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Entropy generation minimization in steady-state and transient diffusional heat conduction processes Part I – Steady-state boundary value problem

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Abstract. An application of the Entropy Generation Minimization principle allows new formulation of the boundary and initial boundary-value problems. Applying Euler-Lagrange variational formalism new mathematical form of heat conduction equation describing steady-state processes have been derived. Mathematical method presented in the paper can also be used for any diffusion heat and mass transfer process. Linear and non-linear problems with internal heat sources have been analyzed.

Key words: heat conduction, entropy generation, boundary-value problems, Euler-Lagrange variational equation.

Nomenclature

c – constants,
c_p – specific heat capacity [kJ/kg·K],
i – internal,
k – heat conduction coefficient [W/m·K],
L_0 – Wiedemann-Franz constant $[\mathrm{W/A{\cdot}K}]^2$,
$\dot{m}$ – mass flow rate [kg/h],
$\dot{q}$ – heat flux [$\mathrm{W/m^2}$],
$\dot{q}_v$ – intensity of internal heat source [$\mathrm{W/m^3}$],
$\dot{s}$ – entropy flux [$\mathrm{W/m^2{\cdot}K}$],
$\dot{S}_{gen}$ – entropy generation due to the process irreversibility [$\mathrm{W/m^3{\cdot}K}$],
T – absolute temperature [K],
x, y – Cartesian coordinates,
Θ – transformed temperature,
Ω – domain,
∇ – operator nabla.

Indexes

He – helium,
in – entering,
o – environment,
out – leaving,
t – total,
x, y – partial derivatives with respects to x and y.

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1. Introduction

According to the Second law of Thermodynamics the measure of energy dissipation in irreversible processes is entropy generation, $\dot{S}_{gen}$, and the Gouy-Stodola theorem states that destroyed exergy $\delta\dot{B}$ is directly proportional to $\dot{S}_{gen}$

$$\delta\dot{B} = T_0\dot{S}_{gen},$$

where T_0 is surroundings temperature, and

$$\dot{S}_{gen} = \frac{d_iS}{d\tau}$$

determines entropy production rate, resulting from irreversibilities of the processes taking place in the system, τ – is time. The Second Law of Thermodynamics says that always and in each elementary domain of the thermodynamic system

$$\dot{S}_{gen} > 0$$

which means that entropy generation is **path dependent** and should not be confused with the entropy as the state function. It is basic feature of this function allowing optimization of any physical and chemical process.

1.1. Steady-state processes. Classical boundary-value problems are usually formulated on the basis of the First Law of Thermodynamics. When heat conduction coefficient depends on temperature, $k = k(T)$, steady state temperature field results from the solution of a differential equation [1]

$$\mathrm{div}\,[k(T)\mathrm{grad}\,T(x_i)] + \dot{q}_v(x_i,T) = 0, \qquad i = 1,2,3$$

with respect to the required boundary conditions, where k is thermal conductivity coefficient and $\dot{q}_v$ represents intensity of internal heat sources.

The different formulation of the classical boundary-value problem is obtained when a heat conduction equation is derived from the minimum entropy generation principle. According to the thermodynamics of irreversible processes [2] entropy generation at steady state is at minimum. Introducing expression for local entropy generation [2, 3]

$$\dot{S}_{gen}(T, T_{x_i}) \equiv \frac{d_i S}{d\tau} = \frac{k(T)}{T^2}(\mathrm{grad}\, T)^2 + \frac{\dot{q}_v}{T}, \tag{1}$$

where $T = T(x_i)$ and T_{x_i} denotes gradient components $\partial T/\partial x_i$, the problem can be formulated in the following way: *find such a function* $T = T(x_i)$ *which satisfying required boundary conditions minimizes simultaneously integral*

$$\dot{S}_{gen,t} = \int\limits_{\Omega} \dot{S}_{gen}(T, T_{x_i}) d\Omega \qquad (i = 1, 2, 3)$$

over the whole domain Ω. Here $\dot{S}_{gen,t}$ represents global entropy production of the process.

Using variational calculus, the function $T(x_i)$, for which $\dot{S}_{gen,t}$ reaches minimum, satisfies the Euler equation [4].

$$\frac{\partial \dot{S}_{gen}}{\partial T} - \sum\nolimits_i \frac{\partial}{\partial x_i}\left(\frac{\partial \dot{S}_{gen}}{\partial T_{x_i}}\right) = 0, \tag{2}$$

where $\dot{S}_{gen}(T, T_x)$ is given by Eq. (1).

