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CRITICAL CASES IN NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS, ARISING FROM HYDRAULIC ENGINEERING

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Abstract. This paper starts from several applications described by initial/boundary value problems for $1D$ (time and one space variable) hyperbolic partial differential equations whose basic properties and stability of equilibria are studied throughout the same properties for certain associated neutral functional differential equations. It is a common fact that asymptotic stability for neutral functional differential equations is normally obtained under the assumption of asymptotic stability of the difference operator associated to the aforementioned neutral functional differential equations. However the physically meaningful applications presented in the paper have the associated difference operator(s) in critical cases (their stability is, generally speaking, non-asymptotic). Consequently the stability of the considered application models is either non-asymptotic or fragile (in a sense introduced in the paper). The models represent an overview gathered from various fields, processed here in order to emphasize the associated neutral functional differential equations which, consequently, are a challenge to the usual approaches. In the concluding part there are suggested possible ways to overcome these difficulties.

Keywords: $1D$ hyperbolic partial differential equations, neutral functional differential equations, difference operator, critical case.

Mathematics Subject Classification: 34K20, 34K40, 35B35, 35L50.

1. INTRODUCTION AND PROBLEM DESCRIPTION

The mathematical aspects of the present paper can be described starting from several points of view. We shall start from the mathematical source of them. In the classification of the differential equations with deviated arguments, the neutral F(unctional) D(ifferential) E(quations) (which appeared firstly in a classification due to L.E. El'sgol'ts [13], then in another classification due to G.A. Kamenskii [26]) have a "neutral" position since their solutions are not smoothed for increasing argument (in fact their smoothness is preserved) while they can be constructed both forwards and backwards.

We shall not list here all books and reference papers presenting Neutral FDE (the reference lists of [14] and [24]) but present below a comprehensive and valuable source of FDE. Starting from the papers of A.D. Myshkis and his co-workers [1, 32] on one side and of K.L. Cooke and his co-worker [6, 7] on the other side, it was established that PDE of $1D$ hyperbolic type can generate equations with deviated argument by integration of the Riemann invariants along the characteristics. To illustrate this we shall reproduce a result of Cooke [6], rigorously and completely proven in [44].

Consider the following nonstandard boundary value problem with initial and derivative boundary conditions

$$\begin{aligned}
&\frac{\partial u^+}{\partial t} + \tau^+(\lambda,t)\frac{\partial u^+}{\partial \lambda} = \Phi^+(\lambda,t),\\
&\frac{\partial u^-}{\partial t} + \tau^-(\lambda,t)\frac{\partial u^-}{\partial \lambda} = \Phi^-(\lambda,t), \quad 0 \le \lambda \le 1,\ t \ge t_0,\\
&\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(0,t) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(0,t)\right] = f_0(t),\\
&\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(1,t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(1,t)\right] = f_1(t),\\
&u^\pm(\lambda,t_0) = \omega^\pm(\lambda)\ , \quad 0 \le \lambda \le 1
\end{aligned} \tag{1.1}$$

with $\tau^+(\lambda,t) > 0$, $\tau^-(\lambda,t) < 0$. Observe that the two equations for the Riemann invariants are decoupled; a coupling is realized through the boundary conditions only. Consider the two families of characteristics

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{\tau^\pm(\lambda,t)}, \quad \tau^+(\lambda,t) > 0, \quad \tau^-(\lambda,t) < 0 \tag{1.2}$$

and let $t^\pm(\sigma;\lambda,t)$ the two characteristic curves crossing some point (λ,t) of the strip $[0,1]\times[t_0,t_1)$. Define

$$T^+(t) := t^+(1;0,t) - t, \quad T^-(t) := t^-(0;1,t) - t \tag{1.3}$$

as *propagation times along the characteristics* or *forward and backward* propagation time respectively. We write down the "progressive (forward) wave" $u^+(\lambda,t)$ along the increasing characteristic $t^+(\sigma;\lambda,t)$ - extendable "to the right" up to $\sigma = 1$, and the "reflected (backward) wave" $u^-(\lambda,t)$ along the decreasing characteristic $t^-(\sigma;\lambda,t)$ and integrate from λ to 1 and from λ to 0 respectively, to obtain

$$\begin{aligned}
u^+(\lambda,t) &= u^+(1,t^+(1;\lambda,t)) - \int_\lambda^1 \frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t))}{\tau^+(\sigma,t^+(\sigma;\lambda,t))}\mathrm{d}\sigma,\\
u^-(\lambda,t) &= u^-(0,t^-(0;\lambda,t)) + \int_0^\lambda \frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t))}{\tau^-(\sigma,t^-(\sigma;\lambda,t))}\mathrm{d}\sigma.
\end{aligned} \tag{1.4}$$

For those cases when $t^+(\sigma;\lambda,t)$ can be extended "to the left" up to $\sigma = 0$ and the decreasing characteristic $t^-(\sigma;\lambda,t)$ - "to the right" up to $\sigma = 1$ (1.4) becomes

$$\begin{aligned} u^+(0,t) &= u^+(1,t+T^+(t)) - \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma, \\ u^-(1,t) &= u^-(0,t+T^-(t)) + \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \end{aligned} \tag{1.5}$$

with $T^\pm(t)$ defined by (1.3). In this way (1.5) define certain functional relations between the boundary values of the two waves. Denoting

$$\begin{aligned} y^+(t) &:= u^+(1,t)\ ,\ \Psi^+(t) := \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma, \\ y^-(t) &:= u^-(0,t)\ ,\ \Psi^-(t) := \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \end{aligned} \tag{1.6}$$

we find that $(y^+(t), y^-(t))$ thus defined satisfy the following system of differential equations with deviated argument

$$\begin{aligned} \sum_{k=0}^m \left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t+T^+(t)) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t)\right] &= f_0(t) + \sum_{k=0}^m a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^+(t), \\ \sum_{k=0}^m \left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t+T^-(t))\right] &= f_1(t) - \sum_{k=0}^m b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^-(t). \end{aligned} \tag{1.7}$$

Its solutions can be constructed by steps for $t > t_0 + \max\{T^-(t_0), T^+(t_0)\}$ provided initial conditions are given; these initial conditions can be obtained by considering those characteristics which cannot be extended on the entire segment $[0,1]$ since they cross the axis $t = t_0$ instead of $\sigma = 0$ for $t^+(\cdot;\lambda,t)$ or of $\sigma = 1$ for $t^-(\cdot;\lambda,t)$. If $\omega^\pm(\lambda)$ are the given initial conditions for (1.1) i.e $u^\pm(\lambda,t_0) = \omega^\pm(\lambda)$, then the initial conditions for (1.7) are

$$y_0^+(t^+(1;\lambda,t_0)) = \omega^+(\lambda) + \int_\lambda^1 \frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t_0))}{\tau^+(\sigma,t^+(\sigma;\lambda,t_0))}\mathrm{d}\sigma, \tag{1.8}$$

where $0 \le \lambda \le 1 \;\Leftrightarrow\; t_0 \le t^+(1;\lambda,t_0) \le t_0 + T^+(t_0)$ and

$$y_0^-(t^-(0;\lambda,t_0)) = \omega^-(\lambda) - \int_0^\lambda \frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t_0))}{\tau^-(\sigma,t^-(\sigma;\lambda,t_0))}\mathrm{d}\sigma, \tag{1.9}$$

where $0 \le \lambda \le 1 \;\Leftrightarrow\; t_0 \le t^-(0;\lambda,t_0) \le t_0 + T^-(t_0)$.

Next, the converse relations, suggested by (1.4), namely

$$\begin{aligned} u^+(\lambda, t) &= y^+(t^+(1;\lambda,t)) - \int\limits_\lambda^1 \frac{\Phi^+(\sigma, t^+(\sigma;\lambda,t))}{\tau^+(\sigma, t^+(\sigma;\lambda,t))}\mathrm{d}\sigma, \\ u^-(\lambda, t) &= y^-(t^-(0;\lambda,t)) + \int\limits_0^\lambda \frac{\Phi^-(\sigma, t^-(\sigma;\lambda,t))}{\tau^-(\sigma, t^-(\sigma;\lambda,t))}d\sigma \end{aligned} \tag{1.10}$$

may be viewed as representation formulae for the solutions of (1.1). The following result is true

Theorem 1.1. *Consider the boundary value problem* (1.1). *If* $u^{\pm}(\lambda, t)$ *is a solution satisfying the equations as well as the initial and the boundary conditions, then* $y^{\pm}(t)$ *defined by* (1.6) *are a solution of* (1.7) *with the initial conditions defined by* (1.8) *and* (1.9). *Conversely, let* $y^{\pm}(t)$ *be a sufficiently smooth solution of* (1.7) *with some initial conditions* $y^{\pm}(t)$ *defined on* $t_0 \leq \lambda \leq t_0 + T^{\pm}(t_0)$. *Then* $u^{\pm}(\lambda, t)$ *defined by* (1.10) *is a solution of* (1.1) *with the initial conditions* $\omega^{\pm}(\lambda)$ *defined also by* (1.10) *computed at* $t = t_0$.

Theorem 1.1 ascertains a one to one correspondence between the solutions of two mathematical objects describing some dynamic processes. In this way all properties and mathematical results obtained for one of them are valid for the other. Since in the field of System Theory and Automatic Control the equations with deviated argument are better studied and known, this aspect has guided us in considering the aforementioned approach along several decades – see e.g. [35–43, 47], and also [9, 10, 45, 46].

Define the integers

$$\begin{gathered} L^+ = \max\{k : a_k^+(t) \neq 0\}, \quad L^- = \max\{k : b_k^-(t) \neq 0\}, \\ K^+ = \max\{k : b_k^+(t) \neq 0\}, \quad K^- = \max\{k : a_k^-(t) \neq 0\}, \\ M = L^+ + L^- - (K^+ + K^-) \end{gathered} \tag{1.11}$$

According to the sign of M system (1.7) belongs to one of the three classes of systems with deviated argument: if $M > 0$ it is of delayed type; if $M < 0$ it is of advanced type; if $M = 0$ it is of neutral type. This assertion follows in a straightforward way from the definitions of [4] and is consistent with the classification of [13, 14, 26].

