ZESZYTY NAUKOWE AKADEMII MARYNARKI WOJENNEJ
SCIENTIFIC JOURNAL OF POLISH NAVAL ACADEMY

2016 (LVII) 2 (205)

DOI: 10.5604/0860889X.1219978

Hubert Wysocki*

A DISCRETE NON-CLASSICAL OPERATIONAL CALCULUS MODEL WITH THE HORADAM DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of the non-classical Bittner operational calculus, in which the derivative S related to Horadam sequences is understood as a difference operation $S\{x(k)\} := \{x(k+2) - a\,x(k+1) - b\,x(k)\}$.

Key words:
operational calculus, derivative, integrals, limit conditions, Horadam difference.

INTRODUCTION

For any functions $\{f(t)\} \in C^0((\alpha,\beta),\mathbb{R}), \{x(t)\} \in C^1((\alpha,\beta),\mathbb{R})$ as well as for every $t_0 \in (\alpha,\beta) \subset \mathbb{R}$ and $t \in (\alpha,\beta)$ the fundamental theorems of the integral calculus apply [1]:

$$\frac{d}{dt}\int_{t_0}^{t} f(\tau)\,d\tau = f(t), \qquad \int_{t_0}^{t} x'(\tau)\,d\tau = x(t) - x(t_0).$$

Using linear operations

$$S\{x(t)\} := \{x'(t)\}, \quad T_{t_0}\{f(t)\} := \left\{\int_{t_0}^{t} f(\tau)\,d\tau\right\}, \quad s_{t_0}\{x(t)\} := \{x(t_0)\}, \tag{1}$$

we can present the above theorems as follows:

* Polish Naval Academy, Faculty of Mechanical and Electrical Engineering, Śmidowicza 69 Str., 81-127 Gdynia, Poland; e-mail: h.wysocki@amw.gdynia.pl

$$S\,T_{t_0}f = f, \quad T_{t_0}S\,x = x - s_{t_0}x, \tag{2}$$

where $f = \{f(t)\}, x = \{x(t)\}$[1].

Apart from the model (1) with the classical *ordinary derivative* $S = d/dt$, there exist other continuous and discrete models in which, for appropriately determined operations S, T_q, s_q, properties (2) hold. These models constitute particular cases (representations) of the so-called non-classical Bittner operational calculus [2–5].

Broadly speaking, the *Bittner operational calculus* is a system

$$CO(L^0, L^1, S, T_q, s_q, Q)^2, \tag{3}$$

in which L^0 and L^1 are linear spaces (over the same scalar field Γ) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$ such that $S\,T_q f = f, f \in L^0$ and $s_q x = x - T_q S\,x$, $x \in L^1$. These operations are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\operatorname{Ker} S$ is a set of elements understood as *constants* for the derivative S. The limit conditions $s_q, q \in Q$ are projections of L^1 on the subspace $\operatorname{Ker} S$.

Beside the continuous model (2), we frequently use a classical discrete model with the derivative S understood as the *forward difference* Δ.

Let $\mathbb{N}_0$ and $\mathbb{C}$ mean the set of non-negative integers and the set of complexes, respectively. Moreover, let $L^0 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k \in \mathbb{N}_0}$ with usual sequences addition and sequences multiplication by complexes. In [2, 3, 5] Bittner considered a model with the derivative

$$S\,x \equiv \Delta x := \{x(k+1) - x(k)\}$$

and its corresponding integral

$$T_0 x := \begin{cases} 0 & \text{for} \quad k = 0 \\ \sum\limits_{i=0}^{k-1} x(i) & \text{for} \quad k > 0 \end{cases}, \quad k \in \mathbb{N}_0$$

and limit condition

$$s_0 x := \{x(0)\},$$

where $x = \{x(k)\} \in L^1 = L^0$.

---

[1] $\{f(t)\}$ stands for the symbol of the function f, i.e. $f = \{f(t)\}$, whereas $f(t)$ denotes the value of the function $\{f(t)\}$ at point t. This notation is derived from J. Mikusiński [15].

