Newtonian nanofluid. The effects of the various physical parameters of the system, namely, the concentration Rayleigh number, Prandtl number, capacity ratio, Lewis number and kinematics visco-elasticity coefficient on the stability of the system has been numerically investigated. In addition, sufficient conditions for the non-existence of oscillatory convection are also derived.

Key words: nanofluid, oscillatory convection, Rayleigh-Bénard convection, viscoelasticity, Walters' (model B') fluid.

List of symbols

a	– wave number,
c	– specific heat,
d	– thickness of the horizontal layer,
D_B	– diffusion coefficient (m^2/s),
D_T	– thermophoretic diffusion coefficient,
g	– acceleration due to gravity (m/s^2),
F	– kinematic visco-elasticity parameter,
k	– thermal conductivity (w/mK),
Le	– Lewis number,
n	– growth rate of disturbances,
N_A	– modified diffusivity ratio,
N_B	– modified particle-density ratio,
p	– pressure (Pa),
Pr	– Prandtl number,
$\boldsymbol{q}$	– Darcy velocity vector (m/s),
Ra	– Rayleigh number,
Ra_c	– critical Rayleigh number,
Rm	– density Rayleigh number,
Rn	– concentration Rayleigh number,
t	– time (s),
T	– temperature (K),
(u,v,w)	– Darcy velocity components,
(x,y,z)	– space co-ordinates (m).

Greek symbols

α	– thermal expansion coefficient (1/K),
ϕ	– nanoparticles volume fraction,
κ	– thermal diffusivity,
μ	– viscosity ($\mu = \rho\nu$g),
μ'	– visco-elasticity,
ν	– kinematic viscosity,
ρ	– density of fluid (kg/m^3),
ρ_p	– nanoparticle mass density (kg/m^3),
ω	– dimensional frequency.

Superscripts

$'$	– non-dimensional variables,
$''$	– perturbed quantity.

Subscripts

p	– particle,
f	– fluid,
b	– basic state,
0	– lower boundary,
1	– upper boundary,
H	– horizontal plane.

1. Introduction

Rayleigh-Bénard convection is an important phenomenon that has applications to many different areas such as geophysics, food processing, oil reservoir modeling, thermal insulator design and nuclear reactors, among others. Thermal convection problems related to different types of fluids and geometric configurations have been extensively studied. The thermal instability of a Newtonian fluid under a wide range of hydrodynamics and hydromagnetic assumptions was discussed in detail by Chandrasekhar [1]. The thermal instability of a Maxwellian visco-elastic fluid in the presence of a magnetic field was analyzed by Bhatia and Steiner [2]. Scanlon and Segel [3] investigated the effect of suspended particles on the onset of convection in a horizontal layer uniformly heated from below and they found that the critical Rayleigh number decreases when suspended particles are present. These authors concluded that the destabilization effect of suspended particles was due to the fact that the heat capacity of the pure fluid was supplemented by the particles.

Much research in recent years has focused on the study of nanofluids with a view to applications in several industries such as the automotive, pharmaceutical or energy sup-

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ply industries. A nanofluid is a colloidal suspension of nano sized particles, that is, particles the size of which is below 100 nm, in a base fluid. Common fluids such as water, ethanol or engine oils are typically used as base fluids in nanofluids. Among the variety of nanoparticles that have been used in nanofluids it can be found oxide ceramics such as Al_2O_3 or CuO, nitride ceramics such as AlN or SiN and several metals such as Al or Cu. Since the term nanofluid was coined by Choi [4] the understanding of the so-called anomalous increase in thermal conductivity of nanofluids has generated considerable research interest.

Buongiorno [5] proposed that the absolute nanoparticle velocity can be viewed as the sum of the base fluid velocity and a relative slip velocity. After analyzing the effect of the following seven slips mechanisms: inertia, Brownian diffusion, thermophoresism, diffusiophoresis, Magnus effect, Fluid drainage and gravity, he concluded that in the absence of turbulent eddies Brownian diffusion and thermophoresis are the dominant slip mechanisms.The onset of convection in a horizontal layer heated from below (Bénard problem) for a nanofluid was studied by Tzou [6].

Alloui et al. [7] performed an analytical and numerical study of a natural convection problem in a shallow cavity filled with a nanofluid and heated from below. These authors reported that the presence of nanoparticles in a fluid reduced the strength of flow field, being these reductions especially relevant at low values of the Rayleigh number. Furthermore, they found that there is an optimum nanoparticle volume fraction, which depends on both the type of nanoparticle and the Rayleigh number, at which the heat transfer through the system is maximum. Dhananjay et al. [8] used a Galerkin method to study the onset of convection in the Rayleigh-Bénard problem with nanofluids and they reported that the joint behavior of Brownian motion and thermophoresis of nanoparticles reduced the critical Rayleigh number by one order of magnitude as compared to fluids without nanoparticles. A considerable number of thermal instability problems in a horizontal layer of porous medium saturated by a nanofluid have also been numerically and analytically investigated by Kuznetsov and Nield [9-11] and Nield and Kuznetsov [12–16]. Furthermore, the effect of rotation on thermal convection in the nanofluid layer saturating a Darcy-Brinkman porous medium has been reported by Chand and Rana [17, 18].

All the studies referred above deal with Newtonian nanofluids. However, the growing importance of non-Newtonian fluids in geophysical fluid dynamics, chemical technology and petroleum industry attracted widespread interest in the study on non-Newtonian nanofluids. Although experiments performed by Tom et al. [19] revealed that the behavior of a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the theoretical model of Oldroyd [20], it is widely known that there are many elastico-viscous fluids that cannot be characterized neither by Maxwell's constitutive relations nor by Oldroyd's constitutive relations. One of such type of fluids, which has relevance in chemical technology, is the Walters' (model B') elastico-viscous fluid. Walters' [21] reported that the mixture of polymethyl methacrylate and pyridine at 25°C containing 30.5 g of polymer per litre with density 0.98 g per litre behaves very nearly as the Walters' (model B') elastico-viscous fluid. Walters' (model B') elastico-viscous fluid form the basis for the manufacture of many important polymers and useful products.