With reference to our previous papers dedicated to applications, e.g. [10, 36, 37, 44–46], we can state that most systems with deviated arguments associated to the boundary value problems for $1D$ hyperbolic partial differential equations are of neutral type. For this reason the next section refers to NFDE.

2. ON NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

2.1. BASIC IDEAS

To illustrate the basic ideas we shall follow [24, Chapter 9], where a "particular" (nevertheless rather present in applications) class of equations is considered

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t)x_t = \mathcal{L}(t)x_t + h(t), \tag{2.1}$$

where $x_t(\vartheta) := x(t+\vartheta)$, $-r \leq \vartheta \leq 0$, $h \in \mathcal{L}^2_{loc}(0,\infty;\mathbb{R}^n)$ and $\mathcal{D}(t) : \mathcal{C}(-r,0;\mathbb{R}^n) \mapsto \mathbb{R}^n$, $\mathcal{L}(t) : \mathcal{C}(-r,0;\mathbb{R}^n) \mapsto \mathbb{R}^n$ being linear operators given by

$$\mathcal{D}(t)\phi = \phi(0) - \int_{-r}^{0} \mathrm{d}[\mu(t,\vartheta)]\phi(\vartheta),\ \ \mathcal{L}(t)\phi = \int_{-r}^{0} \mathrm{d}[\eta(t,\vartheta)]\phi(\vartheta). \tag{2.2}$$

This representation appeared for the first time in [23]and, clearly, standard linear NFDE can be viewed as belonging to (2.1). Observe that if $\mathcal{D}(t)\phi = \phi(0)$ then (2.1) becomes a standard equation of retarded type. This simple remark guided J.K. Hale to develop (or, better said, to adapt) Lyapunov stability theory from retarded to neutral equations provided $\mathcal{D}$ is a stable difference operator. To be more specific, for autonomous (time invariant), with linear difference operator, of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}x_t = f(x_t) \tag{2.3}$$

with $f : \Omega \subseteq \mathcal{C}(-r,0;\mathbb{R}^n) \mapsto \mathbb{R}^n$, Ω being an open subset of $\mathcal{C}(-r,0;\mathbb{R}^n)$, and $\mathcal{D}$ stable, i.e. with $\lim_{t\to\infty} y(t) = 0$ for all solutions of $\mathcal{D}y_t = 0$, theorems of Lyapunov and Razumikhin type were given (Theorems 9.8.1 and 9.8.4 of [24]). Also a theorem of the Barbashin–Krasovskii–LaSalle type (Theorem 9.8.2 of [24]) is given, thus ensuring an instrument for asymptotic stability for the case when the derivative of the Lyapunov functional is only non-positive definite (the so called "weak" Lyapunov function(al)s after Četaev).

To obtain the aforementioned theorems turned to be quite simple – just introduce the difference $\mathcal{D}\phi$ instead of ϕ in the Kamke–Massera functions estimating the Lyapunov functional. The simplicity of the theoretical approach, doubled by the usefulness of the results, lead to many applications of them to those models characterized by NFDE with stable difference operator. Other monographs dedicated to NFDE, e.g. [27–29] mention a slightly different conditions which might be equivalent to the difference operator stability.

2.2. CHALLENGES

We consider that the most important challenges to the aforementioned assumptions arise from applications. In [44] we described certain applications originating from Mechanical Engineering which are in a critical case with respect to the stability of the difference operator.

Consider for instance the dynamic model of the drillstring with controlled rotating speed. The model, deduced using the generalized variational principle of Hamilton reads

$$\begin{aligned}
&\rho(s)I_p(s)\theta_{tt}(s,t) + c(s)(\theta_t(s,t) - \bar{\omega}) - (G(s)I_p(s)(\theta_s(s,t) - \bar{\theta}_s(s)))_s = 0,\\
&c_\ell(\dot{\theta}_m - \bar{\omega}_m) + G(0)I_p(0)(\theta_s(0,t) - \bar{\theta}_s(0)) = 0,\\
&J_m\ddot{\theta}_m + c_0(\dot{\theta}_m - \bar{\omega}_m) + g_0(\dot{\theta}_m(t) - \bar{\omega}_m) + c_\ell(\dot{\theta}(0,t) - \bar{\omega}) = 0,\\
&J_b\ddot{\theta}(L,t) + T(\dot{\theta}(L,t)) - T(\bar{\omega}) + G(L)I_p(L)(\theta_s(L,t) - \bar{\theta}_s(L)) = 0.
\end{aligned} \tag{2.4}$$

Assume a homogeneous material of the drillstring and a zero distributed damping to obtain

$$\begin{aligned}
&\rho\theta_{tt} - G(\theta_{ss} - \bar{\theta}_{ss}(s)) = 0,\\
&c_\ell(\dot{\theta}_m - \bar{\omega}_m) + GI_p(\theta_s(0,t) - \bar{\theta}_s(0)) = 0,\\
&J_m\ddot{\theta}_m + c_0(\dot{\theta}_m - \bar{\omega}_m) + g_0(\dot{\theta}_m - \bar{\omega}_m) + c_\ell(\dot{\theta}(0,t) - \bar{\omega}) = 0,\\
&J_b\ddot{\theta}(L,t) + [T(\dot{\theta}(L,t)) - T(\bar{\omega})] + GI_p(\theta_s(L,t) - \bar{\theta}_s(L)) = 0,
\end{aligned} \tag{2.5}$$

where $\bar{\theta}(s)$, $\bar{\omega}$ and $\bar{\omega}_m$ are steady state values. The nonlinear functions $g_0(\sigma)$ and $T(\sigma)$ are sector restricted

$$\underline{\gamma}\sigma^2 \leq g_0(\sigma)\sigma \leq \bar{\gamma}\sigma^2, \quad \underline{\delta}\sigma^2 \leq T(\sigma + \bar{\omega}) - T(\bar{\omega}) \leq \bar{\delta}\sigma^2 \tag{2.6}$$

and the deviations from the steady state are pointed out. Observe also that the angular variables $\theta(s,t)$, θ_m are cyclic variables. We follow the procedure exposed in Section 1. Introduce first the new state variables

$$\begin{aligned}
&\varpi(s,t) := \theta_t(s,t) - \bar{\omega}, \quad w(s,t) := \theta_s(s,t) - \bar{\theta}_s(s),\\
&\varpi_m := \dot{\theta}_m - \bar{\omega}_m
\end{aligned} \tag{2.7}$$

to obtain the partial differential equations in the symmetric Friedrichs form, together with the boundary conditions; eliminate also the static boundary conditions

$$\begin{aligned}
&\rho\varpi_t - Gw_s = 0, \ w_t - \varpi_s = 0,\\
&-\tfrac{GI_p}{c_\ell}J_m\tfrac{\mathrm{d}}{\mathrm{d}t}w(0,t) - \tfrac{GI_p}{c_\ell}w(0,t) + g_0\left(\tfrac{GI_p}{c_\ell}w(0,t)\right) + c_\ell\varpi(0,t) = 0,\\
&J_b\tfrac{\mathrm{d}}{\mathrm{d}t}\varpi(L,t) + [T(\varpi(L,t) + \bar{\omega}) - T(\bar{\omega})] + GI_pw(0,t) = 0.
\end{aligned} \tag{2.8}$$

Next, we introduce the Riemann invariants of the problem - $r^+(s,t)$ for the forward wave and $r^-(s,t)$ for the backward wave

$$r^\pm(s,t) := \frac{1}{2}\Big[\varpi(s,t) \mp \sqrt{\frac{G}{\rho}}w(s,t)\Big] \tag{2.9}$$

and rewrite (2.8) in the Riemann invariants

$$\begin{aligned}
&r_t^{\pm} \pm \sqrt{\frac{G}{\rho}} r_s^{\pm} = 0, \\
&J_m \frac{\mathrm{d}}{\mathrm{d}t}(r^+(0,t) - r^-(0,t)) + \frac{c_\ell}{I_p\sqrt{\rho G}} g_0 \left(\frac{I_p\sqrt{\rho G}}{c_\ell}(r^+(0,t) - r^-(0,t)) \right) \\
&+ c_0(r^+(0,t) - r^-(0,t)) + \frac{c_\ell^2}{I_p\sqrt{\rho G}}(r^+(0,t) - r^-(0,t)) = 0, \\
&J_b \frac{\mathrm{d}}{\mathrm{d}t}(r^+(L,t) + r^-(L,t)) + [T(\bar{\omega} + r^+(L,t) + r^-(L,t)) - T(\bar{\omega})] \\
&+ I_p\sqrt{\rho G}(r^-(L,t) - r^+(L,t)) = 0.
\end{aligned} \tag{2.10}$$

The characteristic lines are given by

$$t^{\pm}(\sigma; s, t) = t \pm \sqrt{\rho/G}(\sigma - s) \tag{2.11}$$

with the propagation times $T^{\pm} = L\sqrt{\rho/G}$. Denoting and integrating along the characteristics it follows that

$$\begin{aligned}
y^+(t) &:= r^+(L,t) \;\Rightarrow\; r^+(0,t) = y^+(t + T^+) = y^+(t + L\sqrt{\rho/G}), \\
y^-(t) &:= r^-(0,t) \;\Rightarrow\; r^-(L,t) = y^-(t + T^-) = y^-(t + L\sqrt{\rho/G}).
\end{aligned}$$

Introducing finally the new functions

$$\eta^{\pm}(t) := y^{\pm}(t + L\sqrt{\rho/G})$$

we obtain the final form of a nonlinear system of NFDE:

$$\begin{aligned}
&J_m \frac{\mathrm{d}}{\mathrm{d}t}(\eta^+(t) - \eta^-(t - L\sqrt{\rho/G})) + \frac{c_\ell}{I_p\sqrt{\rho G}} g_0 \left(\frac{I_p\sqrt{\rho G}}{c_\ell}(\eta^+(t) - \eta^-(t - L\sqrt{\rho/G})) \right) \\
&+ c_0(\eta^+(t) - \eta^-(t - L\sqrt{\rho/G})) + \frac{c_\ell^2}{I_p\sqrt{\rho G}}(\eta^+(t) + \eta^-(t - L\sqrt{\rho/G})) = 0, \\
&J_b \frac{\mathrm{d}}{\mathrm{d}t}(r^+(L,t) + r^-(L,t)) + [T(\bar{\omega} + r^+(L,t) + r^-(L,t)) - T(\bar{\omega})] \\
&+ I_p\sqrt{\rho G}(r^-(L,t) - r^+(L,t)) = 0
\end{aligned} \tag{2.12}$$

While nonlinear, this system is of the form (2.3). Its difference operator is defined by

$$\mathcal{D}\begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = \begin{pmatrix} \varphi^+(0) \\ \varphi^-(0) \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi^+(-L\sqrt{\rho/G}) \\ \varphi^-(-L\sqrt{\rho/G}) \end{pmatrix} \tag{2.13}$$

and, according to [24], it is stable provided the eigenvalues of the matrix D in the right hand side of (2.13) are inside the unit disk of $\mathbb{C}$. However these eigenvalues are $\pm\imath$, the difference operator thus being marginally (critically) stable. The analysis of the drillstring dynamics reproduces the presentation of [44].

