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A QUANTITATIVE MANAGEMENT SUPPORT MODEL FOR A CERTAIN PRODUCTION SUPPLY SYSTEM IN NON-EXTREME STATES

The paper is devoted to building a probabilistic method of analyzing the operation of a certain production supply system. The analysis is carried out for non-extreme states of the level in store, into which two separate streams of production (the product) are directed. A system of partial differential equations describing this case was derived which is satisfied by the joint density function defining the probabilities of states of the three-dimensional process characterizing the system's functioning.

Keywords: *production supply system, process, non-extreme state, system of differential equations*

1. Introduction

Production supply systems are the subject of research and analyses in various publications (e.g. [1, 3–7, 9–17]). The paper is a continuation of the research described in [3–7, 9–12, 15, 16]. It is devoted to constructing a new probabilistic method of analysing a production supply work system in which the flow of production (the product) is sent to the recipient through a warehouse by using two transport networks. The paper

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contains a description of how the system operates and the theoretical characteristics of its functioning. We present a system of differential equations, which determine the probability density functions defining the states of the three-dimensional process describing the functioning of system when the store is neither empty nor full.

2. Description of the system

The recipient E (e.g. power station) has steady demand for a product (e.g. coal), which it receives from two sources, T_1 and T_2. The production stream $y_1(t)$ from the subsystem T_1 and the production stream $y_2(t)$ from the subsystem T_2 are delivered in a continuous manner (e.g. with the use of belt conveyors, pipelines, transmission lines).

Random changes in the processes $y_1(t)$, $y_2(t)$ and unplanned interruptions (failures) of the transport subsystems T_1 and T_2 are factors diminishing the efficiency of the system's functioning. This efficiency can be increased – and at the same time the probability that the demand of the recipient E is not satisfied can be reduced – by using a store/container M with volume V. The product streams $y_1(t)$, $y_2(t)$ are collected in the subsystem M, when the level $z(t)$ in store does not exceed V and when $y_1(t) > a$ or $y_2(t) > a$. If the level in store is equal to V and $y_1(t) >$ a or $y_2(t) >$ a, then the size of the streams $y_1(t)$, $y_2(t)$ is limited to a. When the store M is empty and $y_1(t) <$ a and $y_2(t) <$ a, then the situation is unfavourable to the recipient E. The determination of the probability of this event is of practical significance. The layout of the subsystems T_1, T_2, M and E is presented in Fig. 1.

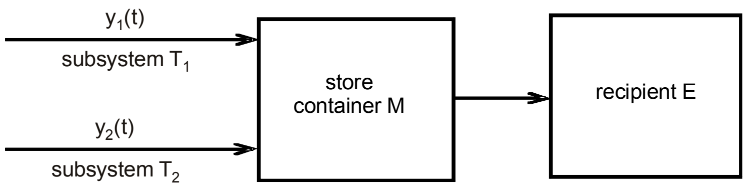

Fig. 1. A general diagram of the production supply system considered

3. Theoretical characteristics of the system's functioning

The operation of the system considered is described by a three-dimensional process ($y_1(t)$, $y_2(t)$, $z(t)$). We assume that the subsystems T_1 and T_2 act independently, and the processes $y_1(t)$ and $y_2(t)$, controlling the level $z(t)$ in store, are Markov processes with a finite number of states. Let us denote the possible states of the product stream $y_1(t)$ into the subsystem M as: $y_{11}, y_{12}, \ldots, y_{1n}$, and the possible states of the product stream $y_2(t)$ as: $y_{21}, y_{22}, \ldots, y_{2m}$.

The intensities of transitions between states (levels of product delivery) of the processes $y_1(t)$ and $y_2(t)$ are denoted by $\pi_{jk}^{(1)}$ and $\pi_{sk}^{(2)}$, respectively, which is schematically written in the form:

$$y_{1j} \xrightarrow{\pi_{jk}^{(1)}} y_{1k} \qquad \text{for} \qquad j \neq k \tag{1}$$

$$y_{2s} \xrightarrow{\pi_{si}^{(2)}} y_{2i} \qquad \text{for} \qquad s \neq i \tag{2}$$

Managing the system considered requires the designation of the joint probability distribution of the system, $P(y_1(t), y_2(t), z(t))$, describing the probability that at a fixed time t the supply stream of the product $y_1(t)$ will be in the state

$$y_1 : y_1 \in \{y_{11},\ y_{12},\ \ldots,\ y_{1n}\} \tag{3}$$

and the supply stream $y_2(t)$ in the state

$$y_2 : y_2 \in \{y_{21},\ y_{22},\ \ldots,\ y_{2m}\} \tag{4}$$

and at the same time the level in the store (container) M will equal z.

For each specific z, $0 < z < V$, this probability is equal to zero:

$$P\big(y_1(t), y_2(t), z(t) = 0\big) = 0$$

because the possible values of $z \in (0, V)$ are uncountable.