Rayleigh-Bénard convection problems in an elastico-viscous Walters' (model B') fluid under a considerable amount of different hydrodynamic and hydromagnetic assumptions has been studied by Sharma and Rana [22], Gupta and Aggarwal [23], Rana and Kumar [24], Rana and Kango [25], Rana and Sharma [26]. Nield [27] discussed the thermal instability problem in a porous medium saturated by non-Newtonian fluids which were modeled by using a power law fluid. Recently, Shivkumara et al. [28] studied the effect of thermal modulation on the onset of thermal convection in Walters' B viscoelastic fluid in a porous medium while Sheu [29] used the Oldroyd-B fluid model to describe the rheological behavior of the nanofluid. More recently, Rana et al. [30] studied the thermosolutal convection in compressible Walters' (model B') fluid permeated with suspended particles in a Brinkman porous medium.

The growing number of applications of nanofluids and Walters' (model B') fluid, which include several industrial and medical fields, such as cancer therapy, motivated the current study. Our main aim is to study the Rayleigh-Bénard convection problem in a horizontal layer of an elastico-viscous Walters' (model B') nanofluid.

2. Mathematical formulations

We consider an infinite horizontal layer of an elastico-viscous Walters' (model B') nanofluid of thickness d, bounded by the planes $z = 0$ and $z = d$. The layer is heated from below with gravity force $\boldsymbol{g} = (0, 0, -g)$ aligned in the z direction, as shown in Fig. 1. The temperature, T, and the volumetric fraction of nanoparticles, φ, at the lower (upper) boundary is assumed to take constant values T_0 and $\boldsymbol{\varphi}_0$ ($\mathbf{T}_1$ and φ_1), respectively. We know that keeping a constant volume fraction

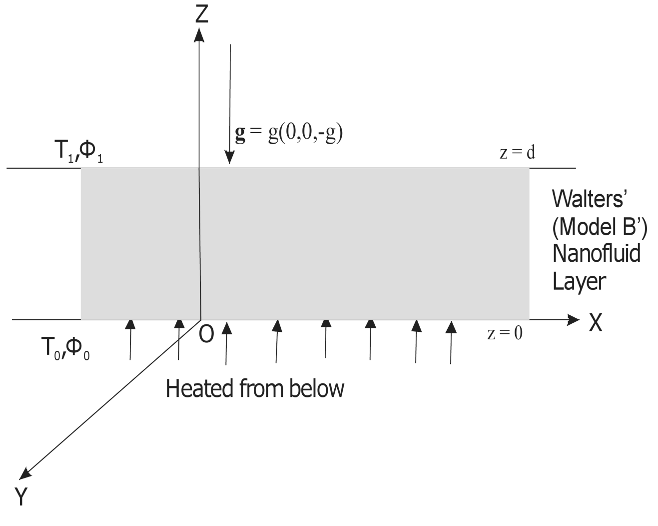

Fig. 1. Physical configuration

of nanoparticles at the horizontal boundaries will be almost impossible in a realistic situation. However, we assumed these conditions, which have also been previously adopted by several authors (Tzou [6], Nield and Kuznetsov [12], Sheu [29]), because they allow the linear stability analysis to be analytically performed.

2.1. Assumptions. The mathematical equations describing the physical model are based upon the following assumptions

i) All thermo physical properties, except for the density in the buoyancy term, are constant (Boussinesq hypothesis);
ii) Base fluid and nano particles are in thermal equilibrium state;
iii) Nano particles are spherical;
iv) Nanofluid is incompressible and laminar;
v) Negligible radiative heat transfer

2.2. Governing equations. The conservation equations of mass and momentum for an incompressible Walters' (model B') elastico-viscous fluid (Walters' [21], Chandrasekhar [1], Sharma and Rana [22] and Rana et al. [30]) are

$$\nabla \cdot \mathbf{q} = 0, \tag{1}$$

$$\rho \frac{d\mathbf{q}}{\mathrm{dt}} = -\nabla p + \rho \mathbf{g} + \left(\mu - \mu' \frac{\partial}{\partial t}\right) \nabla^2 \mathbf{q}, \tag{2}$$

where $\frac{d}{\mathrm{dt}} = \frac{\partial}{\partial t} + (\mathbf{q}\nabla)$ stands for convection derivative, $\mathbf{q} = (u, v, w)$ is the Darcy velocity vector, p is the hydrostatic pressure, $\mathbf{g} = (0, 0, -g)$ is the acceleration due to gravity and μ and μ' are the viscosity and visco-elasticity, respectively .

The ρ density of the nanofluid can be written as (Buongiorno [5])

$$\rho = \varphi \rho_p + (1 - \varphi)\, \rho_f, \tag{3}$$

where φ is the volume fraction of nano particles, ρ_p is the density of nano particles and ρ_f is the density of base fluid. Following Tzou [6] and Kuznetsov and Nield [27] we approximate the density of the nanofluid by that of the base fluid that is we consider $\rho = \rho_f$. Now introducing the Boussinesq approximation for the base fluid, the specific weight, ρg in Eq. (2) becomes

$$\rho \mathbf{g} \cong (\varphi \rho_p + (1 - \varphi)\{\rho (1 - \alpha (T - T_0))\})\, g, \tag{4}$$

where α is the coefficient of thermal expansion.

Ignoring the advective term, $\mathbf{q}$. $\nabla \mathbf{q}$ and using Eq. (4), the equation of motion (2) for an elastico-viscous Walters' nanofluid can be written as

$$\rho \frac{\partial \mathbf{q}}{\partial t} = -\nabla p + (\varphi \rho_p + (1 - \varphi)\{\rho (1 - \alpha (T - T_0))\})\, g + \left(\mu - \mu' \frac{\partial}{\partial t}\right) \nabla^2 \mathbf{q}. \tag{5}$$

The continuity equation for the nanoparticles (Buongiorno [3]) is

$$\frac{\partial \varphi}{\partial t} + \mathbf{q} \cdot \nabla \varphi = D_B \nabla^2 \varphi + \frac{D_T}{T_1} \nabla^2 T. \tag{6}$$

The energy equation for the nanofluid is (Buongiorno [3]) is

$$\rho_c \left(\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T\right) = k \nabla^2 T + (\rho_c)_p \left(D_B \nabla \phi \cdot \nabla T + \frac{D_T}{T_1} \nabla T \cdot \nabla T\right), \tag{7}$$

where ρ_c is the heat capacity of the base fluid, ρ_p, C_p is the heat capacity of nanoparticles and k is the nanofluid thermal conductivity.