2. 1D Boundary value problem

Consider one dimensional (1D) problem of heat conduction in plane wall without internal heat source with first kind boundary conditions (Fig. 1). Local entropy generation is

$$S_{gen}(\mathrm{T}, \mathrm{T}_x) = \frac{k(\mathrm{T})}{T^2(\mathrm{x})}\left(\frac{dT(\mathrm{x})}{dx}\right)^2$$

and its global value to be minimized is given by integral

$$\dot{S}_{gen,t} = \int\limits_0^1 \frac{k(T)}{T^2(x)}\left(\frac{dT(x)}{dx}\right)^2 A dx,$$

where A is unit surface area perpendicular to the heat flux vector ($A = 1$ m^2). Assuming k =const, Euler Eq. (2) becomes in the final form

$$\frac{d^2 T}{dx^2} - \frac{1}{T}\left(\frac{dT}{dx}\right)^2 = 0 \qquad T = T(x) \tag{3}$$

and is different from classical Laplace heat conduction equation

$$\frac{d^2 T}{dx^2} = 0.$$

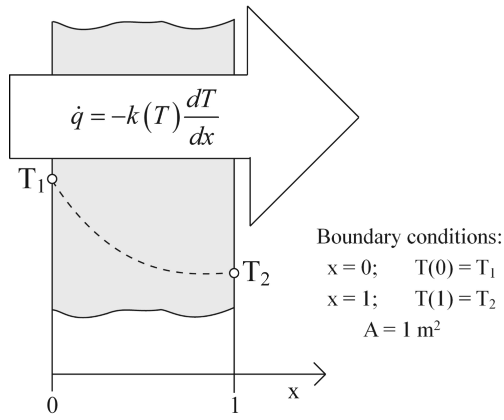

Fig. 1. Plane wall system

For boundary conditions of the first kind, $T(0) = T_1$ and $T(1) = T_2$, (see Fig. 2) solution to Eq. (3) is:

$$T(x) = T_1\left(\frac{T_2}{T_1}\right)^x. \tag{4}$$

Using (4), it is easy to show that the law of energy conservation in a classical form is not satisfied as

$$\mathrm{div}(\dot{q}(x)) \neq 0,$$

where

$$\dot{q}(x) = -k\frac{dT(x)}{dx}.$$

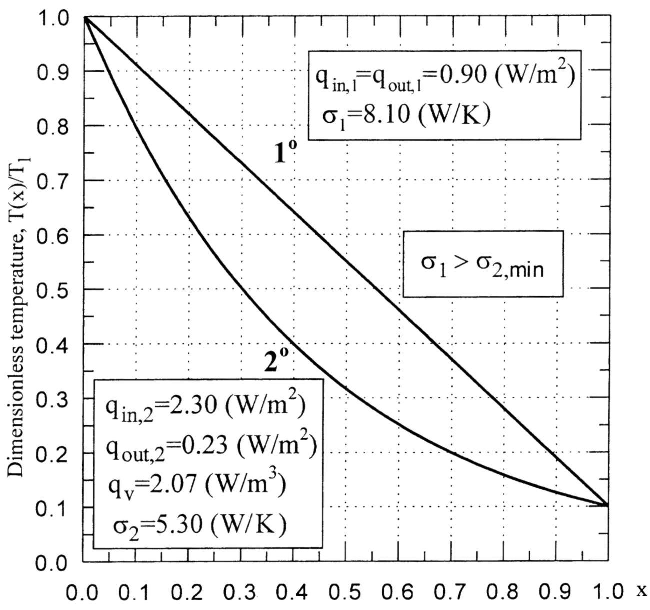

Fig. 2. Temperature distribution in the plane wall: 1 – classical solution, 2 – entropy generation minimization

Further calculations leads to the next confusing result that

$$\mathrm{div}\left(\frac{\dot{q}(x)}{T(x)}\right) = \mathrm{div}(\dot{s}) = 0.$$

Explanation comes directly from Eq. (3). Interpreting its second term as the additional internal heat source

$$\dot{q}_{v,a}(x) = -\frac{k}{T}\left(\frac{dT(x)}{dx}\right)^2 < 0$$

it is easy to prove that the First Law of Thermodynamics is satisfied and entropy increase of the whole process is positive and equal to

$$\dot{S}_{gen} = -\int_0^1 \frac{\dot{q}_v(x)}{T(x)} dx = k\left(\ln\frac{T_2}{T_1}\right)^2 > 0.$$

The same results have been obtained by Bejan [5] using different mathematical approach. Comparison to the classical solution is given in Table 1. Numerical example for $T_2/T_1 = 0.1$ and $k = 1.0$ W/mK is shown in Fig. 2.