Our survey [44] contains several applications, where the difference operators display a single delay and are critically stable, their matrix having the eigenvalues on the unit circle. Other applications of the same class can be found in [45, 46]. In the next section we shall focus on another important class of applications, displaying the same type of criticality, but within another framework.

3. A CRITICAL DYNAMICS ARISING FROM HYDRAULIC ENGINEERING

Here we shall consider the standard (“benchmark”) structure in hydroelectric engineering (designated as such in [25] but discussed in earlier references also, e.g. [2], or in more recent [33, 34]) – see Figure 1(a).

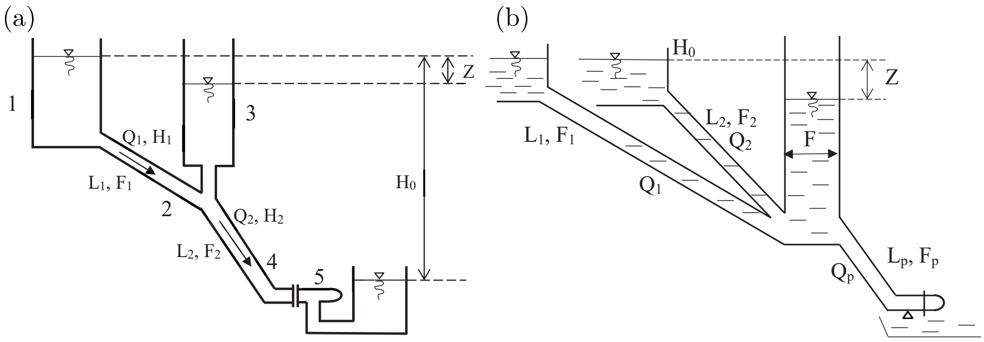

Fig. 1. Benchmark hydroelectric plant structure (a), hydroelectric plant with two tunnels and common lake (b)

The dynamics of this structure is analyzed under normal transients – during the so called frequency/megawatt control by the turbine speed controller – and under abnormal transients – during the water hammer “ignited” by a sudden large load discharge of the turbine. We shall explain in brief both cases based on the model we shall display in the following.

It is a known fact that the model has several time scales which can be pointed out by a singular perturbation analysis. For this reason rated variables are introduced, including rated space variables and rated time. The resulting model looks as follows:

a) The tunnel (upstream conduit uniting the water reservoir – “lake” with the surge tank:

$$\begin{aligned}
&\partial_{\xi_1}\left(h_1 + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}q_1^2\right) + \theta_{w1}\partial_\tau q_1 + \frac{\lambda_1 L_1}{D_1}\frac{1}{2}\frac{\theta_{w1}}{\theta_1}q_1|q_1| = 0,\\
&\delta_1^2\theta_{w1}\partial_\tau h_1 + \partial_{\xi_1}q_1 = 0, \quad h_1(0,\tau) \equiv 1.
\end{aligned} \tag{3.1}$$

b) The surge tank, whose role is to regulate the tunnel output water flow and block the backward water wave during water hammer:

$$\begin{aligned} \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} &= q_1(1,\tau) - q_2(0,\tau), \\ h_1(1,\tau) - R_1'|q_1(1,\tau)|q_1(1,\tau) &= 1 + z(\tau) + R_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} \\ &= h_2(0,\tau) - R_2'|q_2(0,\tau)|q_2(0,\tau). \end{aligned} \tag{3.2}$$

c) The penstock (downstream conduit uniting the surge tank with the hydraulic turbine):

$$\begin{aligned} &\partial_{\xi_2}\left(h_2 + \frac{1}{2}\frac{\theta_{w2}}{\theta_2}q_2^2\right) + \theta_{w2}\partial_\tau q_2 + \frac{\lambda_2 L_2}{D_2}\frac{1}{2}\frac{\theta_{w2}}{\theta_2}q_2|q_2| = 0, \\ &\delta_2^2\theta_{w2}\partial_\tau h_2 + \partial_{\xi_2} q_2 = 0, \quad q_2(1,\tau) = (1-k)f_\theta(\tau)\sqrt{h_2(1,\tau)} + k\varphi(\tau). \end{aligned} \tag{3.3}$$

d) The hydraulic turbine with its speed controller:

$$\begin{aligned} &\theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = q_2(1,\tau)h_2(1,\tau) - \nu_g, \\ &\frac{\mathrm{d}x_c}{\mathrm{d}t} = A_c x_c + b_c(\varphi - 1), \quad \sigma = f_c^* x_c + \gamma_c(\varphi - 1), \quad T_s \dot{f}_\theta = F(\sigma). \end{aligned} \tag{3.4}$$

The speed controller is maintaining the rotating speed of the turbine under the variations of the mechanical load ν_g; this is called *frequency/megawatt control* and is realized by measuring the rotated speed φ, comparing it with the reference synchronous speed ($= 1$, the rotating speed Ω being rated to its synchronous value Ω_c) and modifying the cross-section area f_θ of the turbine wicket gates. As already mentioned, the aforementioned analysis relies on the results already published in [11, 12] (where the rated variables were introduced), starting from [20, 33, 34]; *considering the dynamics of the speed controller is new as well as the analysis which follows.*

Our analysis will be concerned with the transient behavior under water hammer. The scenario for water hammer, under load discharge, is described as follows: the speed controller is decoupled, the wicket gates are blocked in a fixed position $\bar{f}_\theta$ (possibly equaling 0, i.e. completely closed - shut down turbine hence with $\varphi = 0$), the hydraulic system remaining under the control of the surge tank. The study of the water hammer consists of the following three dynamics analyses:

a) *Water mass oscillations* – small and large amplitude; they concern the dynamics within the tunnel – the longest conduit – under various boundary conditions at its output – see [2, 5, 25, 33, 34]. For the small amplitude oscillations, the lumped parameters of the tunnel are sufficient; the model is obtained from (3.1) by singular perturbations as follows: in the second equation we take $\delta_1^2\theta_{w1} = 0$ (here $\delta_1 = \theta_{p1}/\theta_{w1}$), hence $q_1(\xi_1,\tau) \equiv q_1(\tau)$, i.e. it is independent of the space variable ξ_1. Integrating the first equation of (3.1) from 0 to 1 it follows that

$$\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + h_1(1,\tau) - h_1(0,\tau) = 0. \tag{3.5}$$

Now $h_1(0,\tau) \equiv 1$ while $h_1(1,\tau)$ follows from the boundary condition at $\xi_1 = 1$. Usually the literature furnishes information on various hydraulic resistors which have to be tested in small amplitude mass oscillations - see the citations above.

For the large amplitude water mass oscillations, equations (3.1) are considered under the following assumptions: the dynamic head variations are negligible, i.e. $\partial_{\xi_1} q_1^2 \approx 0$, according to registered real data; also the distributed Darcy Weisbach hydraulic losses are neglected, i.e. $(1/2)(\theta_{w1}/\theta_1)\lambda_1 L_1/D_1 \approx 0$ - this assumption being covering from the engineering point of view since the only energy dissipator will remain the aforementioned hydraulic resistor at $\xi_1 = 1$.

Water mass oscillations are not considered in the present paper.

b) *Inherent stability of the surge tank* is a problem arising from the engineering philosophy in the sense that a stabilizing device or construction should be itself stable. The model for this study results as follows. There is taken the water column dynamics in the tunnel - the case of the small amplitude water mass oscillations, i.e. (1.5) with the boundary conditions

$$h_1(0,\tau) = 1,\ h_1(1,\tau) - R_1'|q_1(\tau)|q_1(\tau) = 1 + z(\tau) \tag{3.6}$$

and the resulting equation is coupled with the equation of the surge tank

$$\begin{aligned} &\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + \left(\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + R_1'\right)|q_1|q_1 + z = 0, \\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 - q_2(0,\tau). \end{aligned} \tag{3.7}$$

The flow $q_2(0,\tau)$ is considered to be the load of the dynamics described (3.7). The engineering requirement for the surge tank is to be stable when *following a constant load*. To obtain a constant load we have to consider in (3.4) all derivatives set to zero. Also all losses are neglected since they can be assumed as incorporated in the final constant load ν_g

$$\begin{aligned} &h_2(0,\tau) \equiv 1 + z(\tau), \quad q_2(\xi_2,\tau) = q_2(1,\tau) = q_2(0,\tau) = q_2(\tau), \\ &h_2(\xi_2,\tau) = h_2(0,\tau) = h_2(1,\tau) = h_2(\tau), \quad q_2(1,\tau)h_2(1,\tau) \equiv \nu_g. \end{aligned}$$

From the above equalities we deduce

$$q_2(0,\tau) = q_2(\tau) = \frac{\nu_g}{1 + z(\tau)}.$$

The stability of the surge tank is thus studied on the second order system

$$\begin{aligned} &\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + \left(\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + R_1'\right)|q_1|q_1 + z = 0, \\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 - \frac{\nu_g}{1 + z(\tau)}. \end{aligned} \tag{3.8}$$

A final mention for the models (3.7) and (3.8) – see also (3.6) in comparison to (3.2). The term in R_s' – accounting for the throttling of the surge tank – was neglected. There are several reasons for this: firstly, there are very few surge tanks displaying throttling since it just complicates the construction and the improvements are relatively weak. On the other hand, any surge tank has an input hydraulic resistance which finally may be assimilated to a throttling and incorporated in the overall local hydraulic resistance.