We introduce non-dimensional variables as

$$(x', y', z') = \frac{1}{d}(x, y, z), \qquad (u', v', w') = \frac{d}{\kappa}(u, v, w),$$

$$t' = \frac{\kappa}{d^2} t, \qquad p' = \frac{d^2}{\mu \kappa} p,$$

$$\varphi' = \frac{(\varphi - \varphi_0)}{(\varphi_1 - \varphi_0)}, \qquad T' = \frac{(T - T_0)}{(T_0 - T_1)},$$

where $\kappa = \dfrac{k}{\rho_c}$ is the thermal diffusivity of the fluid.

From now on we eliminate the primes for simplicity. Using the scales defined above and using Boussinesq approximation, temperature gradients in the dilute suspension of nanoparticles are small enough to linearize Eq. (5) by neglecting the term involving the product of φ and T (Tzou [6], Sheu [29]), Eqs. (1), (5), (6) and (7) in non-dimensional form can be written as

$$\nabla \cdot \mathbf{q} = 0, \tag{8}$$

$$\frac{1}{Pr} \frac{\partial \mathbf{q}}{\partial t} = -\nabla p + \left(1 - F \frac{\partial}{\partial t}\right) \nabla^2 \mathbf{q} - \mathrm{Rm} \widehat{e}_z + \mathrm{Ra} T \widehat{e}_z - \mathrm{Rn} \varphi \widehat{e}_z, \tag{9}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{q} \cdot \nabla \varphi = \frac{1}{Le} \nabla^2 \varphi + \frac{N_A}{Le} \nabla^2 T, \tag{10}$$

$$\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T = \nabla^2 T + \frac{N_B}{Le} \nabla \varphi \cdot \nabla T + \frac{N_A N_B}{Le} \nabla T \cdot \nabla T, \tag{11}$$

where non-dimensional parameters are:

Prandtl number

$$\mathrm{Pr} = \frac{\mu}{\rho \kappa};$$

Lewis number

$$\mathrm{Le} = \frac{\kappa}{D_B};$$

Kinematic visco-elasticity parameter

$$F = \frac{\mu' \kappa}{\mu d^2};$$

Rayleigh Number

$$\mathrm{Ra} = \frac{g \rho \alpha d^3 (T_0 - T_x 1)}{\mu \kappa};$$

Density Rayleigh number

$$\mathrm{Rm} = \frac{\rho_p \varphi_0 + \rho (1 - \varphi_0)\, g d^3}{\mu \kappa};$$

Concentration Rayleigh number

$$\mathrm{Rn} = \frac{(\rho_p - \rho)(\varphi_1 - \varphi_0)\, g d^3}{\mu \kappa};$$

Modified diffusivity ratio

$$N_A = \frac{D_T\left(T_0 - T_1\right)}{D_B T_1\left(\varphi_1 - \varphi_0\right)};$$

Modified particle-density ratio

$$N_B = \frac{\rho_p c_p\left(\varphi_1 - \varphi_0\right)}{\rho c}.$$

The dimensionless boundary conditions are

$$w = 0, \quad T = 1, \quad \varphi = 0, \quad \text{at} \quad z = 0, \tag{12}$$

$$w = 0, \quad T = 0, \quad \varphi = 1 \quad \text{at} \quad z = 1. \tag{13}$$

2.3. Basic solutions. Following Kuznetsov and Nield [9–11] and Sheu [29] we assume a quiescent basic state that verifies

$$u = v = w = 0, \quad p = p(z), \quad T = T_b(z), \quad \varphi = \varphi_b(z). \tag{14}$$

Therefore, when the basic state defined in (14) is substituted into Eqs. (9)–(12) these equations reduce to

$$0 = -\frac{dp_b(z)}{dz} - \mathrm{Rm} + \mathrm{Ra}T_b(z) - \mathrm{Rn}\varphi_b(z), \tag{15}$$

$$\frac{d^2\varphi_b(z)}{dz^2} + N_A\frac{d^2T_b(z)}{dz^2} = 0, \tag{16}$$

$$\frac{d^2T_b(z)}{dz^2} + \frac{N_B}{\mathrm{Le}}\frac{d\varphi_b(z)}{dz}\frac{dT_b(z)}{dz} + \frac{N_A N_B}{\mathrm{Le}}\left(\frac{dT_b(z)}{dz}\right)^2 = 0. \tag{17}$$

Using boundary conditions (12) and (13) the solution of Eq. (16) is

$$\varphi_b(z) = -N_A T_b + \left(1 - N_A\right)z + N_A. \tag{18}$$

Substituting expression (18) into Eq. (17), we get

$$\frac{d^2T_b(z)}{dz^2} + \frac{\left(1 - N_A\right)N_B}{\mathrm{Le}}\frac{dT_b(z)}{dz} = 0. \tag{19}$$

The solution of differential Eq. (19) with boundary conditions (12) and (13) is

$$T_b(z) = \frac{1 - e^{-(1-N_A)N_B(1-z)/\mathrm{Le}}}{1 - e^{-(1-N_A)N_B/\mathrm{Le}}}. \tag{20}$$

Since, according to Sheu [29] the exponents in Eq. (20) are small for most typical nanofluids the exponential function in Eq. (20) is expanded into the power series and all the terms except for the first order one are neglected. Therefore, a good approximation for the basic state solution is

$$T_b = 1 - z \quad \text{and} \quad \varphi_b = z.$$

2.4. Perturbation solutions. To study the stability of the system, we superimposed infinitesimal perturbations on the basic state, so that

$$\begin{gathered}\mathbf{q}(u, v, w) = 0 + q''(u, v, w), \\ T = (1 - z) + T'', \qquad \varphi = z + \varphi'', \\ p = p_b + p''.\end{gathered} \tag{21}$$