Hence, the probability density function described above can be denoted $f_w^r(z,t)$ and is defined by the formula

$$\mathrm{P}\big(y_1(t) = r, \qquad y_2(t) = w, \qquad a_1 < z(t) < b_1\big) = \int_{a_1}^{b_1} f_w^r(z,t)\,dz \tag{5}$$

where $0 \leq a_1 < b_1 \leq V$.

Density function $f_w^r(z,t)$ can be treated as a function of two variables, z and t, labelled by states r and w, according to the processes $y_1(t)$ and $y_2(t)$, respectively.

Analysis of the system considered needs to study the following three cases:

1. A partially filled store

$$0 < z(t) < V \tag{6}$$

2. An empty store

$$z(t) = 0 \tag{7}$$

3. A full store

$$z(t) = V \tag{8}$$

These cases should be dealt with individually because they correspond to different conditions in which the system works.

In order to solve many problems associated with improving the efficiency of the system considered (Fig. 1), together with obtaining appropriate forecasts, it is sufficient to derive probabilities of the following forms:

$$P\left(a_1 < z(t) < b_1, \quad y_1(t) = x_{1k}, \quad y_2(t) = X_{2i}\right) = \int_{a_1}^{b_1} f_{x_{2i}}^{x_{1k}}(z,t)\,dz \tag{9}$$

$$P\left(z(t) = 0, \quad x_1(t) = x_{1k}, \quad x_2(t) = X_{2i}\right) \tag{10}$$

$$P\left(z(t) = V, \quad x_1(t) = x_{1k}, \quad x_2(t) = X_{2i}\right) \tag{11}$$

where x_{1k} is the k-th possible state of the process $x_1(t) = y_1(t) - a$ ($x_{1k} = y_{1k} - a$, $k = 1, 2, \ldots, n$) and x_{2i} denotes the i-th possible state of the process $x_2(t) = y_2(t) - a$ ($x_{2i} = y_{2i} - a$, $i = 1, 2, \ldots, m$).

Formula (9) expresses the probability that, at a fixed time t, the level of stocks in the store M belongs to the interval (a_1, b_1), while the states of the processes $x_1(t)$, $x_2(t)$ are x_{1k} and x_{2i}, respectively. Similarly, Eqs. (10), (11) are the probabilities of the process $x_1(t)$ being in the state x_{1k}, the process $x_2(t)$ being in the state x_{2i} and the store being empty and full, respectively, at time t.

The probability $P(0 < z(t) < V, x_1(t) = x_{1k}, x_2(t) = x_{2i}))$ will be found from the formula

$$\begin{aligned} P\left(0 < z(t) < V, x_1(t) = x_{1k}, x_2(t) = x_{2i}\right) \\ = P\left(a_1 < z(t) < b_1, x_1(t) = x_{1k}, x_2(t) = x_{2i}\right) \end{aligned} \tag{12}$$

To calculate probabilities of the form in Eq. (9), a method of determining the density function $f_{x_{2i}}^{x_{1k}}(z,t)$ should be described. The probabilities given in Eqs. (10) and (11) will be denoted by $Q_{x_{2i}}^{x_{1k}}(\{0\},t)$, $Q_{x_{2i}}^{x_{1k}}(\{V\},t)$, respectively.

The paper presents a system of differential equations, which satisfy the density functions $f_{x_{2i}}^{x_{1k}}(z,t)$. Such a system of equations determining the functions $Q_{x_{2i}}^{x_{1k}}(\{0\},t)$ and $Q_{x_{2i}}^{x_{1k}}(\{V\},t)$ will be described elsewhere.

In order to derive these systems of equations for differentiable functions, we use the well known Taylor's formula

$$h(z+\Delta z,t)=h(z,t)+\frac{\partial h(z,t)}{\partial z}\Delta z+o(\Delta z) \tag{13}$$

where $o(\Delta z)$ denotes a term of degree higher than Δz:

$$\lim_{\Delta z\to 0}\frac{o(\Delta z)}{\Delta z}=0 \tag{14}$$

The systems of equations obtained enable us to obtain quantitative characteristics of the system investigated which may be used by a decision-making body in order to increase the effectiveness of the system's functioning.

3. A partially filled store

Analysis of the system will be conducted for the first case, i.e. when the stock level $z(t)$ in the container M satisfies the condition:

$$0<z(t)<V \tag{15}$$

Under this scenario, the transport subsystems T_1 and T_2 can be used to fill the store, and the receiver E has its present demand guaranteed.