The stability of (3.8) was studied by linearization and also on the nonlinear model by using a suitable Lyapunov function [19, 21, 33]. For this reason, inherent stability of the surge tank will remain outside this paper.

c) *Water mass oscillations overall quenching during water hammer*: this analysis makes use of the entire model (3.1)-(3.3); as already mentioned, the wicket gates of the turbine are blocked in a fixed position $\bar{f}_\theta$, possibly equaling 0 i.e completely closed.

The dynamic heads and the distributed Darcy–Weisbach losses are neglected. Physically speaking, under these circumstances, the water mass oscillations are quenching through energy dissipation (local hydraulic resistors and, possibly, the surge tank throttling) and also through surge tank water level damped oscillations. In fact, as it will appear via the mathematical model, the surge tank will turn to be the only stabilizing device under the aforementioned assumptions.

4. STABILITY ANALYSIS ON A LINEAR MODEL

If dynamic heads, Darcy–Weisbach distributed losses and local hydraulic losses are neglected, the following model is obtained

$$\begin{aligned}
&\partial_{\xi_i} h_i + \theta_{wi}\partial_\tau q_i = 0,\ \delta_i^2\theta_{wi}\partial_\tau h_i + \partial_{\xi_i} q_i = 0, \quad i = 1,2,\\
&h_1(0,\tau) \equiv 1,\ h_1(1,\tau) = 1 + z(\tau) + R_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau),\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau), \quad q_2(1,\tau) = 0.
\end{aligned} \tag{4.1}$$

It was considered $\bar{f}_\theta = 0$ in order to have a completely linear model; in this way, however, all local hydraulic resistors are eliminated.

4.1. REPRESENTATION OF THE SOLUTIONS AND BASIC THEORY

We shall apply to (4.1) the methodology of Section 1 starting by pointing out the steady state of (4.1)

$$\bar{q}_1(\xi_1) \equiv \bar{q}_2(\xi_2) = \bar{q} = 0, \quad \bar{h}_1(\xi_1) \equiv \bar{h}(\xi_2) = \bar{h} = 1, \quad \bar{z} = 0 \tag{4.2}$$

and introducing the deviations $\chi_i(\xi_i,\tau) = h_i(\xi_i,\tau) - 1$. The system in deviations becomes

$$\begin{aligned}
&\partial_{\xi_i}\chi_i + \theta_{wi}\partial_\tau q_i = 0,\ \delta_i^2\theta_{wi}\partial_\tau\chi_i + \partial_{\xi_i} q_i = 0, \quad i = 1,2,\\
&\chi_1(0,\tau) \equiv 0,\ \chi_1(1,\tau) = z(\tau) + R_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_2(0,\tau),\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,\tau) - q_2(0,\tau), \quad q_2(1,\tau) = 0.
\end{aligned} \tag{4.3}$$

Associate next the Riemann invariants

$$r_i^{\pm} = \frac{1}{2}(\delta_i \chi_i \pm q_i) \;\Leftrightarrow\; q_i = r_i^+ - r_i^-, \quad \chi_i = \frac{1}{\delta_i}(r_i^+ + r_i^-) \tag{4.4}$$

and express (4.3) in the Riemann invariants

$$\begin{aligned}
&\delta_i \theta_{wi} \partial_\tau r_i^{\pm} \pm \partial_{\xi_i} = 0, \quad r_1^+(0,\tau) + r_1^-(0,\tau) \equiv 0,\\
&\frac{1}{\delta_1}(r_1^+(1,\tau) + r_1^-(1,\tau)) = z(\tau) + R_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{\delta_2}(r_2^+(0,\tau) + r_2^-(0,\tau)),\\
&\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = r_1^+(1,\tau) - r_1^-(1,\tau) - r_2^+(0,\tau) + r_2^-(0,\tau), \quad r_2^+(0,\tau) - r_2^-(0,\tau) \equiv 0.
\end{aligned} \tag{4.5}$$

The two families of characteristics are as follows

$$\tau_i^{\pm}(\sigma; \xi_i, \tau) = \tau \pm \delta_i \theta_{wi}(\sigma - \xi_i). \tag{4.6}$$

Based on the fact that the Riemann invariants are constant along the characteristics – r_i^+ along τ_i^+ and r_i^- along τ_i^- – the representation formulae are deduced

$$r_i^+(\xi_i,\tau) = r_i^+(1, \tau + \delta_i \theta_{wi}(1 - \xi_i)), \; r_i^-(\xi_i,\tau) = r_i^+(0, \tau + \delta_i \theta_{wi} \xi_i), \quad i = 1,2 \tag{4.7}$$

and, defining $y_i^+(\tau) := r_i^+(1,\tau)$, $y_i^-(\tau) := r_i^-(0,\tau)$,

$$\begin{aligned}
r_i^+(0,\tau) &= r_i^+(1, \tau + \delta_i \theta_{wi}) = y_i^+(\tau + \delta_i \theta_{wi}),\\
r_i^-(1,\tau) &= r_i^-(0, \tau + \delta_i \theta_{wi}) = y_i^-(\tau + \delta_i \theta_{wi}).
\end{aligned} \tag{4.8}$$

Substituting in (4.5) we obtain

$$\begin{aligned}
&y_1^+(\tau + \delta_1 \theta_{w1}) + y_1^-(\tau) = 0,\\
&\frac{1}{\delta_1}(y_1^+(\tau) + y_1^-(\tau + \delta_1 \theta_{w1})) = z(\tau) + R_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{\delta_2}(y_2^+(\tau + \delta_2 \theta_{w2}) + y_2^-(\tau)),\\
&\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = y_1^+(\tau) - y_1^-(\tau + \delta_1 \theta_{w1}) - y_2^+(\tau + \delta_2 \theta_{w2}) + y_2^-(\tau),\\
&y_2^+(\tau) - y_2^-(\tau + \delta_2 \theta_{w2}) = 0.
\end{aligned}$$

Next we introduce the more "conventional" way of writing equations with deviated argument by denoting $w_i^{\pm}(\tau) := y_i^{\pm}(\tau + \delta_i \theta_{wi})$:

$$\begin{aligned}
&w_1^+(\tau) + w_1^-(\tau - \delta_1 \theta_{w1}) = 0,\\
&\frac{1}{\delta_1}(w_1^-(\tau) + w_1^+(\tau - \delta_1 \theta_{w1})) = z(\tau) + R_s' \frac{\mathrm{d}z}{\mathrm{d}\tau}\\
&= \frac{1}{\delta_2}(w_2^+(\tau) + w_2^+(\tau - \delta_2 \theta_{w2})), \quad w_2^-(\tau) - w_2^+(\tau - \delta_2 \theta_{w2}) = 0,\\
&\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = w_1^+(\tau - \delta_1 \theta_{w1}) - w_1^-(\tau) - w_2^+(\tau) + w_2^-(\tau - \delta_2 \theta_{w2}).
\end{aligned} \tag{4.9}$$

Now, for stability studies which are made for large $\tau > 0$, it is sufficient to take $\tau > \max\{\delta_1\theta_{w1}, \delta_2\theta_{w2}\}$ and eliminate $w_1^+(\tau)$ and $w_1^-(\tau)$ to obtain

$$\begin{aligned}\frac{1}{\delta_1}(w_1^-(\tau) - w_1^-(\tau - 2\delta_1\theta_{w1})) &= z(\tau) + R'_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\\ &= \frac{1}{\delta_2}(w_2^+(\tau) + w_2^+(\tau - \delta_2\theta_{w2})), \qquad (4.10)\\ \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= -w_1^-(\tau - 2\delta_1\theta_{w1}) - w_1^-(\tau) - w_2^+(\tau) + w_2^+(\tau - 2\delta_2\theta_{w2}).\end{aligned}$$

Consider separately two difference equations, where $\mathrm{d}z/\mathrm{d}\tau$ is substituted from the differential equation. After some manipulation we obtain the following equations in vector matrix form:

$$\begin{pmatrix}1+\delta_1 R"_s & \delta_1 R"_s\\ \delta_2 R"_s & 1+\delta_2 R"_s\end{pmatrix}\begin{pmatrix}w_1^-(\tau)\\ w_2^+(\tau)\end{pmatrix} = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}z(\tau) + \begin{pmatrix}1-\delta_1 R"_s & \delta_1 R"_s\\ -\delta_2 R"_s & -(1-\delta_2 R"_s)\end{pmatrix}$$
$$\times\begin{pmatrix}w_1^-(\tau - 2\delta_1\theta_{w1})\\ w_2^+(\tau - 2\delta_2\theta_{w2})\end{pmatrix}, \quad R"_s := R'_s/\theta_s.$$

The non-singular left hand side matrix can be inverted and, after an additional substitution in the differential equation of $z(\tau)$, system (4.10) will be given the form

$$\begin{aligned}\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1+(\delta_1+\delta_2)R"_s}[-(\delta_1+\delta_2)z - 2w_1^-(\tau - 2\delta_1\theta_{w1}) + 2w_2(\tau - 2\delta_2\theta_{w2})],\\ w_1^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2)R"_s}[\delta_1 z(\tau) + (1+(\delta_2-\delta_1)R"_s)w_1^-(\tau - 2\delta_1\theta_{w1})\\ &\quad + 2\delta_1 R"_s w_2^+(\tau - 2\delta_2\theta_{w2})], \qquad (4.11)\\ w_2^+(\tau) &= \frac{1}{1+(\delta_1+\delta_2)R"_s}[\delta_2 z(\tau) - 2\delta_2 R"_s w_1^-(\tau - 2\delta_1\theta_{w1})\\ &\quad - (1+(\delta_1-\delta_2)R"_s)w_2^+(\tau - 2\delta_2\theta_{w2})].\end{aligned}$$

