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# A CERTAIN APPROACH TO KRIPKE SEMANTICS FOR NORMAL MODAL LOGICS

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**Abstract**
In this paper the authors propose a method of verifying formulae in normal modal logics. In order to show that a formula $\alpha$ is a thesis of a normal modal logic, a set of decomposition rules for any formula is given. These decomposition rules are based on the symbols of assertion and rejection of formulae.

As it is known, validity of formulae in modal systems may be checked by semantic tableaux (see [1], [3]) or by using the classical values 0 and 1 and signed modal symbols (see [2], [7]). Today, tableaux are usually defined as trees, with formulae occuring in their nodes. Recall that in [1] the author presents labelled tableau calculi for a wide class of modal logics that are based on Fitting's idea. In Fitting's method any formula occurs with a prefix which is represented by a finite sequence of positive integers. This prefix is connected with the Kripke semantic, i.e. there is a name of a possible world and the accessibility relation is encoded in the structure of the prefixes. In our approach we will give a modification of the method presented in [6]. In this paper, we propose to verify formulae using the method *reductio ad absurdum* and the rules of assertion and rejection.

Several normal modal systems will be considered. These systems are built on the basis of the classical propositional calculus by adding new modal connectives $(L, M)$, modal axioms (the monotonicity axiom $CLCpqCLpLq$) and rules of deduction (Gödel's rule $(\frac{\alpha}{L\alpha})$)[1].

The following modal systems will be considered (see [4], [5], [6])[2]:

$$\mathbf{K} = Cn(Ax \cup \{CLCpqCLpLq\}),$$

$$\mathbf{K4} = Cn(\mathbf{K} \cup \{CLpLLp\}),$$

$$\mathbf{T} = Cn(\mathbf{K} \cup \{CLpp\}),$$

$$\mathbf{S4} = Cn(\mathbf{K} \cup \{CLpp, CLpLLp\}),$$

$$\mathbf{B} = Cn(\mathbf{T} \cup \{CNpLNLp\}),$$

$$\mathbf{S5} = Cn(\mathbf{S4} \cup \{CNLpLNLp\}),$$

$$\mathbf{S4.2} = Cn(\mathbf{S4} \cup \{CMLpLMp\}),$$

$$\mathbf{S4.3} = Cn(\mathbf{S4} \cup \{ALCLpqLCLqp\}).$$

For the above systems relational structures $(U, R)$, where $U \neq \emptyset$, $R \subseteq U \times U$, are considered. Let $S$ be the set of all meaningful formulae of an arbitrary modal system. We define a valuation $V : S \times \ U \rightarrow \{0, 1\}$ on a structure $(U, R)$ as follows (see [2]):

$$V(p_i, m) = 0 \vee V(p_i, m) = 1, p_i \in S,$$

$$V(N\alpha, m) = 1 \Leftrightarrow V(\alpha, m) = 0,$$

$$V(A\alpha\beta, m) = 1 \Leftrightarrow (V(\alpha, m) = 1 \vee V(\beta, m) = 1),$$

$$V(K\alpha\beta, m) = 1 \Leftrightarrow (V(\alpha, m) = 1 \wedge V(\beta, m) = 1),$$

$$V(C\alpha\beta, m) = 1 \Leftrightarrow (V(\alpha, m) = 1 \Rightarrow V(\beta, m) = 1),$$

[1] The basic notation of this paper will be the Łukasiewicz parenthesis free notation: $A$ (disjunction), $K$ (conjunction), $C$ (implication), $N$ (negation), $L$ (necessity), $M$ (possibility).

[2] $Ax$ denotes the set of all axioms in classical propositional calculus, $Cn$ denotes a consequence operation based on the following rules: substitution, modus ponens and Gödel's rule.

$$V(L\alpha, m) = 1 \Leftrightarrow \forall n(mRn \Rightarrow V(\alpha, n) = 1),$$

$$V(M\alpha, m) = 1 \Leftrightarrow \exists n(mRn \wedge V(\alpha, n) = 1),$$

for every $m \in U$.