[2] CO stands for the French 'calcul opératoire' (operational calculus).

Later, in [6] there appeared a model with the forward difference $S \equiv \Delta$, integrals

$$T_{k_0}x := \begin{cases} -\sum_{i=k}^{k_0-1} x(i) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \sum_{i=k_0}^{k-1} x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{N}_0$$

and limit conditions

$$s_{k_0}x := \{x(k_0)\},$$

where $k_0 \equiv q \in Q := \mathbb{N}_0$.

Notice that the integrals T_{k_0} can be shown as follows

$$T_{k_0}x = \left\{\sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i)\right\}^3.$$

In this paper, we shall discuss other discrete models of the Bittner operational calculus related to the operation

$$S\{x(k)\} := \{x(k+2) - a\,x(k+1) - b\,x(k)\}, \tag{4}$$

where $a, b \in \mathbb{C}$ and $b \neq 0$.

We will consider two cases:

$$D := a^2 + 4b \neq 0 \quad \text{and} \quad D = 0.$$

In literature (e.g. [7, 12, 14, 16]), each element c belonging to the kernel of the operation (4) is called a *Horadam sequence* [8, 9].

In particular, the Horadam sequence $c = \{c(k)\} \in \operatorname{Ker} S$, i.e. a solution of the equation

$$c(k+2) = a\,c(k+1) + b\,c(k), \quad k \in \mathbb{N}_0, \tag{5}$$

can be [10]:

– the *Fibonacci sequence* $\{\mathcal{F}(k)\}$ (for $a = b = 1, \mathcal{F}(0) = 0, \mathcal{F}(1) = 1$)

$$\{\mathcal{F}(k)\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\};$$

3 Given the definition of T_{k_0}, we assume that $\sum_{i=0}^{-1} x(i) := 0$.

– the *Lucas sequence* $\{\mathcal{L}(k)\}$ (for $a = b = 1$, $\mathcal{L}(0) = 2$, $\mathcal{L}(1) = 1$)

$$\{\mathcal{L}(k)\} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \ldots\};$$

– the *Pell sequence* $\{\mathcal{P}(k)\}$ (for $a = 2, b = 1, \mathcal{P}(0) = 0, \mathcal{P}(1) = 1$)

$$\{\mathcal{P}(k)\} = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \ldots\};$$

– the *Pell-Lucas sequence* $\{p(k)\}$ (for $a = 2, b = 1, p(0) = 2, p(1) = 2$)

$$\{p(k)\} = \{2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, \ldots\};$$

– the *Jacobsthal sequence* $\{\mathcal{J}(k)\}$ (for $a = 1, b = 2, \mathcal{J}(0) = 0, \mathcal{J}(1) = 1$)

$$\{\mathcal{J}(k)\} = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, \ldots\};$$

– the *Jacobsthal-Lucas sequence* $\{j(k)\}$ (for $a = 1, b = 2, j(0) = 2, j(1) = 1$)

$$\{j(k)\} = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, 4097, 8191, \ldots\}.$$

Another interesting example is also an *anti-forward Fibonacci sequence* $\{f(k)\}$, for which $a = -1, b = 1, f(0) = 0, f(1) = 1$. Then, we have

$$f(k+2) = -[\underbrace{f(k+1) - f(k)}_{\Delta f(k)}] \iff f(k) = f(k+1) + f(k+2), \quad k \in \mathbb{N}_0,$$

from which we obtain

$$\{f(k)\} = \{0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, \ldots\}.$$

We also have

$$f(k) = (-1)^{k+1}\mathcal{F}(k), \quad k \in \mathbb{N}_0. \tag{6}$$

In [13] Kalman and Mena presented two-term recurrences (5) and related Horadam sequences in the operational approach, using classical difference operations.

A MODEL WITH THE HORADAM DIFFERENCE, WHEN $D \neq 0$

In what follows, we shall call the operation (4) a *Horadam derivative* or *difference*.