The solution of this system of coupled delay differential and difference equations can be constructed by steps on intervals of the form $(2m\delta_1\theta_{w1}, 2(m+1)\delta_1\theta_{w1})$. For this construction we need initial conditions. Starting from a certain solution $(z(\tau), \chi_i(\xi_i,\tau), q_i(\xi_i,\tau))$ of (4.3) with the initial conditions $(z(0), \chi_i^0(\xi_1), q_i^0(\xi_i))$, $0 \le \xi_i \le 1$, $i = 1, 2$, the initial conditions for the associated Riemann invariants, $r_{i0}^{\pm}$, $i = 1, 2$, are obtained, based on (4.4). Next, we make use of formulae (1.8) for the initial conditions, also of the definition for $w_i^{\pm}(\cdot)$ to obtain

$$w_{i0}^+(\tau) = r_{i0}^+(-\tau/(\delta_i\theta_{wi})), \quad w_{i0}^-(\tau) = r_{i0}^-(1+\tau/(\delta_i\theta_{wi})), \quad -\delta_i\theta_{wi} \le \tau < 0. \quad (4.12)$$

On the other hand, in (4.11) initial conditions are needed also on $(-2\delta_i\theta_{wi}, -\delta_i\theta_{wi})$. Since,

$$w_1^+(\tau) = -w_1^-(\tau - \delta_1\theta_{w1}), \quad w_2^-(\tau) = w_2^+(\tau - \delta_2\theta_{w2})$$

and deduce after some straightforward manipulation

$$\begin{aligned}w_{10}^-(\tau) &= -r_{10}^+(-1 - \tau/(\delta_1\theta_{w1})), \quad -2\delta_1\theta_{w1} \le \tau < -\delta_1\theta_{w1},\\ w_{20}^+(\tau) &= r_{20}^-(2 + \tau/(\delta_2\theta_{w2})), \quad -2\delta_2\theta_{w2} \le \tau < -\delta_2\theta_{w2}.\end{aligned} \qquad (4.13)$$

Obviously, $z(0)$ "migrates" from (4.3) to (4.11).

Conversely, let $(z(\tau), w_1^-(\tau), w_2^+(\tau))$ be a solution of (4.11) defined by some initial conditions $(z(0), w_{10}^-(\tau), w_{20}^+(\tau))$, $w_{10}^-(\cdot)$ and $w_{20}^+(\cdot)$ being given on $(-2\delta_1\theta_{w1}, 0)$ and $(-2\delta_2\theta_{w2}, 0)$ respectively.

Using the representation formulae (4.7) we define

$$\begin{aligned} r_i^+(\xi_i, \tau) &= y_i^+(\tau + \delta_i\theta_{wi}(1-\xi_i)) = w_i^+(\tau - \delta_i\theta_{wi}\xi_i), \\ r_i^-(\xi_i, \tau) &= y_i^-(\tau + \delta_i\theta_{wi}\xi_i) = w_i^-(\tau + \delta_i\theta_{wi}(\xi_i - 1)). \end{aligned} \tag{4.14}$$

Making also use of (4.9) we can write

$$r_1^+(\xi_1, \tau) = -w_1^-(\tau - \delta_1\theta_{w1}(\xi_1 + 1)), \quad \tau > \delta_1\theta_{w1} \tag{4.15}$$

and, for $0 < \tau < \delta_1\theta_{w1}$, $r_1^+(\xi_1, \tau)$ is defined by the initial condition for $w_1^-(\cdot)$, i.e. by $w_{10}^-(\cdot)$. In particular, $r_{10}^+(\xi_1) = -w_{10}^-(-\delta_1\theta_{w1}(\xi_1 + 1))$, the initial condition for $w_1^-(\cdot)$ on $(-2\delta_1\theta_{w1}, -\delta_1\theta_{w1})$. Further, $r_1^-(\xi_1, \tau)$ is defined by (4.14) for $\tau > 0$ and, for $\tau = 0$ by $w_{10}^-(\cdot)$ on $(-\delta_1\theta_{w1}, 0)$.

A similar construction holds for $r_2^\pm(\xi_2, \tau)$. Namely

$$r_2^+(\xi_2, \tau) = w_2^+(\tau - \delta_2\theta_{w2}), \quad \tau > 0, \quad 0 \le \xi_2 \le 1 \tag{4.16}$$

with $r_2^+(\xi_2, 0) = r_{20}^+(\xi_2) = w_{20}^+(-\delta_2\theta_{w2}\xi_2)$, using the values of $w_{20}^+(\cdot)$ on $(-\delta_2\theta_{w2}, 0)$. Further

$$r_2^-(\xi_2, \tau) = w_2^+(\tau + \delta_2\theta_{w2}(\xi_2 - 2)), \ \tau > \delta_2\theta_{w2} \tag{4.17}$$

while for $0 < \tau < \delta_2\theta_{w2}$, $r_2^-(\xi_2, \tau)$ is defined by the initial condition $w_{20}^+(\cdot)$. In particular, $r_2^-(\xi_2, 0) = w_{20}^+(\delta_2\theta_{w2}(\xi_2 - 2))$, the initial condition for $w_2^+(\cdot)$ on $(-2\delta_2\theta_{w2}, -\delta_2\theta_{w2})$.

Summarizing, we have obtained the following *new* result, in the line of Theorem 1.1

Theorem 4.1. *Consider the systems* (4.5) *and* (4.11). *Let* $\{z(\tau), r_i^\pm(\xi_i, \tau)\}$ *be a classical solution of* (4.5) *with the initial conditions* $\{z(0), r_{i0}^\pm(\xi_i, \tau), \quad 0 \le \xi_1 \le 1\}$. *Associate system* (4.11) *following the procedure described by* (4.6)–(4.10). *Then* $\{z(\tau), w_1^-(\tau), w_2^+(\tau)\}$ *is a solution of* (4.11) *with the initial conditions defined by* (4.12)-(4.13). *The functions* $\{w_1^-(\tau), w_2^+(\tau)\}$ *have the smoothness of their initial conditions and possible discontinuities at* $\tau = 2(m_1\delta_1\theta_{w1} + m_2\delta_2\theta_{w2})$ *with positive integers* m_1, m_2.

Conversely, let $\{z(\tau), w_1^-(\tau), w_2^+(\tau)\}$ *be a solution of* (4.11) *defined by the initial conditions* $\{z(0), w_{10}^-(\cdot), w_{20}^+(\cdot)\}$ *with* $w_{10}^-(\cdot)$, $w_{20}^+(\cdot)$ *sufficiently smooth, given on* $(-2\delta_1\theta_{w1}, 0)$ *and* $(-2\delta_2\theta_{w2}, 0)$ *respectively. Then* $\{z(\tau), r_i^\pm(\xi_i, \tau)\}$ *is a (possibly) discontinuous classical solution of* (4.5), *where* $r_i^\pm(\xi_i, \tau)$ *are defined by* (4.14)–(4.17), *the initial conditions resulting by taking* $\tau = 0$ *in the aforementioned representation formulae.*

The significance of this theorem is identical to that of Theorem 1.1: it is proven a one-to-one correspondence between the solutions of the two mathematical objects – the non-standard initial boundary value problem (4.5) – and via formulae (4.4), also the initial boundary value problem (4.3) – and the system of coupled delay differential and difference equations (4.11). In this way all properties obtained for one mathematical object are automatically projected back on the other. This assertion will be illustrated in what follows.

4.2. ENERGY IDENTITY AND THE LYAPUNOV FUNCTIONAL

For system (4.3) we write down the energy identity [16]

$$\frac{1}{2}\theta_{wi}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1[q_i^2(\xi_i,\tau)+\delta_i^2\chi_i^2(\xi_i,\tau)]\mathrm{d}\xi_i+q_i(\xi_i,\tau)\chi_i(\xi_i,\tau)|_0^1\equiv 0,\quad i=1,2 \tag{4.18}$$

which, for (4.3), reads via (4.4) as

$$\delta_i\theta_{wi}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1[r_i^+(\xi,\tau)^2+r_i^-(\xi,\tau)^2]\mathrm{d}\xi_i+[r_i^+(\xi,\tau)^2-r_i^-(\xi,\tau)^2]|_0^1\equiv 0. \tag{4.19}$$

The energy identity suggests the following Lyapunov functional for (4.1)

$$\mathcal{V}(z,\phi_i(\cdot),\psi_i(\cdot))=\frac{1}{2}\left\{\theta_s z^2+\sum_1^2\theta_{wi}\int_0^1[\phi_i^2(\xi_i)+\delta_i^2\psi_i^2(\xi_i)]\mathrm{d}\xi_i\right\} \tag{4.20}$$

written as a functional on the state space $\mathbb{R}\times\mathcal{L}^2(0,1;\mathbb{R}^4)$. For (4.3) this functional will be

$$\mathcal{V}(z,\phi_i^+(\cdot),\phi_i^-(\cdot))=\frac{1}{2}\theta_s z^2+\sum_1^2\delta_i\theta_{wi}\int_0^1[\phi_i^+(\xi_i)^2+\phi_i^-(\xi_i)^2]\mathrm{d}\xi_i. \tag{4.21}$$

We write down (4.20) along the solutions of (4.3), differentiate it and take into account the energy identity and the boundary conditions in (4.3)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{V}(z(\tau),q_i(\cdot,\tau),\chi_i(\cdot,\tau))=-R_s'\theta_s\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\leq 0 \tag{4.22}$$