The set $U$ is called a set of levels ("possible worlds", states), the relation $R$ is an accessibility relation. For any formula $\alpha \in S$ we say that $\alpha$ is valid in the structure $(U, R)$ if $V(\alpha, m) = 1$ for every valuation $V$ and for every $m \in U$.

For the mentioned systems the following completeness theorems hold (see [3], [4], [7]):

**Theorem 1.** *A formula $\alpha$ is a thesis:*

- *of the system* **K** *if and only if $\alpha$ is valid in any relational structure;*
- *of the system* **K4** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is transitive;*
- *of the system* **T** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive;*
- *of the system* **S4** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive and transitive;*
- *of the system* **B** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive and symmetrical;*
- *of the system* **S5** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive, symmetrical and transitive;*
- *of the system* **S4.2** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive, transitive and convergent;*[3]
- *of the system* **S4.3** *if and only if $\alpha$ is valid in any class of relational structures, where the relation $R$ is reflexive, transitive and coherent.*

[3]The properties of convergent and coherent relations are the following:

$R$ in convergent $\Leftrightarrow \forall x, y, z[(xRy \wedge xRz) \Rightarrow \exists u(yRu \wedge zRu)]$,

$R$ is coherent $\Leftrightarrow \forall x, y, z[(xRy \wedge xRz) \Rightarrow (yRz \vee zRy)]$.

By the symbols $\vdash^m \alpha$ and $\dashv^m \alpha$ we denote the facts that a formula $\alpha$ is accepted on the $m$th level (in the $m$th "world") and a formula $\alpha$ is rejected on the $m$th level (in the $m$th "world"), respectively.

We adopt the following rules of assertion and rejection of formulae:

for negation: $(N^{\vdash})\dfrac{\vdash^m N\alpha}{\dashv^m \alpha}$, $(N^{\dashv})\dfrac{\dashv^m N\alpha}{\vdash^m \alpha}$,

for disjunction: $(A^{\vdash})\dfrac{\vdash^m A\alpha\beta}{\vdash^m \alpha \quad \vdash^m \beta}$, $(A^{\dashv})\dfrac{\dashv^m A\alpha\beta}{\begin{array}{c}\dashv^m \alpha\\ \dashv^m \beta\end{array}}$,

for conjunction: $(K^{\vdash})\dfrac{\vdash^m K\alpha\beta}{\begin{array}{c}\vdash^m \alpha\\ \vdash^m \beta\end{array}}$, $(K^{\dashv})\dfrac{\dashv^m K\alpha\beta}{\dashv^m \alpha \quad \dashv^m \beta}$,

for implication: $(C^{\vdash})\dfrac{\vdash^m C\alpha\beta}{\dashv^m \alpha \quad \vdash^m \beta}$, $(C^{\dashv})\dfrac{\dashv^m C\alpha\beta}{\begin{array}{c}\vdash^m \alpha\\ \dashv^m \beta\end{array}}$,

for necessity: $(L^{\vdash})\dfrac{\vdash^m L\alpha}{\forall n(mRn \Rightarrow \vdash^n \alpha)}$, $(L^{\dashv})\dfrac{\dashv^m L\alpha}{\exists n(mRn \wedge \dashv^n \alpha)}$,

for possibility: $(M^{\vdash})\dfrac{\vdash^m M\alpha}{\exists n(mRn \wedge \vdash^n \alpha)}$, $(M^{\dashv})\dfrac{\dashv^m M\alpha}{\forall n(mRn \Rightarrow \dashv^n \alpha)}$.

Using these rules, one can build a tree decomposition for any expression of the form: $\dashv^m \alpha$, $(\alpha \in S)$. We specify two kinds of trees decomposition: the alternative trees $(A^{\vdash}, K^{\dashv}, C^{\vdash})$ and conjunctive trees (others).