Introducing Eq. (21) into Eqs. (8)–(11), linearizing the resulting equations by neglecting nonlinear terms that are product of prime quantities and dropping the primes ($''$) for convenience, the following equations are obtained

$$\nabla.\mathbf{q} = 0, \tag{22}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial\mathbf{q}}{\partial t} = -\nabla p + \left(1 - F\frac{\partial}{\partial t}\right)\nabla^2\mathbf{q} + \mathrm{Ra}T\widehat{e}_z - \mathrm{Rn}\varphi\widehat{e}_z, \tag{23}$$

$$\frac{\partial\varphi}{\partial t} + w = \frac{1}{\mathrm{Le}}\nabla^2\varphi + \frac{N_A}{\mathrm{Le}}\nabla^2 T, \tag{24}$$

$$\frac{\partial T}{\partial t} - w = \nabla^2 T + \frac{N_B}{\mathrm{Le}}\left(\frac{\partial T}{\partial z} - \frac{\partial\varphi}{\partial z}\right) - \frac{2N_A N_B}{\mathrm{Le}}\frac{\partial T}{\partial z}. \tag{25}$$

Boundary conditions for Eqs. (22)–(25) are

$$w = 0, \quad T = 0, \quad \varphi = 0 \quad \text{at} \quad z = 0, 1. \tag{26}$$

Note that as the parameter *Rm* is not involved in Eqs. (22)–(26) it is just a measure of the basic static pressure gradient. The six unknowns u, v, w, p, T and φ can be reduced to three by operating Eq. (23) with $e_z.curl\ curl$, which yields

$$\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\nabla^2 w - \left(1 - F\frac{\partial}{\partial t}\right)\nabla^4 w = \mathrm{Ra}\nabla_H^2 T - \mathrm{Rn}\nabla_H^2\varphi, \tag{27}$$

where ∇_H^2 is the two-dimensional Laplace operator on the horizontal plane, that is

$$\nabla_H^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}.$$

3. Normal modes

We express the disturbances into normal modes of the form

$$[w, T, \varphi] = [W(z), \Theta(z), \varphi(z)]\exp\left(ik_x x + ik_y y + nt\right), \tag{28}$$

where k_x, k_y are the wave numbers in the x and y direction, respectively, and n is the growth rate of the disturbances.

Substituting the identities in (28) into Eqs. (24)–(27) we obtain the following eigen value problem

$$\left(D^2 - a^2\right)\left((1 - nF)\left(D^2 - a^2\right) - \frac{n}{\mathrm{Pr}}\right)W - a^2\mathrm{Ra}\Theta + a^2\mathrm{Rn}\varphi = 0, \tag{29}$$

$$W - \frac{N_A}{Le}\left(D^2 - a^2\right)\Theta - \left(\frac{1}{\mathrm{Le}}\left(D^2 - a^2\right) - n\right)\varphi = 0, \tag{30}$$

$$W + \left(D^2 - a^2 - n + \frac{N_B}{\mathrm{Le}}D - \frac{2N_A N_B}{\mathrm{Le}}D\right)\Theta - \frac{N_B}{\mathrm{Le}}D\varphi = 0, \tag{31}$$

$$W = 0, \quad D^2W = 0, \quad \Theta = 0, \quad \varphi = 0 \quad \text{at} \quad z = 0 \tag{32}$$

and

$$W = 0 \quad D^2W = 0, \quad \Theta = 0, \quad \varphi = 0 \quad \text{at} \quad z = 1. \tag{33}$$

where $D = \frac{d}{dz}$ and $a^2 = k_x^2 + k_y^2$ is the dimensionless horizontal wave number.

4. Linear stability analysis

The eigen functions $f_j(z)$ corresponding to the eigen value problem (29)–(33) are $f_j = \sin(j\pi z)$. Considering solutions W, Θ and Φ of the form

$$W = W_0 \sin(\pi z), \quad \Theta = \Theta_0 \sin(\pi z), \quad \Phi = \Phi_0 \sin(\pi z). \tag{34}$$

Substituting (33) into Eqs. (29)–(31) and integrating each equation from $z = 0$ to $z = 1$, we obtain the following linear system

$$\begin{pmatrix} -J\left((1-nF)J + \dfrac{n}{\mathrm{Pr}}\right) & a^2\mathrm{Ra} & -a^2\mathrm{Rn} \\ 1 & -(J+n) & 0 \\ 1 & \dfrac{N_A}{\mathrm{Le}}J & \dfrac{J}{\mathrm{Le}} + n \end{pmatrix} \cdot \begin{pmatrix} W_0 \\ \Theta_0 \\ \varphi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $J = \pi^2 + a^2$.

The above linear system has a non-trivial solution if and only if

$$\mathrm{Ra} = \frac{1}{a^2}\left((J+n)\left(J^2 + \left(-J^2F + \frac{J}{\mathrm{Pr}}\right)n\right)\right) - \frac{\left(\dfrac{N_A}{\mathrm{Le}}J + J\right) + n}{\dfrac{J}{\mathrm{Le}} + n}\mathrm{Rn}. \tag{35}$$

Setting $n = i\omega$, (where ω is a real dimensional frequency) in Eq. (35), we get

$$\mathrm{Ra} = \Delta_1 + i\omega\Delta_2, \tag{36}$$

where

$$\Delta_1 = \frac{J}{a^2}\left(J^2 + \left(JF - \frac{1}{\mathrm{Pr}}\right)\omega^2\right) - \frac{\dfrac{J^2}{\mathrm{Le}^2}(N_A + \mathrm{Le}) + \omega^2}{\left(\dfrac{J}{\mathrm{Le}}\right)^2 + \omega^2}\mathrm{Rn}, \tag{37}$$

and

$$\Delta_2 = \frac{J^2}{a^2}\left(1 - JF + \frac{1}{\mathrm{Pr}}\right) - \frac{\frac{J}{\mathrm{Le}} - J\left(\frac{N_A}{\mathrm{Le}} + 1\right)}{\left(\frac{J}{\mathrm{Le}}\right)^2 + \omega^2}\mathrm{Rn}. \tag{38}$$

Since Ra is a physical quantity, it must be real. Hence, from Eq. (36) we can conclude that either $\omega = 0$ (exchange stability, steady onset) or $\Delta_2 = 0$ (or $\omega \neq 0$, oscillatory onset).