Let

$$\Phi_{a,b} := \frac{a + \sqrt{D}}{2}, \quad \varphi_{a,b} := \frac{a - \sqrt{D}}{2}, \quad D \neq 0.$$

Then, we have

$$\Phi_{a,b}^2 - a\,\Phi_{a,b} - b = 0, \quad \varphi_{a,b}^2 - a\,\varphi_{a,b} - b = 0$$

and

$$\Phi_{a,b} + \varphi_{a,b} = a, \quad \Phi_{a,b} - \varphi_{a,b} = \sqrt{D}, \quad \Phi_{a,b}\varphi_{a,b} = -b.$$

We will prove the following

Theorem 1. *The system* (3), *where* $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{N}_0$ *and*

$$S\,x := \{x(k+2) - a\,x(k+1) - b\,x(k)\}, \tag{7}$$

$$T_{k_0}x := \frac{1}{\sqrt{D}} \begin{cases} -\sum\limits_{i=k-1}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) & for \quad k < k_0 \\ 0 & for \quad k = k_0, k_0+1 \\ \sum\limits_{i=k_0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) & for \quad k > k_0+1 \end{cases}, \quad k \in \mathbb{N}_0, \tag{8}$$

$$s_{k_0}x := \left\{ \frac{1}{\sqrt{D}} [\, b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0) + (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0+1)\,] \right\}, \tag{9}$$

forms a discrete model of the Bittner operational calculus with the Horadam difference (7), *when* $D \neq 0$.

Proof. It is obvious that (7) – (9) are linear operations. It is also easy to verify that T_{k_0} can be presented in the form of

$$T_{k_0}x := \left\{ \frac{1}{\sqrt{D}} [\, \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i)\,] \right\},$$

where $\sum_{i=0}^{j} x(i) := 0$ for $j = -2, -1$.

Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Hence

$$S\{y(k)\} = \{y(k+2) - a\,y(k+1) - b\,y(k)\}$$

$$= \Bigg\{ \frac{1}{\sqrt{D}} \Big([\, \sum_{i=0}^{k} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i)\,]$$

$$-a[\, \sum_{i=0}^{k-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i)\,]$$

$$-b[\, \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i)\,] \Big) \Bigg\}$$

$$= \left\{ \frac{1}{\sqrt{D}} \Big((\Phi_{a,b} - \varphi_{a,b}) x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b})) x(k-1) \right.$$

$$+ \sum_{i=0}^{k-2} [(\Phi_{a,b}^2 - a\,\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\,\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}] x(i)$$

$$\left. - \sum_{i=0}^{k_0-1} [(\Phi_{a,b}^2 - a\,\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\,\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}] x(i) \Big) \right\} = \{x(k)\},$$

so $S\,T_{k_0}\{x(k)\} = \{x(k)\}$ holds.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+2) - a\,x(k+1) - b\,x(k)\}$. Then

$$T_{k_0} S\{x(k)\} = T_{k_0}\{f(k)\}$$

$$= \left\{ \frac{1}{\sqrt{D}} \Big[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) f(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) f(i) \Big] \right\}$$

$$= \left\{ \frac{1}{\sqrt{D}} \Big[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i+2) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i+2) \right.$$

$$-a \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i+1) + a \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i+1)$$

$$\left. -b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \Big] \right\}$$

$$= \left\{ \frac{1}{\sqrt{D}} \Big[\sum_{i=2}^{k} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i}) x(i) - \sum_{i=2}^{k_0+1} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i}) x(i) \right.$$

$$-a \sum_{i=1}^{k-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) + a \sum_{i=1}^{k_0} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i)$$

$$\left. -b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \Big] \right\}$$

$$= \left\{ \frac{1}{\sqrt{D}} \Big[(\Phi_{a,b} - \varphi_{a,b}) x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b})) x(k-1) \right.$$

$$+ \sum_{i=2}^{k-2} [(\Phi_{a,b}^2 - a\,\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\,\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}] x(i)$$

$$-(\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0}) x(k_0+1)$$

$$- \sum_{i=2}^{k_0} [(\Phi_{a,b}^2 - a\,\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\,\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}]x(i)$$

$$-b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0)\,]\Big\}$$

$$= \{x(k)\} - \Big\{ \frac{1}{\sqrt{D}} [\, b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0) + (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0+1)\,]\Big\}.$$