**Theorem 2.** *A formula $\alpha$ is a thesis of a given modal system if and only if a tree decomposition for the expression $\dashv^m \alpha$ is closed (i.e. on all branches of that tree there is a contradiction).*

It is obvious, according to theorem 1, that in the case of a given modal system, the relation $R$ has to fulfil suitable conditions. Recall that in Priest's account (see [6]) in a node of a tableau (tree) there is either an expression of the form $A, i$ ($A$ is a formula, $i$ is a natural number which represents the name of a possible world, in which the formula $A$ is true) or an expression of the form $irj$ (such expressions are connected with the accessibility relation in a Kripke model). There are also rules concerning the modalities and rules corresponding to the properties of the accessibility relation (in order to introduce expressions of the form $irj$).

**Example 1.** *Show that the formula $LCNLpLNLp$ is a thesis of the system* **S5**.

Consider any relational structure $(U, R)$ with reflexive, symmetrical and transitive relation $R$. Assume also that the formula $LCNLpLNLp$ is rejected on the $m$th level (for a some $m \in U$).

| | | |
|---|---|---|
| 1. | $\dashv^m LCNLpLNLp$ | { assumption }, |
| 2. | $\exists n(mRn \wedge \dashv^n CNLpLNLp)$ | $\{L^{\dashv}, 1\}$, |
| 3. | $mRn_1$ | $\{2\}$ |
| 4. | $\dashv^{n_1} CNLpLNLp$ | |
| 5. | $\dashv^{n_1} NLp$ | $\{C^{\dashv}, 4\}$ |
| 6. | $\dashv^{n_1} LNLp$ | |
| 7. | $\dashv^{n_1} Lp$ | $\{N^{\vdash}, 5\}$ |
| 8. | $\exists l(n_1Rl \wedge \dashv^l p)$ | $\{L^{\dashv}, 7\}$ |
| 9. | $n_1Rl_1$ | $\{8\}$ |
| 10. | $\dashv^{l_1} p$ | |
| 11. | $\exists k(n_1Rk \wedge \dashv^k NLp)$ | $\{L^{\dashv}, 6\}$ |
| 12. | $n_1Rk_1$ | $\{11\}$ |
| 13. | $\dashv^{k_1} NLp$ | |
| 14. | $\vdash^{k_1} Lp$ | $\{N^{\dashv}, 13\}$ |
| 15. | $\forall u(k_1Ru \Rightarrow \vdash^u p)$ | $\{L^{\vdash}, 14\}$ |
| 16. | $k_1Rn_1$ | $\{12, R$ - symmetrical$\}$ |
| 17. | $k_1Rl_1$ | $\{16, 9, R$ - transitive$\}$ |
| 18. | $\vdash^{l_1} p$ | $\{17, 15\}$ |
| | contradiction | $\{10, 18\}$ |

The tree decomposition is closed, so the examined formula is a thesis of the system **S5**.

**Example 2.** *Show that the formula* $CLKpqKLpLq$ *is a thesis of the system* **K**.

We shall examine this formula in any relational structure $(U, R)$, where $R$ is an arbitrary relation. Moreover, we assume that $CLKpqKLpLq$ is rejected on the $m$th level (in the $m$th "possible world") for some $m \in U$.

| | | |
|---|---|---|
| 1. | $\dashv^m CLKpqKLpLq$ | {assumption} |
| 2. | $\vdash^m LKpq$ | $\{C^{\dashv}, 1\}$ |
| 3. | $\dashv^m KLpLq$ | |
| 4. | $\forall n(mRn \Rightarrow \vdash^n Kpq)$ | $\{L^{\vdash}, 2\}$ |

| | | | | | |
|---|---|---|---|---|---|
| 5.1 | $\dashv^m Lp$ | $\{K^{\dashv}, 3\}$ | 5.2 | $\dashv^m Lq$ | $\{K^{\dashv}, 3\}$ |
| 6.1 | $\exists u(mRu \wedge \dashv^u p)$ | $\{L^{\dashv}, 5.1\}$ | 6.2 | $\exists v(mRv \wedge \dashv^v q)$ | $\{L^{\dashv}, 5.2\}$ |
| 7.1 | $mRu_1$ | $\{6.1\}$ | 7.2 | $mRv_1$ | $\{6.2\}$ |
| 8.1 | $\dashv^{u_1} p$ | | 8.2 | $\dashv^{v_1} q$ | |
| 9.1 | $\vdash^{u_1} Kpq$ | $\{7.1, 4\}$ | 9.2 | $\vdash^{v_1} Kpq$ | $\{7.2, 4\}$ |
| 10.1 | $\vdash^{u_1} p$ | $\{K^{\vdash}, 9.1\}$ | 10.2 | $\vdash^{v_1} p$ | $\{K^{\vdash}, 9.2\}$ |
| 11.1 | $\vdash^{u_1} q$ | | 11.2 | $\vdash^{v_1} q$ | |
| | contradiction | $\{8.1, 11.1\}$ | | contradiction | $\{8.2, 11.2\}$ |