Table 1
Comparison of the results

	Classical approach	Entropy generation minimization
Conduction equation	$\frac{d^2T}{dx^2} = 0$	$\frac{d^2T}{dx^2} - \frac{1}{T}\left(\frac{dT}{dx}\right)^2 = 0$
Minimized function	$k\int_0^1 \left(\frac{dT}{dx}\right)^2 dx = 0$	$k\int_0^1 \frac{1}{T^2}\left(\frac{dT}{dx}\right)^2 dx = 0$
Temperature distribution for first kind boundary conditions	$T(x) = T_1 - (T_1 - T_2)x$	$T(x) = T_1\left(\frac{T_2}{T_1}\right)^x$
Entropy increase of the whole heat conduction process	$\sigma_1 = k(T_1 - T_2)\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$	$\sigma_2 = k\left(\ln\frac{T_2}{T_1}\right)^2$

In the case when heat conduction coefficient depends on temperatures, the problem is formulated in the following way [6]:

- *local entropy generation rate*

$$\dot{S}_{gen}(x) = \frac{k(T)}{T^2(x)}\left(\frac{dT(x)}{dx}\right)^2,$$

  where $T = T(x)$,
- *global entropy generation to be minimized*

$$\dot{S}_{gen,t} = \int_0^1 \frac{k(T)}{T^2(x)}\left(\frac{dT(x)}{dx}\right)^2 dx \to \min$$

and Euler-Lagrange equation is

$$k(T)\frac{d^2T}{dx^2} + \left(\frac{1}{2}\frac{dk}{dT} - \frac{k(T)}{T}\right)\left(\frac{dT}{dx}\right)^2 = 0$$

(for comparison-classical heat conduction equation)

$$k(T)\frac{d^2T}{dx^2} + \frac{dk}{dT}\left(\frac{dT}{dx}\right)^2 = 0$$

or for more clear physical interpretation

$$k(T)\frac{d^2T}{dx^2} + \frac{dk}{dT}\left(\frac{dT}{dx}\right)^2 - \left(\frac{1}{2}\frac{dk}{dT} + \frac{k(T)}{T}\right)\left(\frac{dT}{dx}\right)^2 = 0. \tag{5}$$

Comparison of (5) with the classical form leads to the expression for additional internal heat source

$$\dot{q}_{v,a}(x) = -\left(\frac{1}{2}\frac{dk}{dT} + \frac{k(T)}{T}\right)\left(\frac{dT}{dx}\right)^2.$$

2.1. Heat conduction coefficient depends on temperature – nonlinear 1D solution. Assuming frequently used dependence [2]

$$k(T) = k_1\left(\frac{T(x)}{T_1}\right)^n, \tag{6}$$

where $k_1 = k(T_1)$ and n is arbitrary constant, Euler-Lagrange heat conduction equation takes the form

$$\frac{d^2T}{dx^2} + \frac{n-2}{2}\frac{1}{T}\left(\frac{dT}{dx}\right)^2 = 0. \tag{7}$$

Assuming first kind boundary conditions, the solution of Eq. (7) becomes in the form

$$T(x)_1^{n/2} = T_1^{n/2} - \left(T_1^{n/2} - T_2^{n/2}\right)x. \tag{8}$$

The local entropy generation rate

$$\dot{S}_{gen}(x) = \frac{4k_1}{n^2}\left[\left(\frac{T_2}{T_1}\right)^{n/2} - 1\right] = \text{const}$$

and is equal to its global value $\dot{S}_{gen,t}$. For comparison, solution of the classical problem is

$$T(x)^{n+1} = T_1^{n+1} - \left(T_1^{n+1} - T_2^{n+1}\right)x \qquad (\text{for } n \neq -1).$$

Local entropy generation

$$\dot{S}_{gen}(x) = \frac{k_1}{n+1}\frac{1}{T_1^n}\left(T_2^{n+1} - T_1^{n+1}\right)^2 T(x)^{-(n+2)}$$

and its global value

$$\dot{S}_{gen}(x) = \int_0^1 \dot{S}_{gen}(x)dx = \frac{k_1}{n+1}\left[1 - \left(\frac{T_2}{T_1}\right)^{n+1}\right]\left(\frac{T_1}{T_2} - 1\right).$$

Direct comparison gives

$$N = \frac{\dot{S}_{gen,t}}{\dot{S}_{gen,\min}} = \frac{n^2}{4(n+1)}\frac{\left[1 - (T_2/T_1)^{n+1}\right](1 - T_2/T_1)}{(T_2/T_1)\left[(T_2/T_1)^{n/2} - 1\right]^2}$$

which is in agreement with Bejan results [5]. Relationship

$$N = N\left(\frac{T_2}{T_1}, n\right)$$

is shown in Fig. 3. Using solution (8), additional internal heat source intensity can be calculated from

$$\dot{q}_{v,a} = \int_0^1 q_{v,a}(x)dx = -\int_0^1 \left(\frac{1}{2}\frac{dk}{dT} + \frac{k(T)}{T}\right)\left(\frac{dT}{dx}\right)^2 dx.$$