5. Stationary convection

For stationary convection $n = 0$ ($\omega = 0$), Eq. (35) reduces to

$$(\mathrm{Ra})_s = \frac{\left(\pi^2 + a^2\right)^3}{a^2} - \mathrm{Rn}\left(\mathrm{Le} + N_A\right). \tag{39}$$

Equation (39) expresses the Rayleigh number as a function of the dimensionless horizontal wave number a^2 and the parameters Rn, Le, N_A. Since the elastico-viscous parameter F is not present in Eq. (39) it may be concluded that in the stationary case ($n = 0$) the Walters' (model B') elastico-viscous fluid behaves like an ordinary Newtonian fluid. Note that Eq. (39) is identical to that obtained by Sheu [29] in the absence of porous medium and Chand and Rana [17] in the absence of Darcy-Brinkman porous medium.

The critical cell size at the onset of instability is obtained by minimizing Ra with respect to a. Thus, the critical cell size must satisfy

$$\left(\frac{\partial \mathrm{Ra}}{\partial a}\right)_{a=a_c} = 0,$$

which gives

$$2(a_c^2)^3 + 3\pi^2 a_c^2 - \pi^6 = 0. \tag{40}$$

From Eq. (39), we get

$$a_c = \frac{\pi}{\sqrt{2}} \simeq 2.22144. \tag{41}$$

And the corresponding critical Rayleigh number Ra_c for steady onset is

$$(\mathrm{Ra}_c)_s = \frac{27\pi^2}{4} - R_n(L_e + N_A). \tag{42}$$

6. Oscillatory convection

For oscillatory convection ($\omega \neq 0$), we must have $\Delta_2 = 0$. Hence the frequency of the oscillations is

$$\omega^2 = \frac{a^2}{J}\left(\frac{\dfrac{1}{\mathrm{Le}} - \left(\dfrac{N_A}{\mathrm{Le}} + 1\right)}{1 - JF + \dfrac{1}{\mathrm{Pr}}}\right)\mathrm{Rn} - \frac{J^2}{\mathrm{Le}^2}. \tag{43}$$

Equation (36) with $\Delta_2 = 0$ gives the following thermal oscillatory Rayleigh number

$$(\mathrm{Ra})_{osc} = \frac{J}{a^2}\left(J^2 + \left(JF - \frac{1}{\mathrm{Pr}}\right)\omega^2\right) - \frac{\dfrac{J^2}{\mathrm{Le}^2}(\mathrm{Le} + \mathrm{N_A}) + \omega^2}{\left(\dfrac{J}{\mathrm{Le}}\right)^2 + \omega^2}\mathrm{Rn}. \tag{44}$$

Since oscillatory convection will only exist when positive values for ω^2 in Eq. (44) can be obtained, the following conditions

$$\begin{aligned} &\mathrm{Rn} < 0, 1 + \frac{1}{\mathrm{Pr}} > JF \quad \text{and} \quad \frac{1}{\mathrm{Le}} > \left(\frac{N_A}{\mathrm{Le}} + 1\right) \\ \text{or} \quad &\mathrm{Rn} > 0, 1 + \frac{1}{\mathrm{Pr}} < JF \quad \text{and} \quad \frac{1}{\mathrm{Le}} < \left(\frac{N_A}{\mathrm{Le}} + 1\right) \end{aligned} \tag{45}$$

are sufficient conditions for the non-existence of oscillatory convection. Note that the violation of (45) does not necessarily imply the occurrence of oscillatory convection.

In order to analyze the effect of the physical parameters on the onset of oscillatory convection we use the following procedure:

i) we fix a set of values for the physical parameters F, Le, N_A, Pr and Rn and we check that sufficient conditions (45) for the non-existence of oscillatory convection are not satisfied for the selected parameters, otherwise the set is discarded;
ii) we insert the selected values into Eqs. (43) and (44);
iii) we compute the corresponding neutral curve $(\mathrm{Ra})_{osc}(a)$ by increasing/decreasing the value of a stepwise. In order to ensure the existence of oscillatory convection. along the computation of the neutral curve we check that the value of ω_2 in Eq. (43) is positive;
iv) we obtain the critical oscillatory Rayleigh number by finding numerically the minimum of the neutral curve obtained in step iv);
v) we repeat steps i) to v) for several different sets of values for the physical parameters F, Le, N_A, Pr and Rn.

7. Results and discussions

Critical Rayleigh numbers for onset of steady and oscillatory convection are given by Eqs. (42) and (43), (44), respectively. The critical Rayleigh value obtained for the onset of steady convection in the current Walters (model B') elastico-viscous fluid problem does not depend on viscoelastic parameters and it takes the same value that the one obtained for an ordinary Newtonian fluid. Furthermore, the critical wave number, a_c, defined by Eq. (41) at the onset of steady convection coincides with those reported by Tzou [6], Kuznetsov and Nield [9] and Chand and Rana [17]. Note that this critical value does not depend on any thermo physical property of the nanofluid. Consequently, the interweaving behaviors' of Brownian motion and thermophoresis of nanoparticles does not change the cell size at the onset of steady instability and the critical cell size (a_c) is identical to the well known result for Bénard instability with a regular fluid [1] . In the absence of nanoparticles, that is $\mathrm{Rn} = N_A = 0$, the critical Rayleigh number takes the value $\mathrm{Ra}_c = 27\pi^2/4$, which is exactly the critical Rayleigh number for regular fluids [1]. Thus the combined effect of Brownian motion and thermophoresis of nanoparticles on the critical Rayleigh number is reflected in the second term in Eq. (41). From Eq. (41), it can be concluded that for the case of top-heavy distribution of nano particles ($\varphi_1 > \varphi$ and $\rho_p > \rho$), which corresponds to positive values of Rn, the value of the steady critical Rayleigh number for the nanofluid is smaller than that for an ordinary fluid, that is, steady convection sets earlier in these kind of nanofluids than in an ordinary fluid. This implies that thermal conductivity of ordinary fluids is higher than that of nanofluids with top-heavy distribution of nano particles. On the contrary, for the case of bottom-heavy distribution of nano particles ($\varphi_1 < \varphi$ and $\rho_p > \rho$), which corresponds to negative values of Rn, the value of the critical Rayleigh number for the nanofluid is larger than that for an ordinary fluid, that is, convection sets earlier in a ordinary fluid than in a nanofluid with bottom-heavy distribution of nano particles. This implies that thermal conductivity of this kind of nanofluids is higher than that of ordinary fluids.