The above tree decomposition is closed, because on each branch there is a contradiction. This implies that $\alpha$ is a thesis of the system **K**.

**Example 3.** *Decide whether the formula* $CLApqALpLq$ *is a thesis of the system* **K**.

We build a tree decomposition for the expression $\dashv^m CLApqALpLq$ assuming that the relation $R$ is arbitrary in any relational structure $(U, R)$, $m \in U$.

$$
\begin{array}{lll}
1. & \dashv^m CLApqALpLq & \{\text{assumption}\} \\
2. & \vdash^m LApq & \multirow{2}{*}{$\{C^{\dashv}, 1\}$} \\
3. & \dashv^m ALpLq & \\
4. & \dashv^m Lp & \multirow{2}{*}{$\{A^{\dashv}, 3\}$} \\
5. & \dashv^m Lq & \\
6. & \forall n(mRn \Rightarrow \vdash^n Apq) & \{L^{\vdash}, 2\} \\
7. & \exists k(mRk \wedge \dashv^k p) & \{L^{\dashv}, 4\} \\
8. & \exists l(mRl \wedge \dashv^l q) & \{L^{\dashv}, 5\} \\
9. & mRk_1 & \multirow{2}{*}{$\{7\}$} \\
10. & \dashv^{k_1} p & \\
11. & mRl_1 & \multirow{2}{*}{$\{8\}$} \\
12. & \dashv^{l_1} q & \\
13. & \vdash^{k_1} Apq & \{9, 6\} \\
14. & \vdash^{l_1} Apq & \{11, 6\}
\end{array}
$$

15.1 $\vdash^{k_1} p$ | 15.2 $\vdash^{k_1} q$ $\{A^{\vdash}, 13\}$

contradiction $\{15.1, 10\}$

16.2.1 $\vdash^{l_1} p$ | 16.2.2 $\vdash^{l_1} q$ $\{A^{\vdash}, 14\}$

no contradiction | contradiction $\{16.2.2, 12\}$

This tree decomposition is not closed, because there is one branch without any contradiction. So, the examined formula is not a thesis of the system **K**. On the basis of the opening branch, we may construct a relational structure in which the formula $CLApqALpLq$ is invalid.

Let $U = \{m, k_1, l_1\}$, $R = \{< m, k_1 >, < m, l_1 >\}$, $V_0(p, l_1) = 1$, $V_0(q, l_1) = 0$, $V_0(p, k_1) = 0, V_0(q, k_1) = 1$. From the definition of valuation, we have:

$V_0(Lp, m) = 0$, because $(mRk_1 \wedge V_0(p, k_1) = 0)$,

$V_0(Lq, m) = 0$, because $(mRl_1 \wedge V_0(q, l_1) = 0)$

$V_0(ALpLq, m) = 0$, because $(V_0(Lp, m) = 0 \wedge V_0(Lq, m) = 0)$,

$V_0(Apq, k_1) = 1$, because $(V_0(p, k_1) = 0 \wedge V_0(q, k_1) = 1)$,

$V_0(Apq, l_1) = 1$, because $(V_0(p, l_1) = 1 \wedge V_0(q, l_1) = 0)$,

$V_0(LApq, m) = 1$, because $\forall s \in \{m, k_1, l_1\}(mRs \Rightarrow V_0(Apq, s) = 1)$,

$V_0(CLApqALpLq, m) = 0$, because $(V_0(LApq, m) = 1 \wedge V_0(ALpLq, m) = 0)$.

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