After integration, when relationship (6) is used

$$\dot{q}_{v,a} = \frac{2k_1}{n}T_1\left[\left(\frac{T_2}{T_1}\right)^{1+n/2} - \left(\frac{T_2}{T_1}\right)^{n+1} - 1 + \left(\frac{T_2}{T_1}\right)^{n/2}\right]$$

and is show in Fig. 4 as a function of n and T_2/T_1.

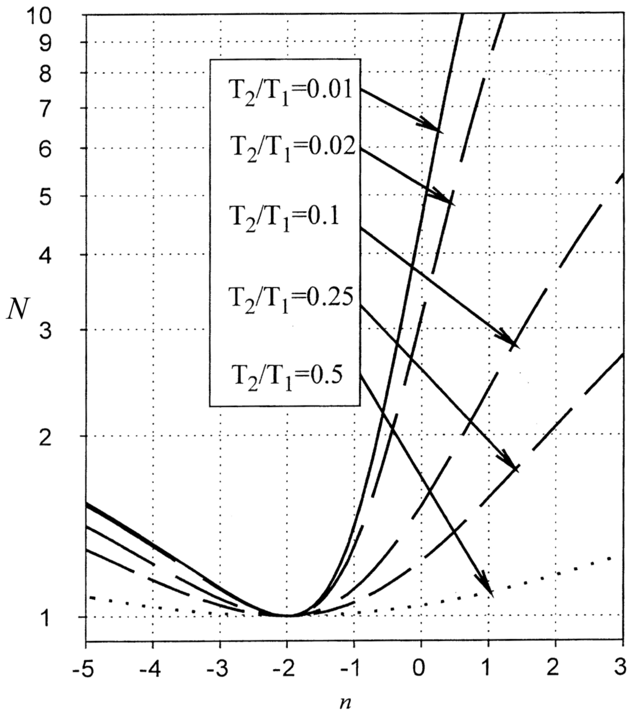

Fig. 3. N vs n for different T_2/T_1

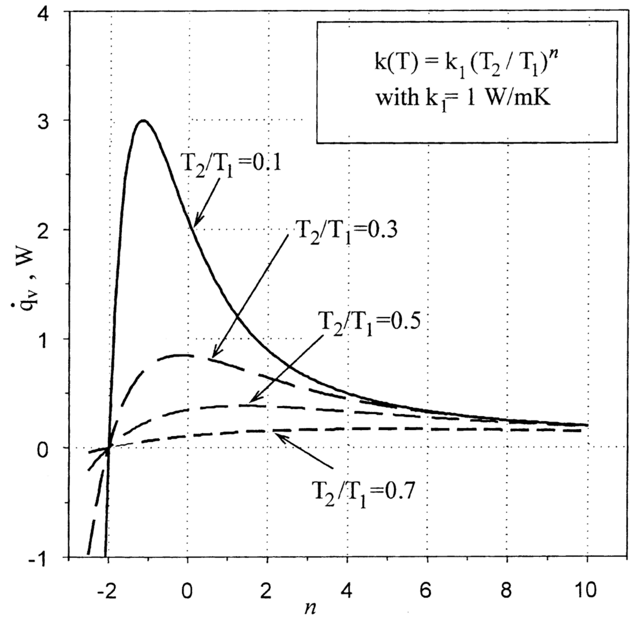

Fig. 4. $\dot{q}_v$ vs n for different T_2/T_1

3. Practical examples

3.1. Helium boil-off minimization. Minimizing helium boil-off rate during cooling of structural support of large superconducting magnet (Fig. 5), has been formulated and discussed by Bejan [5]. Here, the problem is presented in different theoretical way. The problem is to design mechanical support in such a way the helium boil-off rate $\dot{m}_{He}$ is minimized. This is equivalent to minimizing $\dot{q}_{out}$. Neglecting low temperature gradient values in y and z directions, the problem is similar to the minimization of entropy generation for the plane wall (see Fig. 1)