Therefore, $T_{k_0} S\{x(k)\} = \{x(k)\} - s_{k_0}\{x(k)\}$ also holds, which completes the proof. □

Example 1. It is not difficult to check (see Th. 3[5]) that an abstract differential equation

$$S\,x = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = c_{0,q}, \quad c_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = c_{0,q} + T_q f. \tag{10}$$

A. In particular,

$$x(k) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{-1+\sqrt{5}}{2}\Big)^k - \Big(\frac{-1-\sqrt{5}}{2}\Big)^k\Big], \quad k \in \mathbb{N}_0$$

is the solution of the homogeneous difference equation

$$x(k+2) + x(k+1) - x(k) = 0, \quad k \in \mathbb{N}_0$$

with initial conditions

$$x(0) = 0, x(1) = 1,$$

which results from the limit condition form (9) for $a = -1, b = 1$ and $k_0 = 0$.

Hence we have

$$x(k) = (-1)^{k+1} \cdot \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^k - \Big(\frac{1-\sqrt{5}}{2}\Big)^k\Big], \quad k \in \mathbb{N}_0.$$

It is a form (6) of the anti-forward Fibonacci sequence $\{f(k)\}$ general term, where

$$\mathcal{F}(k) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^k - \Big(\frac{1-\sqrt{5}}{2}\Big)^k\Big], \quad k \in \mathbb{N}_0$$

is a well-known Binet formula of the Fibonacci sequence $\{\mathcal{F}(k)\}$ general term.

B. For the Cauchy problem

$$\mathscr{T}(k+2) - \mathscr{T}(k+1) - 2\mathscr{T}(k) = 3, \quad k \in \mathbb{N}_0 \tag{11}$$
$$\mathscr{T}(0) = 0, \mathscr{T}(1) = 1$$

we have $a = 1, b = 2$ and for $k_0 = 0$, on the basis of (10), we obtain

$$\mathscr{T}(k) = \frac{1}{3}[2^k - (-1)^k + 3\sum_{i=0}^{k-2}(2^{k-1-i} - (-1)^{k-1-i})] = \frac{1}{6}(2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.$$

Similarly, if

$$\mathscr{T}(k+2) - \mathscr{T}(k) = 2^{k+2}, \quad k \in \mathbb{N}_0 \tag{12}$$
$$\mathscr{T}(0) = 0, \mathscr{T}(1) = 1,$$

then $a = 0, b = 1$ and for $k_0 = 0$, we get

$$\mathscr{T}(k) = \frac{1}{2}[1 - (-1)^k + \sum_{i=0}^{k-2}(1 - (-1)^{k-1-i}) \cdot 2^{i+2}] = \frac{1}{6}(2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.$$

The sequence $\{\mathscr{T}(k)\}$, defined with the use of Jacobsthal numbers as

$$\mathscr{T}(k) := \begin{cases} \mathcal{J}(0) & \text{for} \quad k = 0 \\ \mathcal{J}(1) & \text{for} \quad k = 1 \\ \sum\limits_{i=2}^{k+1} \mathcal{J}(i) & \text{for} \quad k \in \mathbb{N}_0 \setminus \{0, 1\} \end{cases},$$

was introduced in [11], where Horadam gave a number of its properties, including (11) and (12).

A MODEL WITH THE HORADAM DIFFERENCE, WHEN $D = 0$

If the Horadam difference (4) takes a particular form of

$$S\{x(k)\} := \left\{x(k+2) - a\,x(k+1) + \frac{1}{4}a^2\,x(k)\right\},$$

then $D = 0$. For this case, we will prove.