The variation of the steady thermal Rayleigh number $(\mathrm{Ra})_s$ as a function of the wave number a for different sets of values for the physical parameters Le, Rn and N_A are shown in Figs. 2, 3 and 4. Once the values of N_A and Rn are fixed to 10 and -1, respectively, the variation of $(\mathrm{Ra})_s$ with the wave number for three different values of the Lewis number, namely $\mathrm{Le} = 100$, 500 and 1000, is plotted in Fig. 2. The increase in the Rayleigh number observed in Fig. 2 as the Lewis number is increased reveals that an increase in the Lewis number tends to delay the onset of steady convection. Conversely, the decrease experimented by the steady Rayleigh number in Fig. 3 when the concentration Rayleigh number Rn is decreased and Le and N_A take the values 500 and 10, respectively, indicates that increasing the concentration Rayleigh number accelerates the onset of steady convection. Note that a negative value of Rn is associated to a bottom-heavy distribution of nanoparticles whereas a positive value of Rn indicates a top-heavy distribution of nanoparticles. Figure 3 shows that although the steady Rayleigh number is smaller for top-heavy nanoparticles the onset of steady convection is possible for both bottom-heavy and top-heavy nanoparticle distributions. Figure 4 represents the variation of Ra_s as a function of the wave number for two values of modified diffusivity ratio ($N_A = 1$ and 10) when $\mathrm{Le} = 500$ and $\mathrm{Rn} = -1$. This figure shows that an increase in the value of N_A implies an increase in the value of $(\mathrm{Ra})_s$. Results observed in Figs. 2, 3 and 4 are consistent with those reported by Sheu [29] and Chand and Rana [17].

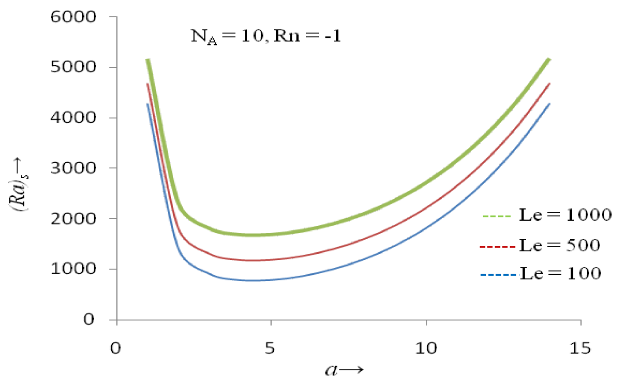

Fig. 2. Variation of stationary Rayleigh number $(\mathrm{Ra})_s$ with wave number a for different values of Lewis number Le

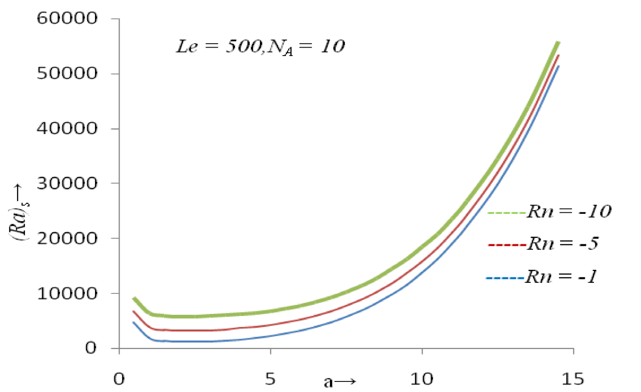

Fig. 3. Variation of stationary Rayleigh number $(\mathrm{Ra})_s$ with wave number a for different values of concentration Rayleigh number Rn

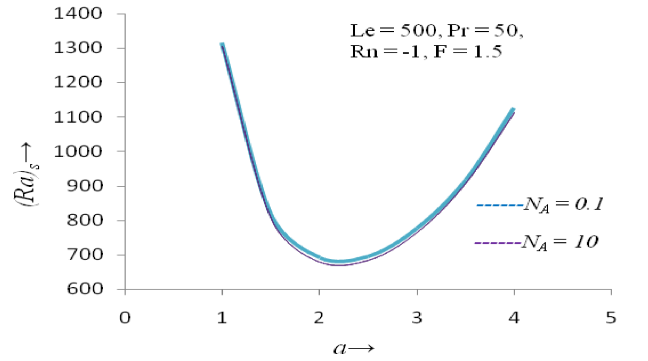

Fig. 4. Variation of stationary Rayleigh number $(\mathrm{Ra})_s$ with wave number a for different values of modified diffusivity ratio N_A

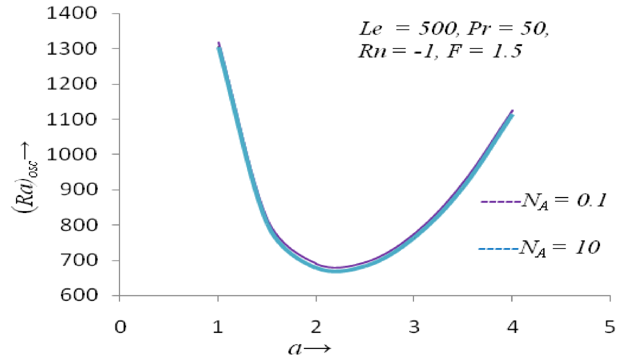

Fig. 6. Variation of oscillatory Rayliegh number $(\mathrm{Ra})_{osc}$ with wave number a for different modified diffusivity ratio N_A

Nield and Kuznetsov [12] and Kuznetsov and Nield [9] showed that oscillatory instability is possible only for bottom-heavy nanoparticle distributions. For heavy nanoparticles ($\rho_p > \rho$), a bottom-heavy nanoparticle distribution is equivalent to a negative values of Rn. In such case the value of N_A will also be negative. From now only negative values of Rn are considered. Figure 5 shows the variation of the oscillatory Rayleigh number Ra_{osc} with the wave number a for three values of the kinematic visco-elasticity parameter F. It is found that Ra_{osc} increases as F is increased. Thus, it can be inferred from Figs. 5–8 and 9 that whereas the kinematic visco-elasticity parameter F has a stabilizing effect on the oscillatory convection, that is the onset of oscillatory convection is retarded when F is increased, the rest of physical parameters, namely N_A, Pr, Rn and Le, have a destabilizing effect on the oscillatiory convection, that is the onset of oscillatory convection is accelerated when one of these parameters is increased. Figure 10 shows the variation of both stationary and oscillatory Rayleigh numbers as the wave number is varied for a set of fixed values of the rest of physical parameters. It can be observed in Fig. 10 that the steady Rayleigh number is higher than the oscillatory Rayleigh number along the whole range of wave numbers investigated. This result suggests that oscillatory convection might set in before steady convection.