$$T(x) = T_1 \left(\frac{T_2}{T_1}\right)^x \qquad x \in (0,1) \tag{9}$$

and

$$\dot{S}_{gen} = k \ln\left(\frac{T_2}{T_1}\right)^2 .$$

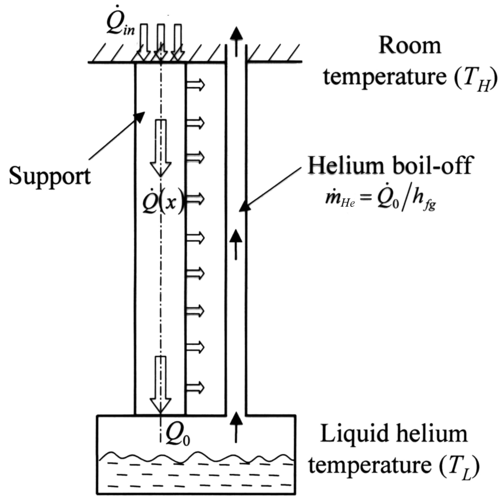

Fig. 5. Cooling of mechanical support (general scheme)

Cooling heat flow rate is exactly equal to the additional heat source term in Eq. (3) equal to

$$\dot{q}_{v,a}(x) = -\frac{k}{T(x)}\left(\frac{dT(x)}{dx}\right)^2 .$$

Using solution (9), $\dot{q}_v(x)$ becomes in the form

$$\dot{q}_{v,a}(x) = -kT_0\left(\frac{T_{\text{He}}}{T_0}\right)^x\left(\ln\frac{T_{\text{He}}}{T_0}\right)^2$$

and after integration its global value is

$$\dot{q}_{v,t} = k\left(T_{\text{He}} - T_0\right)\ln\frac{T_{\text{He}}}{T_0} < 0.$$

This can also be obtained from heat balance equation

$$\dot{q}_{v,t} = \dot{q}_{\text{out}} - \dot{q}_{\text{in}},$$

$$\dot{q}_{\text{out}} = kT_{\text{He}}\ln\frac{T_{\text{He}}}{T_0},$$

$$\dot{q}_{\text{in}} = kT_0\ln\frac{T_{\text{He}}}{T_0}.$$

To design cooling system, helium vapour is heated along the length of the support.

From heat balance equation

$$\dot{m}_{\text{He}}c_{p,\text{He}}dT_{\text{He}} = -kT_0\left(\frac{T_{\text{He}}}{T_0}\right)^x\left(\ln\frac{T_{\text{He}}}{T_0}\right)^2 dx$$

and

$$\frac{dT_{\text{He}}}{dx} = \frac{-kT_0}{\dot{m}_{\text{He}}c_{p,\text{He}}}\left(\frac{T_{\text{He}}}{T_0}\right)^x\left(\ln\frac{T_{\text{He}}}{T_0}\right)^2 .$$

After integration

$$T_{\text{He}}(x) = -\frac{kT_0}{\dot{m}_{\text{He}} c_{p,He}} \left(\ln \frac{T_{\text{He}}}{T_0}\right)^2 \int\limits_0^x \left(\ln \frac{T_{\text{He}}}{T_0}\right)^{x'} dx'. \quad (10)$$

Finally, for boundary condition $T = T_{\text{He}}$ for $x = 1$

$$T_{\text{He}}(x) = T_{\text{He}} + \frac{k}{\dot{m}_{\text{He}} c_{p,\text{He}}} \left[T_0 \left(\frac{T_{\text{He}}}{T_0}\right)^x - T_{\text{He}}\right].$$

Thus, temperature of helium vapour at $x = 0$ is

$$T_{\text{He}}(0) = T_{\text{He}} + \frac{k}{\dot{m}_{\text{He}} c_{p,\text{He}}} (T_0 - T_{\text{He}}).$$

It shows that the cooling system should be designed on the basis of Eq. (10).

3.2. Boundary-value problem with simultaneous heat and electric current flow-nonlinear 1D solution. General scheme of the 1D boundary value problem is shown in Fig. 6. Entropy generation rate is given by expression [5, 6]

$$\dot{S}_{gen} = \int\limits_0^1 \left[\frac{k(T)}{T^2}\left(\frac{dT}{dx}\right)^2 + \frac{i^2\rho(T)}{T}\right] dx \quad (11)$$
$$T = T(x),$$

where i – electric current density, $\rho(T)$ and $k(T)$ electric resistivity and thermal conductivity coefficient, respectively. Minimization of entropy generation rate, given by Eq. (11) leads to the Euler-Lagrange equation [4, 7]

$$\frac{d^2T}{dx^2} + \left[\frac{1}{2k(T)}\frac{dk(T)}{dT} - \frac{1}{T}\right]\left(\frac{dT}{dx}\right)^2 + \frac{i^2}{2k(T)}\left[\rho(T) - T\frac{d\rho(T)}{dT}\right] = 0 \quad (12)$$