Theorem 2. *The system* (3)*, where* $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{N}_0$ *and*

$$S\{x(k)\} := \left\{x(k+2) - a\,x(k+1) + \frac{1}{4}a^2\,x(k)\right\}, \tag{13}$$

$$T_{k_0}x := \left\{\left(\frac{a}{2}\right)^{k-2}\left[\sum_{i=0}^{k-2}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i) - \sum_{i=0}^{k_0-1}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i)\right]\right\}, \tag{14}$$

$$s_{k_0}x := \left\{-\left(\frac{a}{2}\right)^{k-k_0}\left[(k-1-k_0)\,x(k_0) - \frac{2}{a}\,(k-k_0)\,x(k_0+1)\right]\right\}, \tag{15}$$

forms a discrete model of the Bittner operational calculus with the Horadam difference (13).

Proof. Similarly as before, the operations (13)–(15) are linear. Moreover, if $\{y(k)\} := T_{k_0}\{x(k)\}$, then

$$S\{y(k)\} = \left\{y(k+2) - a\,y(k+1) + \frac{1}{4}a^2\,y(k)\right\}$$

$$= \left\{\sum_{i=0}^{k}\left(\frac{a}{2}\right)^{k-i}(k+1-i)\,x(i) - \sum_{i=0}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}(k+1-i)\,x(i)\right.$$

$$-2\sum_{i=0}^{k-1}\left(\frac{a}{2}\right)^{k-i}(k-i)\,x(i) + 2\sum_{i=0}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}(k-i)\,x(i)$$

$$\left.+\sum_{i=0}^{k-2}\left(\frac{a}{2}\right)^{k-i}(k-1-i)\,x(i) - \sum_{i=0}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}(k-1-i)\,x(i)\right\}$$

$$= \left\{x(k) + \sum_{i=0}^{k-2}\left(\frac{a}{2}\right)^{k-i}\Big[(k+1-i) - 2(k-i) + (k-1-i)\Big]\,x(i)\right.$$

$$\left.-\sum_{i=0}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}\Big[(k+1-i) - 2(k-i) + (k-1-i)\Big]\,x(i)\right\} = \{x(k)\}.$$

If, in turn, $\{f(k)\} := S\{x(k)\} = \{x(k+2) - a\,x(k+1) + \frac{1}{4}a^2\,x(k)\}$, then

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\}$$

$$= \left\{\left(\frac{a}{2}\right)^{k-2}\left[\sum_{i=0}^{k-2}\left(\frac{2}{a}\right)^{i}(k-1-i)\,f(i) - \sum_{i=0}^{k_0-1}\left(\frac{2}{a}\right)^{i}(k-1-i)\,f(i)\right]\right\}$$

$$= \left\{\left(\frac{a}{2}\right)^{k-2}\Bigg[\sum_{i=0}^{k-2}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i+2) - \sum_{i=0}^{k_0-1}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i+2)\right.$$

$$-a\sum_{i=0}^{k-2}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i+1) + a\sum_{i=0}^{k_0-1}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i+1)$$

$$\left.+\left(\frac{a}{2}\right)^{2}\sum_{i=0}^{k-2}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i) - \left(\frac{a}{2}\right)^{2}\sum_{i=0}^{k_0-1}\left(\frac{2}{a}\right)^{i}(k-1-i)\,x(i)\Bigg]\right\}$$

$$= \left\{\sum_{i=2}^{k}\left(\frac{a}{2}\right)^{k-i}(k+1-i)\,x(i) - \sum_{i=2}^{k_0+1}\left(\frac{a}{2}\right)^{k-i}(k+1-i)\,x(i)\right.$$

$$-2\sum_{i=1}^{k-1}\left(\frac{a}{2}\right)^{k-i}(k-i)\,x(i) + 2\sum_{i=1}^{k_0}\left(\frac{a}{2}\right)^{k-i}(k-i)\,x(i)$$

$$\left.+\sum_{i=0}^{k-2}\left(\frac{a}{2}\right)^{k-i}(k-1-i)\,x(i) - \sum_{i=0}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}(k-1-i)\,x(i)\right\}$$