Inequality (4.22) gives in fact the Lyapunov stability of the zero solution of (4.3) *in the sense of the metrics induced* by the Lyapunov functional itself, i.e.

$$\begin{aligned}&\tfrac{1}{2}\left\{\theta_s z^2(\tau)+\textstyle\sum_1^2\theta_{wi}\int_0^1[q_i(\xi_i,\tau)^2+\delta_i^2\chi_i(\xi_i,\tau)^2]\mathrm{d}\xi_i\right\}\\ &\leq\frac{1}{2}\left\{\theta_s z^2(0)+\sum_1^2\theta_{wi}\int_0^1[q_{i0}(\xi_i)^2+\delta_i^2\chi_{i0}(\xi_i)^2]\mathrm{d}\xi_i\right\}.\end{aligned} \tag{4.23}$$

Inequality (4.22) shows that asymptotic stability can be obtained via the invariance principle of Barbashin–Krasovskii–LaSalle. For this we shall turn to system (4.11) via system (4.5). Using the representation formulae for the Riemann invariants, hence for $q_i(\xi_i,\tau)$ and $\chi_i(\xi_i,\tau)$, there is obtained, after a change of the integration variable

$$\begin{aligned}
&\frac{1}{2}\theta_s z^2(\tau) + \frac{1}{\delta_1}\int_{-2\delta_1\theta_{w1}}^{0} w_1^-(\tau+\vartheta)^2\mathrm{d}\vartheta + \frac{1}{\delta_2}\int_{-2\delta_2\theta_{w2}}^{0} w_2^+(\tau+\vartheta)^2\mathrm{d}\vartheta \\
&\leq \frac{1}{2}\theta_s z^2(0) + \frac{1}{\delta_1}\int_{-2\delta_1\theta_{w1}}^{0} w_{10}^-(\vartheta)^2\mathrm{d}\vartheta + \frac{1}{\delta_2}\int_{-2\delta_2\theta_{w2}}^{0} w_{20}^+(\vartheta)^2\mathrm{d}\vartheta \qquad (4.24)\\
&= \frac{1}{2}\left\{\theta_s z^2(0) + \sum_1^2 \theta_{wi}\int_0^1 [q_{i0}(\xi_i)^2 + \delta_i^2\chi_{i0}(\xi_i)^2]\mathrm{d}\xi_i\right\}.
\end{aligned}$$

Now, the derivative function of $\mathcal{V}$ vanishes for $\mathrm{d}z/\mathrm{d}\tau = 0$ hence on the set where

$$-(\delta_1+\delta_2)z(\tau) - 2w_1^-(\tau - 2\delta_1\theta_{w1}) + 2w_2^+(\tau - 2\delta_2\theta_{w2}) = 0 \qquad (4.25)$$

(see the right hand side of the first equation in (4.11)). On this set the difference subsystem of (4.11) becomes - by substituting $z(\tau)$ from (4.25)

$$\begin{aligned}
w_1^-(\tau) &= \frac{1}{\delta_1+\delta_2}[(\delta_2-\delta_1)w_1^-(\tau-2\delta_1\theta_{w1}) + 2\delta_1 w_2^+(\tau-2\delta_2\theta_{w2})],\\
w_2^+(\tau) &= \frac{1}{\delta_1+\delta_2}[-2\delta_2 w_1^-(\tau-2\delta_1\theta_{w1}) + (\delta_2-\delta_1)w_2^+(\tau-2\delta_2\theta_{w2})].
\end{aligned} \qquad (4.26)$$

The invariant set of (4.26) is composed of its constant solutions $\{\bar{w}_1^-, \bar{w}_2^+\}$ which are solutions of

$$\bar{w}_1^- - \bar{w}_2^+ = 0,\ \delta_2\bar{w}_1^- + \delta_1\bar{w}_2^+ = 0.$$

But the only solution of the system above is $\bar{w}_1^- = \bar{w}_2^+ = 0$ and this implies $\bar{z} = 0$. The only invariant set included in the set where the derivative of the Lyapunov functional vanishes is the zero solution. The theorem of Barbashin–Krasovskii–LaSalle for system (4.11) would give asymptotic stability and, therefore, asymptotic stability for the zero solution of (4.3) via the representation formulae (4.15), (4.17) and (4.4).

There is however a specific detail for neutral functional differential equations (and (4.11) is indeed of neutral type): for such systems the invariance principle of Barbashin–Krasovskii–LaSalle – Theorem 9.8.2 of [24] – is proven under the assumption that the difference operator is asymptotically stable. We have thus to turn back to (4.11) and consider its difference subsystem; for stability studies we can take $z(\tau) \equiv 0$. Denote, for the simplicity of the analysis,

$$\begin{aligned}
&\rho_1 := \frac{1+(\delta_2-\delta_1)R"_s}{1+(\delta_1+\delta_2)R"_s},\quad \rho_2 := \frac{1+(\delta_1-\delta_2)R"_s}{1+(\delta_1+\delta_2)R"_s},\\
&\vartheta := 2\delta_2\theta_{w2},\quad \nu\vartheta := 2\delta_1\theta_{w1}\ (\nu := (\delta_1\theta_{w1})(\delta_2\theta_{w2})^{-1}).
\end{aligned}$$

The difference system now reads

$$\begin{aligned}
w_1^-(\tau) &= \rho_1 w_1^-(\tau-\nu\vartheta) + (1-\rho_1)w_2^+(\tau-\vartheta),\\
w_2^+(\tau) &= -(1-\rho_2)w_1^-(\tau-\nu\vartheta) - \rho_2 w_2^+(\tau-\vartheta)
\end{aligned} \qquad (4.27)$$

having two delays. Therefore it belongs to the class defined by

$$y(t) = \sum_1^p A_k y(t - r_k) \tag{4.28}$$

Asymptotic stability of (4.28) is equivalent to the location of the roots of the characteristic equation

$$\det\left(I - \sum_1^p A_k \mathrm{e}^{-\lambda r_k}\right) = 0 \tag{4.29}$$

in a left half plane $\{\lambda \in \mathbb{C} | \Re e(\lambda) \leq -\alpha < 0\}$. If the spectral radius of $\sum_1^p A_k \mathrm{e}^{\imath\theta_k}$ is less that 1 for all $\theta_k \in [0, \pi)$, $k = \overline{1,p}$, the difference operator is *strongly stable*. Strong stability is in fact a *robustness property* of the stability since it ensures stability for all possible values of $r_k > 0$, $k = \overline{1,p}$. This property is important due to the sensitivity of (4.28) with respect to delay uncertainties ([24, Theorem 9.6.1]).

For system (4.27) its characteristic equation will be

$$\left(1 - \rho_1 \mathrm{e}^{-\lambda\nu\vartheta}\right)\left(1 + \rho_2 \mathrm{e}^{-\lambda\vartheta}\right) + (1 - \rho_1)(1 - \rho_2)\mathrm{e}^{-\lambda(\nu+1)\vartheta} = 0. \tag{4.30}$$

The two delays are, generally speaking, rationally independent, i.e. ν is, in general, a real number. Consequently (4.30) ought have its roots with $\Re e(\lambda) \leq -\alpha < 0$ for some positive α. Denoting $z := \mathrm{e}^{\lambda\vartheta}$, the aforementioned condition reduces to the condition for the equation

$$(z^\nu - \rho_1)(z + \rho_2) + (1 - \rho_1)(1 - \rho_2) = 0 \tag{4.31}$$

to have all its roots with $|z| < 1$ (inside the unit disk $\mathbb{D}_1 \subset \mathbb{C}$). Let $z = r\mathrm{e}^{\imath\theta}$. For $r > 1$, a straightforward manipulation will give

$$\begin{aligned}
&\left|r^\nu \mathrm{e}^{\imath\nu\theta} - \rho_1\right| \left|r\mathrm{e}^{\imath\theta} + \rho_2\right| > (r^\nu - \rho_1)(r - \rho_2) > (1 - \rho_1)(1 - \rho_2),\\
&(r^\nu - \rho_1)(r + \rho_2) > (1 - \rho_1)(1 - \rho_2) + 2\rho_2(1 - \rho_1) > (1 - \rho_1)(1 - \rho_2),
\end{aligned}$$

hence equation (4.31) cannot have roots with $|z| > 1$. On the other hand, let $z = \mathrm{e}^{\imath\theta}$; if $\nu = p/q$ with p and q having no common divisors and $\theta = \pi$ there are two cases: if p is even, then $z = -1$ is a root of (4.31), another one being $\rho_1 + \rho_2 - 1 \in (-1, 1)$; if p is odd, there is no such possibility having a root on the unit circle. In *this* last case the invariance principle of Barbashin–Krasovskii–LaSalle can be applied and deduce asymptotic stability for (4.11). Through the representation formulae (4.14)–(4.17) asymptotic stability for (4.5) is obtained. Further, from (4.4) the asymptotic stability is obtained for (4.3).

In order to analyze the case of an irrational delay ratio ν, we turn to the problem of the strong stability of the difference operator. Comparison of (4.27) and (4.28) shows that

$$A_1 = \begin{pmatrix} \rho_1 & 0 \\ -(1 - \rho_2) & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 - \rho_2 \\ 0 & -\rho_2 \end{pmatrix}$$

and we have to compute the spectral radius $A_1 \mathrm{e}^{\imath\theta_1} + A_1 \mathrm{e}^{\imath\theta_1}$ and show it to be less than 1. Another straightforward manipulation gives that the aforementioned spectral radius is less than 1 if and only if the roots of the equation

$$(z-\rho_1)\left(z+\rho_2 \mathrm{e}^{\imath\theta}\right) + (1-\rho_1)(1-\rho_2)\mathrm{e}^{\imath\theta} \tag{4.32}$$

are inside the unit disk for all $\theta \in [0, 2\pi)$. This equation looks very much alike (4.31). Being a second degree equation with complex coefficients, we might think to apply the Schur–Cohn criterion. But we can proceed as in the case of (4.31) and deduce there are no roots with $|z| > 1$. Next, if $\theta = \pi$, then $z = 1$ is a root of (4.32), the other one being again $\rho_1 + \rho_2 - 1 \in (-1, 1)$. This is enough to establish that the spectral radius equals 1. Taking into account Theorem 9.6.1 of [24] on strong stability, the equivalence of (i) and (ii), it follows that there will be no asymptotic stability of (4.27) for irrational ν.