or in the general form

$$\frac{d^2T}{dx^2} + F_1(T)\left(\frac{dT}{dx}\right)^2 + F_2(T) = 0$$

where

$$F_1(T) = \frac{1}{2k(T)}\frac{dk(T)}{dT} - \frac{1}{T},$$

$$F_2(T) = \frac{i^2}{2k(T)}\left[\rho(T) - T\frac{d\rho(T)}{dT}\right].$$

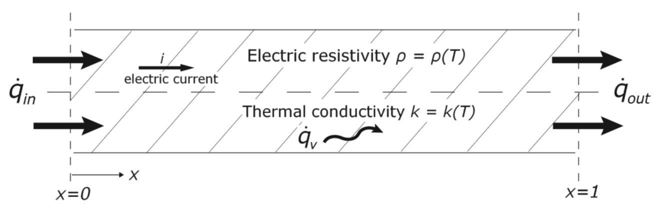

Fig. 6. 1D boundary-value problem

Boundary-value problem is completed with the boundary conditions

$$T(0) = T_1,$$
$$T(1) = T_2.$$

Specific form of the Euler-Lagrange equation when $F_1(T)$ and $F_2(T)$ are both not equal to zero, allows to obtain an analytical solution. One of the methods is described below. Introducing a new variable

$$u = \frac{dT(x)}{dx}$$

and calculated

$$\frac{d^2T}{dx^2} = \frac{du}{dx} = \frac{du}{dT}\frac{dT}{dx} = u\frac{du}{dT}.$$

Thus, Eq. (12) simplifies to the form

$$\frac{du}{dT} + F_1(T)u = -F_2(T)u^{-1}$$

which is Bernoulli differential equation of the first order having general solution in non-explicit form [6]

$$\left(\frac{dT}{dx}\right)^2 = \left[+2\int F_2(T)e^{2\int F_1(T)dT} + c_1\right]e^{-2\int F_1(T)dT}.$$

Finally, the solution is

$$\int\left\{\left[-2\int F_2(T)e^{2\int F_1(T)dT} + c_1\right]e^{-2\int F_1(T)dT}\right\}^{-1/2} dT = x + c_2.$$

Other methods of solution of the Euler-Lagrange Eq. (12) depend on the mathematical form of the functions $F_1(T)$ and$F_2(T)$. Assuming as previously frequently used dependence [5]

$$k(T) = k_1\left(\frac{T(x)}{T_1}\right)^n,$$

where $k_1 = k(T_1)$ and n are arbitrary constants, and from the Widemann-Franz law for the relationship between thermal conductivity and effective resistivity of pure metals

$$k(T)\rho(T) = L_0 T, \qquad L_0 = 2.45 \cdot 10^{-8}\left(\frac{W}{AK}\right)^2$$

the differential Euler-Lagrange Eq. (12) becomes in the form

$$\frac{dw}{dT} + \left(\frac{1}{2k(T)}\frac{dk}{dT} - \frac{1}{T}\right)T = -\frac{i^2}{2}\left(\frac{\rho(T)}{k(T)} - \frac{T}{k(T)}\frac{d\rho}{dT}\right),$$

where

$$w(x) = \frac{dT(x)}{dx}.$$

After calculation

$$\frac{dw}{dT} + \frac{n-2}{2}\frac{1}{T} = -\frac{L_0 T^{2n}}{k_1^2}(2-n)T^{-1}$$

and its solution is

$$T(x) = \left[\frac{c_1}{4}n^2(x+c_2)^2 - \frac{i^2}{2c_1}\frac{L_0 T_1^{2n}}{k_1^2}\right]^{1/n}.$$

Calculation results for n = var, with boundary conditions $T(0) = T_1$ at $x = 0$, $T(1) = T_2$ at $x = 1$ when $\frac{i^2}{2}\frac{L_0 T_1^{2n}}{k_1^2} = 1.0$ are presented in Table 2 and were obtained from equation

$$T(x) = \left[\frac{c_1}{4}n^2(x+c_2)^2 - \frac{1}{c_1}\right]^{1/n}.$$

Table 2
Boundary value problem solution

n	-3	-2	1	2	3
C_1	473.03	121.10	-0.333	3.02	1.778
C_2	$-3.07 \cdot 10^{-2}$	$-4.12 \cdot 10^{-2}$	4.91	-0.669	-0.625

4. 2D boundary value problem

Consider 2D boundary-value problem without an internal heat source when k =const. Global entropy generation is

$$\dot{S}_{gen,t} = \int_0^1 \int_0^1 \dot{S}_{gen,t}(x,y)\, dxdy$$

$$= \int_0^1 \int_0^1 \frac{k}{T^2(x,y)} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] dxdy$$

and its minimization requires the Euler-Lagrange equation to be satisfied

$$\frac{\partial \dot{S}_{gen}}{\partial T} - \frac{\partial}{\partial x} \left(\frac{\partial \dot{S}_{gen}}{\partial T_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \dot{S}_{gen}}{\partial T_y} \right) = 0.$$

After calculation, Euler-Lagrange heat conduction equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{1}{T} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] = 0 \qquad (13)$$

where $T = T(x,y)$.