$$= \left\{x(k) + \sum_{i=2}^{k-2}\left(\frac{a}{2}\right)^{k-i}\Big[(k+1-i) - 2(k-i) + (k-1-i)\Big]\,x(i)\right.$$

$$-\left(\frac{a}{2}\right)^{k-k_0-1}(k-k_0)\,x(k_0+1) - \left(\frac{a}{2}\right)^{k-k_0}(k+1-k_0)\,x(k_0) + 2\left(\frac{a}{2}\right)^{k-k_0}(k-k_0)\,x(k_0)$$

$$\left.-\sum_{i=2}^{k_0-1}\left(\frac{a}{2}\right)^{k-i}\Big[(k+1-i) - 2(k-i) + (k-1-i)\Big]\,x(i)\right\}$$

$$= \{x(k)\} - \left\{-\left(\frac{a}{2}\right)^{k-k_0}\left[(k-1-k_0)\,x(k_0) - \frac{2}{a}(k-k_0)\,x(k_0+1)\right]\right\}$$

$$= \{x(k)\} - s_{k_0}\{x(k)\}.$$

□

REFERENCES

[1] Apostol T. M., *Calculus*, Vol. 1: *One-Variable Calculus, with an Introduction to Linear Algebra*, John Wiley and Sons, New York — London 1967.

[2] Bittner R., *On certain axiomatics for the operational calculus*, 'Bull. Acad. Polon. Sci.' Cl. III, 1959, Vol. 7, No. 1, pp. 1–9.

[3] Bittner R., *Operational calculus in linear spaces*, 'Studia Math.', 1961, 20, pp. 1–18.

[4] Bittner R., *Algebraic and analytic properties of solutions of abstract differential equations*, 'Rozprawy Matematyczne' ['Dissertationes Math.'], 41, PWN, Warszawa 1964.

[5] Bittner R., *Rachunek operatorów w przestrzeniach liniowych*, PWN, Warszawa 1974 [*Operational Calculus in Linear Spaces* — available in Polish].

[6] Bittner R., Mieloszyk E., *About eigenvalues of differential equations in the operational calculus*, 'Zeszyty Naukowe Politechniki Gdańskiej', Matematyka XI, 1978, 285, pp. 87–99.

[7] Čerin Z., *On sums of products of Horadam numbers*, 'Kyungpook Math. J.', 2009, 49, pp. 483–492.

[8] Horadam A. F., *Generating functions for powers of a certain generalised sequence of numbers*, 'Duke Math. J.', 1965, 32 (3), pp. 437–446.

[9] Horadam A. F., *Basic properties of a certain generalized sequence of numbers,* 'Fibonacci Quart.', 1965, 3 (3), pp. 161–176.

[10] Horadam A. F., *Jacobsthal and Pell curves*, 'Fibonacci Quart.', 1988, 26 (1), pp. 77–83.

[11] Horadam A. F., *Jacobsthal representation numbers*, 'Fibonacci Quart.', 1996, 34 (1), pp. 40–54.

[12] Horzum T., Kocer E. G., *On some properties of Horadam polynomials*, 'Int. Math. Forum', 2009, 4 (25), pp. 1243–1252.

[13] Kalman D., Mena R., *The Fibonacci numbers — exposed*, 'Math. Magazine', 2003, 76 (3), pp. 167–181.

[14] Mansour T., *A formula for the generating functions of powers of Horadam's sequence*, 'Australas. J. Combin.', 2004, 30, pp. 207–212.

[15] Mikusiński J., *Operational Calculus*, Pergamon Press, London 1959.

[16] Weisstein E. W., 'Horadam Sequence', From *MathWorld* — A Wolfram Web Resource, [online], http://mathworld.wolfram.com/HoradamSequence.html [access 12.04.2016].

MODEL DYSKRETNY NIEKLASYCZNEGO RACHUNKU OPERATORÓW Z RÓŻNICĄ HORADAMA

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna S, związana z ciągami Horadama, rozumiana jest jako operacja różnicowa $S\{x(k)\} := \{x(k+2) - a\,x(k+1) - b\,x(k)\}$.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica Horadama.