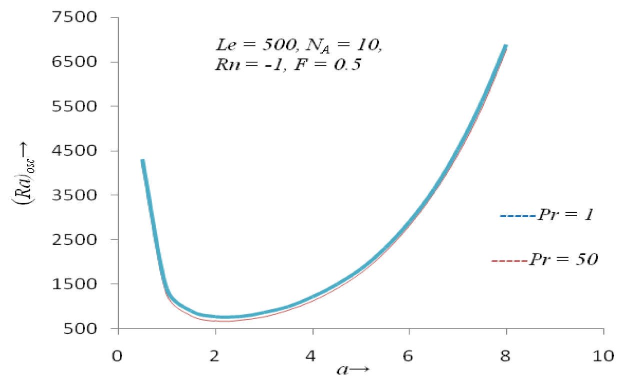

Fig. 7. Variation of oscillatory Rayliegh number $(\mathrm{Ra})_{osc}$ with wave number a for different Prandtl number Pr

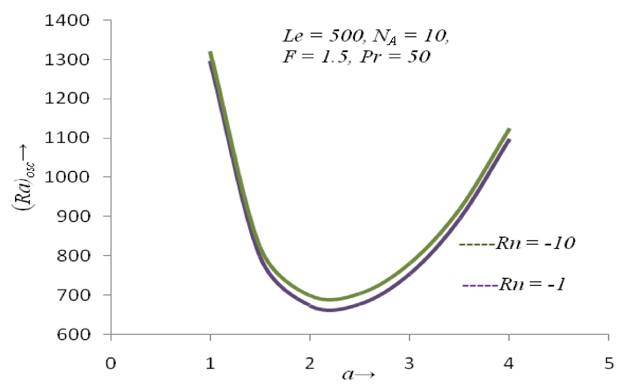

Fig. 8. Variation of oscillatory Rayliegh number $(\mathrm{Ra})_{osc}$ with wave number a for different concentration Rayleigh number Rn

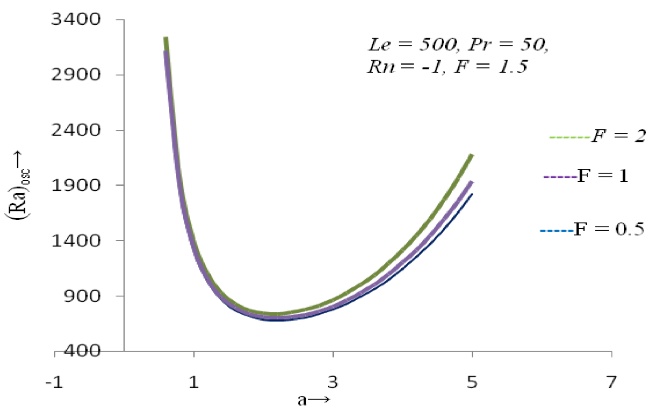

Fig. 5. Variation of oscillatory Rayliegh number $(\mathrm{Ra})_{osc}$ with wave number a for different value of kinamtic visco-elasticity parameter F

The variation of the critical oscillatory Rayleigh number $(\mathrm{Ra})_{osc}$ with the kinematic visco-elaticity parameter F for different sets of values for the physical parameters Le, Rn, N_A and Pr are shown in Figs. 11 to 14. It is worth noting that for the set of physical parameters chosen in all these figures convection sets in with an oscillatory mode and it remains oscillatory until a certain (critical) value of the kinematic visco-elasticity parameter F is reached. At this critical value of F convection ceases to be oscillatory and it becomes steady. This critical value of F, which depends upon the other physical parameters Le, Rn, N_A and Pr determines the boundary between oscillatory and steady convection. Figure 11 shows the effect of the Prandtl number Pr on the variation of the critical oscillatory Rayleigh number as a function or the kinematic visco elasticity parameter F when $\mathrm{Le} = 500$, $N_A = 10$ and $\mathrm{Rn} = 1$.

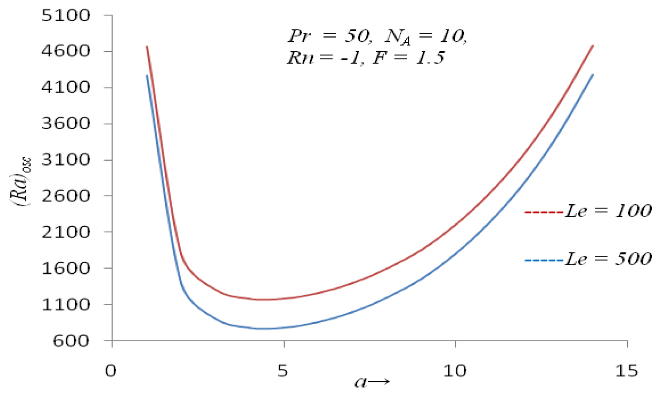

Fig. 9. Variation of oscillatory Rayliegh number $(\mathrm{Ra})_{osc}$ with wave number a for different Lewis number Le

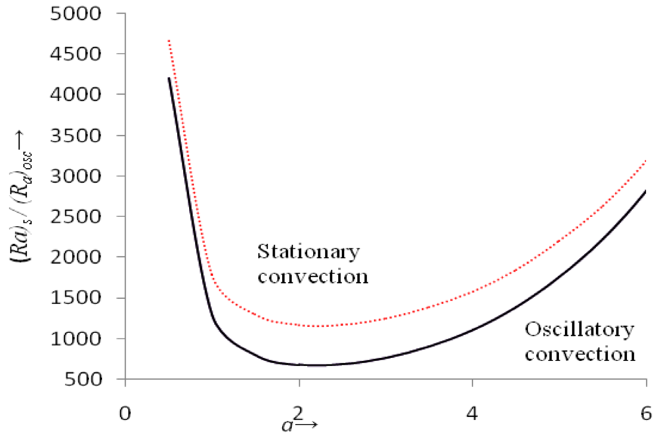

Fig. 10. Variation of stationary Rayleigh number and oscillatory Rayleigh number with wave number a for $\mathrm{Pr} = 50$, $N_A = 10$, $\mathrm{Rn} = -1$, $F = 1.5$