Summarizing, we have pointed out in this application a property of asymptotic stability which is *fragile*: it holds for a *countable set* of rational ratios of the propagation time constants – those ν having an odd numerator. *Fragility* follows from the fact that the set of irrationals is dense and a small uncertainty in the delays can modify ν from rational to irrational. Observe that we have described in fact another critical case for a difference operator – here displaying two delays. The aforementioned analysis is in general, new since it relies on the newly introduced Lyapunov functional (unlike in [20], where other approaches were applied – in fact only sketched there). The entire explicit analysis of the system of neutral functional differential equations (4.11) aiming to apply the Barbashin–Krasovskii–LaSalle invariance principle is new. The properties of the roots of (4.31) and (4.32) are those pointed out in [20, 33, 34]; however they suggest the new idea of *fragile(non-robust)* asymptotic stability.

5. ANOTHER APPLICATION IN WATER HAMMER STABILITY

In [15] there was considered a water hammer stability analysis for the structure in Figure1b, displaying a hydroelectric plant supplied through two independent tunnels starting from the same reservoir (lake). After the publication of [20] a problem was stated: to consider the case of [15] within the same framework of the paper mentioned above – of a linear model with distributed parameters. Consequently, the entire analysis which follows is new. Since all losses were considered, as in the case of [20], negligible, the only energy dissipator remained, as previously, the surge tank throttle. Consequently, the following model resulted:

$$\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i} h_i = 0,\ \delta_i^2 \theta_{wi}\partial_\tau h_i + \partial_{\xi_i} q_i = 0, \quad i = 1, 2, p, \\
&h_1(0,\tau) = h_2(0,\tau) \equiv 1, \\
&h_1(1,\tau) = 1 + z(\tau) + R_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(1,\tau) = h_p(0,\tau), \\
&\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) + q_2(1,\tau) - q_p(0,\tau), \quad q_p(1,\tau) \equiv 0.
\end{aligned} \tag{5.1}$$

The steady state of (5.1) is easily seen to be given by

$$\begin{aligned}&\bar{q}_i(\xi_i) \equiv \bar{q}_i \equiv \text{const}, \quad \bar{h}_i(\xi_i) \equiv \bar{h}_i \equiv \text{const}, \quad \bar{h}_1(1) = \bar{h}_2(1) = \bar{h}_p(0) = 1 + \bar{z},\\ &\bar{h}_1(0) = \bar{h}_2(0) = 1, \quad \bar{q}_p = 0, \quad \bar{q}_1 + \bar{q}_2 = 0,\end{aligned} \tag{5.2}$$

thus resulting $\bar{z} = 0$, $\bar{h}_1 = \bar{h}_2 = \bar{h}_p = 1$, $\bar{q}_p = 0$, $\bar{q}_1 + \bar{q}_2 = 0$.

Clearly $\bar{q}_1$ and $\bar{q}_2$ are not uniquely determined. Admitting $\bar{q}_1 > 0$ it follows that $\bar{q}_2 < 0$ i.e the water will flow upstream and downstream, the circle being "closed" through the reservoir, whose water level will remain constant. This "strange" behavior is a consequence of neglecting all local and distributed losses along the water conduits. It would be quite interesting to see if such a steady state is indeed stable to be really observable (according to the Stability Postulate of N.G. Četaev).

First we introduce the deviations with respect to some steady state defined by $(1, \bar{q}, 1, -\bar{q}, 1, 0, 0)$:

$$\begin{aligned}&\chi_i(\xi_i, \tau) = h_i(\xi_i, \tau) - 1, \quad i = 1, 2, p, \quad \varpi_1(\xi_1, \tau) = q_1(\xi_1, \tau) - \bar{q},\\ &\varpi_2(\xi_2, \tau) = q_2(\xi_2, \tau) + \bar{q}\end{aligned} \tag{5.3}$$

while $q_p(\xi_p, \tau)$ and $z(\tau)$ remain as such since they are deviations from the zero steady state. System (5.1) is replaced by

$$\begin{aligned}&\theta_{wi}\partial_\tau\varpi_i + \partial_{\xi_i}\chi_i = 0,\ \delta_i^2\theta_{wi}\partial_\tau\chi_i + \partial_{\xi_i}\varpi_i = 0, \quad i = 1, 2,\\ &\theta_{wp}\partial_\tau q_p + \partial_{\xi_p}\chi_p = 0,\ \delta_p^2\theta_{wp}\partial_\tau\chi_p + \partial_{\xi_p}q_p = 0,\\ &\chi_1(0, \tau) = \chi_2(0, \tau) \equiv 0,\\ &\chi_1(1, \tau) = z(\tau) + R_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_2(1, \tau) = \chi_p(0, \tau),\\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \varpi_1(1, \tau) + \varpi_2(1, \tau) - q_p(0, \tau), \quad q_p(1, \tau) = 0.\end{aligned} \tag{5.4}$$

We follow now the line of the previous Section 4: introduce the Riemann invariants

$$\begin{aligned}&r_i^\pm = \frac{1}{2}(\delta_i\chi_i \pm \varpi_i) \Leftrightarrow \varpi_i = r_i^+ - r_i^-, \quad \chi_i = \frac{1}{\delta_i}(r_i^+ + r_i^-), \quad i = 1, 2,\\ &r_p^\pm = \frac{1}{2}(\delta_p\chi_p + q_p) \Leftrightarrow q_p = r_p^+ - r_p^-, \quad \chi_p = \frac{1}{\delta_p}(r_p^+ + r_p^-),\end{aligned} \tag{5.5}$$

and express (5.4) in the Riemann invariants

$$\begin{aligned}&\delta_i\theta_{wi}\partial_\tau r_i^\pm \pm \partial_{\xi_i} = 0, \quad i = 1, 2, p,\\ &r_1^+(0, \tau) + r_1^-(0, \tau) = r_2^+(0, \tau) + r_2^-(0, \tau) \equiv 0,\\ &\frac{1}{\delta_1}(r_1^+(1, \tau) + r_1^-(1, \tau)) = \frac{1}{\delta_2}(r_2^+(1, \tau) + r_2^-(1, \tau))\\ &= z(\tau) + R_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{\delta_p}(r_p^+(0, \tau) + r_p^-(0, \tau)),\\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = r_1^+(1, \tau) - r_1^-(1, \tau) + r_2^+(1, \tau) - r_2^-(1, \tau) - r_p^+(0, \tau) + r_p^-(0, \tau),\\ &r_p^+(0, \tau) - r_p^-(0, \tau) \equiv 0.\end{aligned} \tag{5.6}$$

The three families of characteristics are as follows:

$$\tau_i^{\pm}(\sigma;\xi_i,\tau)=\tau\pm\delta_i(\sigma-\xi),\quad i=1,2,p. \tag{5.7}$$

Based on the fact that the Riemann invariants are constant along the characteristics – r_i^+ along τ_i^+ and r_i^- along τ_i^- – the representation formulae are deduced

$$r_i^+(\xi_i,\tau)=r_i^+(1,\tau+\delta_i\theta_{wi}(1-\xi_i)),\quad r_i^-(\xi_i,\tau)=r_i(0,\tau+\delta_i\theta_{wi}\xi_i),\quad i=1,2,p \tag{5.8}$$

and, defining as previously, $y_i^+(\tau):=r_i^+(1,\tau)$, $y_i^-(\tau):=r_i^-(0,\tau)$, it follows that

$$\begin{aligned} r_i^+(0,\tau)&=r_i^+(1,\tau+\delta_i\theta_{wi})=y_i^+(\tau+\delta_i\theta_{wi}),\\ r_i^-(1,\tau)&=r_i^-(0,\tau+\delta_i\theta_{wi})=y_i^-(\tau+\delta_i\theta_{wi}),\quad i=1,2,p.\end{aligned} \tag{5.9}$$

Denote, as previously in Section 4,

$$w_i^{\pm}(\tau):=y_i^{\pm}(\tau+\delta_i\theta_{wi}).$$

Finally, the following system with deviated argument is obtained

$$\begin{aligned} \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}&=w_1^+(\tau-\delta_1\theta_{w1})-w_1^-(\tau)+w_2^+(\tau-\delta_2\theta_{w2})\\ &\quad-w_2^-(\tau)-w_p^+(\tau)+w_p^-(\tau-\delta_p\theta_{wp}),\\ w_1^+(\tau)&=-w_1^-(\tau-\delta_1\theta_{w1}),\quad w_2^+(\tau)=-w_2^-(\tau-\delta_2\theta_{w2}),\\ w_1^-(\tau)+w_1^+(\tau-\delta_1\theta_{w1})&=\delta_1z(\tau)+\delta_1R"_s[w_1^+(\tau-\delta_1\theta_{w1})-w_1^-(\tau)\\ &\quad+w_2^+(\tau-\delta_2\theta_{w2})-w_2^-(\tau)-w_p^+(\tau)+w_p^-(\tau-\delta_p\theta_{wp})],\\ w_2^-(\tau)+w_2^+(\tau-\delta_2\theta_{w2})&=\delta_2z(\tau)+\delta_2R"_s[w_1^+(\tau-\delta_1\theta_{w1})-w_1^-(\tau)\\ &\quad+w_2^+(\tau-\delta_2\theta_{w2})-w_2^-(\tau)-w_p^+(\tau)+w_p^-(\tau-\delta_p\theta_{wp})],\\ w_p^+(\tau)+w_p^-(\tau-\delta_p\theta_{wp})&=\delta_pz(\tau)+\delta_pR"_s[w_1^+(\tau-\delta_1\theta_{w1})-w_1^-(\tau)\\ &\quad+w_2^+(\tau-\delta_2\theta_{w2})-w_2^-(\tau)-w_p^+(\tau)+w_p^-(\tau-\delta_p\theta_{wp})],\\ w_p^-(\tau)&=w_p^+(\tau-\delta_p\theta_{wp}).\end{aligned} \tag{5.10}$$