Equation (13) must be satisfied with imposed boundary conditions. Solution can be easily obtained by transformation

$$\theta(x,y) = \ln T(x,y). \qquad (14)$$

In such a case, heat conduction Eq. (14) takes the form

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \qquad (15)$$

where $\theta = \theta(x,y)$, and entropy generation rates are given by

– local

$$\dot{S}_{gen,t}(x,y) = k \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right],$$

– global (16)

$$\dot{S}_{gen}(x,y) = k \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right] dxdy.$$

The boundary-value problem given by Eq. (15) and arbitrary formulated boundary conditions can be solved with any methods available in heat conduction literature, eg. [1].

4.1. Example of solution. Consider boundary-value problem of heat conduction in square with first kind boundary conditions (Fig. 7), where $T(x,y)$ is dimensionless temperature

$$T(x,y) = \frac{u(x,y)}{u_0}$$

and $u(x,y)$ and u_0 represent temperature (in K) and reference temperature (in K), respectively. Boundary-value problem is:

- *governing equation*

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{1}{T} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] = 0, \qquad (17)$$

$$T = T(x,y),$$

- *boundary conditions*

$$T(x,0) = T(0,y) = T(1,y) = 1,$$

$$T(x,1) = f(x).$$

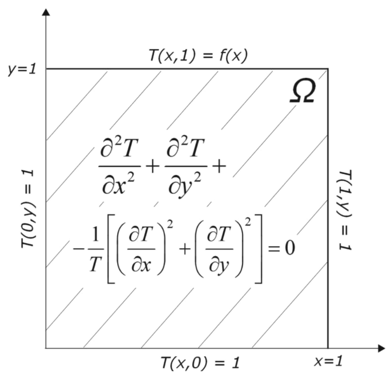

Fig. 7. 2D boundary-value problem

Introducing transformation $\theta(x,y) = \ln T(x,y)$ Eq. (17) becomes

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

with boundary conditions

$$\theta(x,0) = \theta(0,y) = \theta(1,y) = 1,$$

$$\theta(x,1) = \ln f(x).$$

The solution is [4]

$$\theta(x,y) = \sum_{n=1}^{\infty} A_n \frac{\sinh(n\pi y)}{\sinh(n\pi)} \sin(n\pi x),$$

where

$$A_n = 2 \int_0^1 \ln f(x) \sin(n\pi x) dx.$$

For $f(x) = V_0$ =const. the solution becomes

$$\theta(x,y) = \ln T(x,y)$$

$$= \frac{4 \ln V_0}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \frac{\sinh(n\pi y)}{\sinh(n\pi)} \sin(n\pi x).$$

Entropy generation rate is then calculated from Eqs. (16).

5. Conclusions

Minimization of entropy generation in heat conduction processes is always possible by introducing an **additional** internal heat source

$$\dot{q}_{v,a} = -\frac{k}{T}\nabla T \circ \nabla T.$$

For the linear problem ($k =$const.), and

$$\dot{q}_{v,a} = -\left(\frac{1}{2}\frac{dk(T)}{dT} + \frac{k(T)}{T}\right)\nabla T \circ \nabla T$$

when the thermal conductivity coefficient depends on temperature ($k = k(T)$).

As Bejan stated [5] "*Entropy Generation Minimization is the method of thermodynamic optimization of real system that owe their thermodynamic imperfections to heat transfer, fluid flow and mass transfer irreversibilities*".

The wide overview applications are – optimization of heat exchangers, insulation system, optimization of thermodynamic cycles, solar-thermal power generation and many others (see [5]).