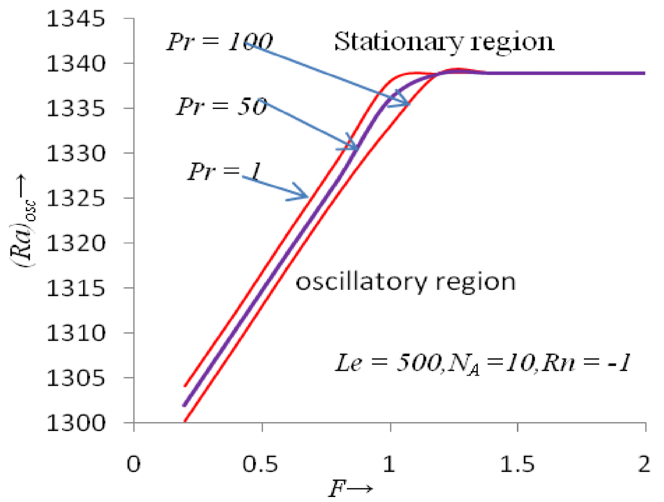

Fig. 11. Variation in the critical oscillatory Rayleigh number $(\mathrm{Ra})_{osc}$ with kinamatic visco-elaticity parameter F for different values of Prantdl number Pr

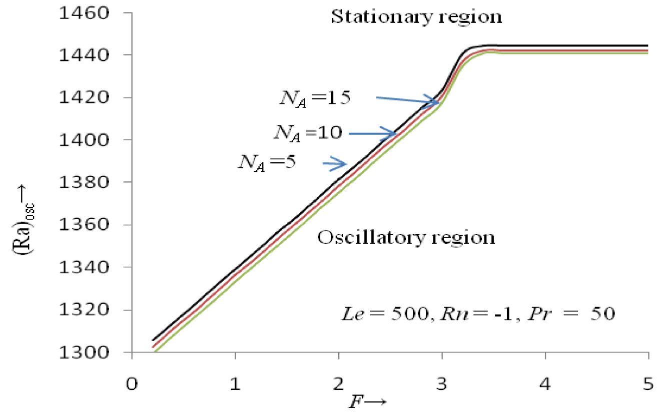

Fig. 12. Variation in the critical oscillatory Rayleigh number $(\mathrm{Ra})_{osc}$ with kinematic visco-elaticity parameter F for different values modified diffusivity ratio N_A

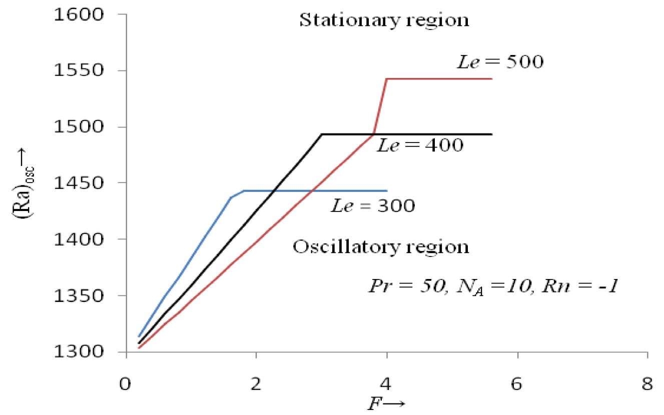

Fig. 13. Variation in the critical oscillatory Rayleigh number $(\mathrm{Ra})_{osc}$ with kinamatic visco-elaticity parameter F for different values Lewis number Le

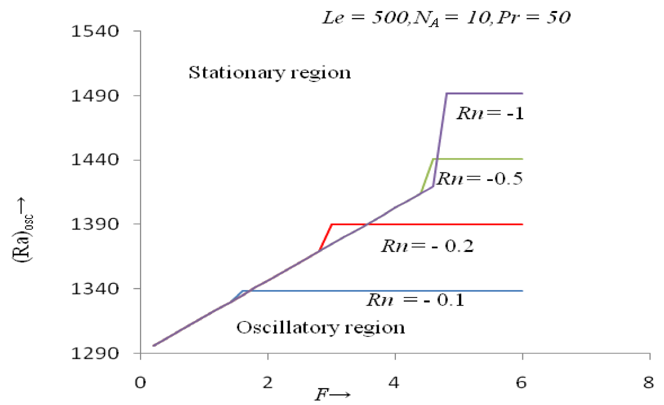

Fig. 14. Variation in the critical oscillatory Rayleigh number $(\mathrm{Ra})_{osc}$ with kinamatic visco-elaticity parameter F for different values concentration Rayleigh number Rn

The effect of the modified diffusivity ratio N_A on the oscillatory Rayleigh number for fixed values of the rest of parameters is shown in Fig. 12. Figures 13 and 14 show, respectively, the effect of the parameters Le and Rn on the oscillatory Rayleigh number when the rest pf parameters are fixed.

The results presented in Figs. 5 to 14 are in good agreement with those obtained by Sheu [27].

8. Conclusions

The onset of both stationary and oscillatory convection for a Walters' (model B') elastico-viscous nanofluid layer heated from below is investigated by using a linear stability analysis. The main conclusions of the current study are:

- The Lewis number Le and modified diffusivity ratio N_A stabilize the stationary convection and destabilize the oscillatory convection.
- The concentration Rayleigh number *Rn* destabilizes both stationary and oscillatory convection.
- The oscillatory convection is possible only for bottom-heavy nanoparticle distributions whereas stationary convection is possible for both bottom and top-heavy distributions of nanoparticles.
- The Prandtl number Pr destabilizes the oscillatory convection and kinematic visco-elasticity parameter F stabilizes and both has no effect on stationary convection.
- Convection intially begins in form of a oscillatory mode and it remains oscillatory until certain (critical) value of the kinematic visco-elasticity parameter F is reached. At this critical value of F convection ceases to be oscillatory and stationary convection occurs. This critical value of F, which depends upon the rest of parameters, namely Le, N_A and Rn, determines the boundary betweeen oscillatory and stationary convection.
- The steady Rayleigh number is higher than the oscillatory Rayleigh number.
- Sufficient conditions for the non-existence of oscillatory convection are given in (45).

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