Calculating the probability

$$P\left(a_1<z(t)<b_1,\,x_1(t)=x_{1k},\,x_2(t)=x_{2i}\right)$$

in accordance with Eq. (9) requires knowledge of the density function $f_{x_{2i}}^{x_{1k}}(z,t)$. We now introduce a set of equations which these functions satisfy. According to the working conditions of the system in configuration (15), we have:

$$\begin{aligned} f_{x_{2i}}^{x_{1k}}(z,t+\tau)\approx{}& f_{x_{2i}}^{x_{1k}}\left(\dot{z}-\left(x_{1k}+x_{2i}+a\right)\tau,t\right)\left(1-\left(\pi_k^{(1)}+\pi_i^{(2)}\right)\tau\right)\\ &+\sum_{k'\neq k} f_{x_{2i}}^{x_{1k'}}\left(z-\left(x'_{1k}+x_{2i}+a\right)\tau,t\right)\left(1-\pi_i^{(2)}\tau\right)\pi_{k'k}^{(1)}\tau\\ &+\sum_{i'\neq i} f_{x'_{2i}}^{x_{1k}}\left(z-\left(x_{1k}+x'_{2i}+a\right)\tau,t\right)\left(1-\pi_i^{(1)}\tau\right)\pi_{i'i}^{(2)}\tau\\ &+\sum_{i'\neq i} f_{x'_{2i}}^{x'_{1k}}\left(z-\left(x'_{1k}+x'_{2i}+a\right)\tau,t\right)\pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau \end{aligned} \tag{16}$$

where:

$$\pi_k^{(1)} = \sum_{l \neq k} \pi_{kl}^{(1)} \tag{17}$$

$$\pi_i^{(2)} = \sum_{l \neq i} \pi_{il}^{(2)} \tag{18}$$

The intuition behind Eq. (16) is as follows: the first element in this equation indicates the probability of remaining in the state (x_{1k}, x_{2i}). This probability is 1 minus the sum of the transition intensities from the state (x_{1k}, x_{2i}) (see, e.g. [2, 8]). In our case, the transition intensity from state x_{1k} is $\pi_k^{(1)}$ (Def. (1), Eq. (17)) and from state x_{2i} is equal to $\pi_i^{(2)}$ (Def. (2), Eq. (18)). This uses the fact that in "simple" processes double changes of states occur at the rate of degree higher than τ. This is taken into account by the asymptotic equality ≈, which allows us to omit the term $o(\tau)$, which satisfies the condition

$$\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0 \tag{19}$$

Regardless the way the states x_{1k} and x_{2i} change, there are changes in the level of stocks z. They are determined both through the process $y_1(t)$ and the flow of production $y_2(t)$. If the state at time t is (x_{1k}, x_{2i}), then τ time units later, the stock level has increased by $(x_{1k} + x_{2i} + a)\tau$. Thus, when at time $t + \tau$ the stock level is z, then at time t it must equal $z - (x_{1k} + x_{2i} + a)\tau$. This fact is reflected in the first term of Eq. (16). The other terms can be interpreted in a similar way.

Equation (16) can be transformed using Taylor's formula (13):

$$\begin{aligned} f_{x_{2i}}^{x_{1k}}(z, t+\tau) \approx \Bigg(& f_{x_{2i}}^{x_{1k}}(z,t) + \frac{\partial f_{x_{2i}}^{x_{1k}}(z,t)}{\partial z}\big(-(x_{1k}+x_{2i}+a)\tau\big) \\ & + o\big(-(x_{1k}+x_{2i}+a)\tau\big)\Bigg)\Big(1-\big(\pi_k^{(1)}+\pi_i^{(2)}\big)\tau\Big) \\ & + \sum_{\substack{k' \neq k \\ i' \neq i}} \Bigg(f_{x'_{2i'}}^{x_{1k'}}(z,t) + \frac{\partial f_{x'_{2i'}}^{x_{1k'}}(z,t)}{\partial z}\big(-(x'_{1k}+x'_{2i}+a)\tau\big) \\ & + o\big(-(x'_{1k}+x'_{2i}+a)\tau\big)\Bigg)\pi_{k'k}^{(1)}\pi_{i'i}^{(2)}\tau^2 \end{aligned} \tag{20}$$

We shall now successively apply the following operations to Eq. (20): (i) move the function $f_{x_{2i}}^{x_{1k}}(z,t)$ to the left hand side of Eq. (20), (ii) divide both sides of the formula obtained by τ, (iii) on both sides take the limit when $\tau \to 0$.

As a result of these operations, the asymptotic equality (20) turns into a simple equality (21) based on Eqs. (19) and (14). Equation (21) presents the system of differential equations which is satisfied by the density functions $f_{x_{2i}}^{x_{1k}}(z,t)$ determining the probabilities given by Eq. (9). The system is of the form:

$$\frac{\partial f_{x_{2i}}^{x_{1k}}(z,t)}{\partial t} = -\frac{\partial f_{x_{2i}}^{x_{1k}}(z,t)}{\partial z}\left(x_{1k} + x_{2i} + a\right) - f_{x_{2i}}^{x_{1k}}(z,t) \tag{21}$$

for $0 < z < V$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, n$.

An analysis of the system's operation when the store is empty or full will be presented elsewhere. The result expressed by Eq. (21), together with the derivations of the probabilities of the states of the system for the two cases above, will create the opportunity to obtain the system characteristics, enabling an increase in its operational efficiency.

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