Appendix 1.
Methods of solution of Euler-Lagrange heat conduction equations

– *1D problem, k = const., without internal heat source*

The heat conduction equation

$$\frac{d^2T}{dx^2} - \frac{1}{T}\left(\frac{dT}{dx}\right)^2 = 0, \qquad T = T(x). \tag{A1}$$

Introducing

$$w(x) = \frac{dT}{dx}.$$

Equation (A1) is reduced to the separable first order ordinary differential equation

$$\frac{dw}{dT} - \frac{w}{T} = 0$$

and its solution is

$$T(x) = c_1 e^{c_2 x},$$

where c_1 and c_2 are calculated from boundary conditions. Simpler way of solution is based on the fact that expression for the entropy generation rate

$$\dot{S}_{gen}(x) = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2, \qquad T = T(x), \tag{A2}$$

does not depend directly on the independent coordinate x. In such a case the Euler-Lagrange equation becomes [4]

$$\dot{S}_{gen} - T_x\frac{\partial \dot{S}_{gen}}{\partial T_x} = \text{ const.}$$

where T_x is equal to $\partial T/\partial x$. Using Eq. (A2), we obtain

$$\frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 = C$$

which can be easily solved.

– *Simultaneous heat and electric current flow*

- *General nonlinear solution*

The entropy generation rate is

$$\dot{S}_{gen} = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 + \frac{i^2\rho(T)}{T} \qquad T = T(x)$$

which leads to the Euler-Lagrange equation [3, 7]

$$\frac{d^2T}{dx^2} + \left[\frac{1}{k(T)}\frac{dk(T)}{dT} - \frac{1}{T}\right]\left(\frac{dT}{dx}\right)^2 + \frac{i^2}{2k(T)}\left(\rho(T) - T\frac{d\rho(T)}{dT}\right) = 0. \tag{A3}$$

Introducing from definition

$$F_1(T) = \frac{1}{k(T)}\frac{dk(T)}{dT} - \frac{1}{T},$$

$$F_2(T) = \frac{i^2}{2k(T)}\left(\rho(T) - T\frac{d\rho(T)}{dT}\right).$$

Equation (A3) becomes

$$\frac{d^2T}{dx^2} + F_1(T)\left(\frac{dT}{dx}\right)^2 + F_2(T) = 0.$$

Which after introducing a new variable

$$u(x) = \frac{dT(x)}{dx}$$

takes the form of the Bernoulli differential equation

$$\frac{du}{dT} + F_1(T)u = -F_2(T)u^{-1}.$$

Its solution is

$$u^2 \exp\left(2\int F_1(T)dT\right) = -2\int F_2(T)\exp\left(2\int F_1(T)dT\right)dT + c_1.$$

The final solution for $T(x)$ is given indirectly from

$$\int \frac{dT}{u(T)} = x + c_2,$$

where c_1 and c_2 are calculated from boundary conditions.

3D boundary-value problems – general nonlinear solutions

The problem is formulated as follows:

– find such a function $T(x_i)$, ($i = 1, 2, 3$), which satisfying boundary conditions minimizes simultaneously integral

$$\dot{S}_{gen,t} = \iiint_\Omega \dot{S}_{gen}(T, T_{x_i})d\Omega \Rightarrow \min$$

over the whole domain Ω''.

Using the Euler-Lagrange equation

$$\frac{\partial}{\partial T}\left(\dot{S}_{gen}(T, T_{x_i})\right) + \sum_{i=1}^{3}\frac{d}{dx_i}\left(\frac{\partial}{\partial T_{x_i}}\dot{S}_{gen}(T, T_{x_i})\right) = 0,$$

where

$$\dot{S}_{gen}\left(T, T_{x_i}\right) = \sum_{i=1}^{3} \left[\frac{k(T)}{T^2} \nabla T \circ \nabla T + \frac{\dot{q}_v}{T} \right]$$

and $\dot{q}_v$ is intensity of internal heat source. After calculation, Euler-Lagrange heat conduction equation takes the form

$$\nabla^2 T + \left[\frac{1}{k(\mathrm{T})} \frac{dk(\mathrm{T})}{dT} - \frac{1}{T} \right] \nabla T \circ \nabla T + \frac{1}{2k(\mathrm{T})} \left(\dot{q}_v - T \frac{\partial \dot{q}_v}{\partial T} \right) = 0, \qquad \text{(A4)}$$

$$T = T(x_i) \qquad (i = 1, 2, 3).$$

Introducing from definition

$$F_1(\mathrm{T}) = \frac{1}{k(\mathrm{T})} \frac{dk(\mathrm{T})}{dT} - \frac{1}{T},$$

$$F_2(\mathrm{T}) = \frac{1}{2k(\mathrm{T})} \left(\dot{q}_v - T \frac{\partial \dot{q}_v}{\partial T} \right).$$

Equation (A4) can be written in the shorter form

$$\nabla^2 T + F_1(\mathrm{T}) \nabla T \circ \nabla T + F_2(\mathrm{T}) = 0.$$

A method of solution depends on the mathematical form of the functions F_1(T) and F_2(T).

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