Again, the stability analysis concerns solutions for large $\tau>0$. Let

$$\tau\geq\max\{\delta_1\theta_{w1},\delta_2\theta_{w2},\delta_p\theta_{wp}\}.$$

Proceeding as in Section 4, the following system is obtained

$$\begin{aligned}
\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1+(\delta_1+\delta_2+\delta_p)R"_s}[-(\delta_1+\delta_2+\delta_p)z - 2w_1^-(\tau-2\delta_1\theta_{w1}) \\
&\quad - 2w_2^-(\tau-2\delta_2\theta_{w2}) + 2w_p^+(\tau-2\delta_p\theta_{wp})], \\
w_1^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2+\delta_p)R"_s}[\delta_1 z(\tau) + (1+(\delta_2+\delta_p-\delta_1))R"_s w_1^-(\tau-2\delta_1\theta_{w1}) \\
&\quad - 2\delta_1 R"_s w_2^-(\tau-2\delta_2\theta_{w2}) + 2\delta_1 R"_s w_p^+(\tau-2\delta_p\theta_{wp})], \\
w_2^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2+\delta_p)R"_s}[\delta_2 z(\tau) - 2\delta_2 R"_s w_1^-(\tau-2\delta_1\theta_{w1}) \\
&\quad + (1+(\delta_1+\delta_p-\delta_2)R"_s) w_2^-(\tau-2\delta_2\theta_{w2}) + 2\delta_2 R"_s w_p^+(\tau-2\delta_p\theta_{wp})], \\
w_p^+(\tau) &= \frac{1}{1+(\delta_1+\delta_2+\delta_p)R"_s}[\delta_p z(\tau) - 2\delta_p R"_s w_1^-(\tau-2\delta_1\theta_{w1}) \\
&\quad - 2\delta_p R"_s w_2^-(\tau-2\delta_2\theta_{w2}) - (1+(\delta_1+\delta_2-\delta_p)R"_s) w_p^+(\tau-2\delta_p\theta_{wp})].
\end{aligned} \tag{5.11}$$

For systems (5.6) and (5.11), a theorem like Theorem 4.1 can be formulated and proven, putting the solutions of the two mathematical objects in a one-to-one correspondence, in the line of Theorem 1.1. We leave aside the statement of this equivalence theorem and focus on the stability problem.

The energy identities will suggest, as in Section 4, the following Lyapunov functional for system (5.4)

$$\begin{aligned}
\mathcal{V}(z,\phi_i(\cdot),\psi_i(\cdot)) = \frac{1}{2}\Bigg\{ & \theta_s z^2 + \sum_1^2 \theta_{wi} \int_0^1 [\phi_i^2(\xi_i) + \delta_i^2 \psi_i^2(\xi)]\mathrm{d}\xi_i \\
& + \theta_{wp} \int_0^1 [\phi_p^2(\xi_p) + \delta_p^2 \psi_p^2(\xi_p)]\mathrm{d}\xi_p \Bigg\}
\end{aligned} \tag{5.12}$$

written as a state function on $\mathbb{R} \times \mathcal{L}^2(0,1;\mathbb{R}^6)$. For (5.6) this functional will be

$$\begin{aligned}
\mathcal{V}(z,\phi_i^+(\cdot),\phi_i^-(\cdot)) = \frac{1}{2}\theta_s z^2 &+ \sum_1^2 \delta_i \theta_{wi} \int_0^1 [\phi_i^+(\xi_i)^2 + \phi_i^-(\xi)^2]\mathrm{d}\xi_i \\
&+ \delta_p \theta_{wp} \int_0^1 [\phi_p^+(\xi_p)^2 + \phi_p^-(\xi_p)]\mathrm{d}\xi_p.
\end{aligned} \tag{5.13}$$

We write down (5.12) along the solutions of (5.4), differentiate it and take into account the energy identity and the boundary conditions in (5.4)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{V}(z(\tau);\varpi_i(\cdot,\tau),\chi_i(\cdot,\tau)) \leq -R_s'\theta_s\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 = R"_s\left(\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \leq 0. \tag{5.14}$$

The derivative function being only negative semi-definite, the zero solution of (5.4) is Lyapunov stable in the sense induced by the Lyapunov functional (5.12) itself.

For the asymptotic stability we turn to system (5.11) and associate to it the Lyapunov functional (5.12), expressed using the representation formulae (5.8). Proceeding as in Section 4, based also on (5.14), we obtain

$$\begin{aligned}&\frac{1}{2}\theta_s z(\tau)^2 + \sum_1^2 \frac{1}{\delta_i} \int\limits_{-2\delta_i\theta_{wi}}^{0} w_i^-(\tau+\vartheta)^2 \mathrm{d}\vartheta + \frac{1}{\delta_p} \int\limits_{-2\delta_p\theta_{wp}}^{0} w_p^+(\tau+\vartheta)^2 \mathrm{d}\vartheta \\ &\leq \mathcal{V}(z(0); \varpi_{i0}(\cdot), \chi_{i0}(\cdot)).\end{aligned} \tag{5.15}$$

Before discussing asymptotic stability, let us remind that the system in deviations was obtained around all steady states satisfying $\bar{q}_p = 0$, $\bar{q}_1 + \bar{q}_2 = 0$ hence any steady state of (5.1) thus results stable in the sense of Lyapunov.

For the asymptotic stability we consider application of the Barbashin–Krasovskii–LaSalle invariance principle to system (5.11). The derivative of the Lyapunov functional (5.14) vanishes on the set where $\frac{\mathrm{d}z}{\mathrm{d}\tau} = 0$, hence for $z \equiv \text{const}$ and for

$$z(\tau) = \frac{2}{\delta_1+\delta_2+\delta_p}[-w_1^-(\tau - 2\delta_1\theta_{w1}) - w_2^-(\tau - 2\delta_2\theta_{w2}) + w_p^+(\tau - 2\delta_p\theta_{wp})]. \tag{5.16}$$

On this set the subsystem of difference equations of (5.11) becomes

$$\begin{aligned}w_1^-(\tau) &= \frac{1}{\delta_1+\delta_2+\delta_p}[(\delta_2+\delta_p-\delta_1)w_1^-(\tau-2\delta_1\theta_{w1}) \\ &\quad - 2\delta_1 w_2^-(\tau-2\delta_2\theta_{w2}) + 2\delta_1 w_p^+(\tau-2\delta_p\theta_{wp})],\\ w_2^-(\tau) &= \frac{1}{\delta_1+\delta_2+\delta_p}[-2\delta_2 w_1^-(\tau-2\delta_1\theta_{w1}) + (\delta_1+\delta_p-\delta_2)w_2^-(\tau-2\delta_2\theta_{w2}) \\ &\quad + 2\delta_2 w_p^+(\tau-2\delta_p\theta_{wp})],\\ w_p^+(\tau) &= \frac{1}{\delta_1+\delta_2+\delta_p}[-2\delta_p w_1^-(\tau-2\delta_1\theta_{w1}) - 2\delta_p w_2^-(\tau-2\delta_2\theta_{w2}) \\ &\quad - (\delta_1+\delta_2-\delta_p)w_p^+(\tau-2\delta_p\theta_{wp})]\end{aligned} \tag{5.17}$$

and its invariant set is given by its constant solutions which are subject to

$$\bar{w}_1^- + \bar{w}_2^- - \bar{w}_p^+ = 0, \quad \delta_p(\bar{w}_1^- + \bar{w}_2^-) + (\delta_1+\delta_2)\bar{w}_p^+ = 0, \tag{5.18}$$

hence $\bar{w}_p^+ = 0$, $\bar{w}_1^- + \bar{w}_2^- = 0$.

The largest invariant set included in the set where the Lyapunov functional vanishes *is not* the zero equilibrium but a set defined by $\{0, \bar{w}^-, -\bar{w}^-, 0\}$, where $\bar{w}^-$ is an arbitrary real constant. It is interesting to see the significance of this set.

Making use of (5.5), (5.8) and of the difference allowing to eliminate w_i^+, $i = 1, 2$, the following representation formulae are obtained

$$\begin{aligned}\varpi_i(\xi_i,\tau) &= -w_i^-(\tau - \delta_i\theta_{wi}(\xi_i+1)) - w_i^-(\tau + \delta_i\theta_{wi}(\xi_i-1)),\\ q_p(\xi_p,\tau) &= w_p^+(\tau - \delta_p\theta_{wp}\xi_p) - w_p^+(\tau + \delta_p\theta_{wp}(\xi_p-2)).\end{aligned} \tag{5.19}$$

For $w_p^+(\cdot) \equiv 0$ we obtain $q_p \equiv 0$. Also, if $w_1^-(\cdot) \equiv \bar{w}^-$ and $w_2^-(\cdot) \equiv -\bar{w}^-$, it follows that $\bar{\varpi}_1 + \bar{\varpi}_2 = 0$.

The result is thus clear: using the one-to-one correspondence between the solutions of (5.6) and (5.11) - in fact also of (5.4) and (5.11) - asymptotic stability of the stationary set defined by $\{0, \bar{w}^-, -\bar{w}^-, 0\}$ or $\{0, \bar{q}, -\bar{q}, 0\}$ was obtained via the Barbashin–Krasovskii–LaSalle invariance principle applied to (5.11), provided the fulfilment of the condition on (strong) stability for the difference operator of the difference subsystem of (5.11). This difference operator displays three delays ordered as $\delta_1\theta_{w1} > \delta_2\theta_{w2} > \delta_p\theta_{wp}$ (based on the real data for certain hydroelectric power plants). For the stability of the difference operator we take $z(\tau) \equiv 0$ in the difference subsystem of (5.11) and make the simplifying notations

$$\rho_1 := \frac{1 + (\delta_2 + \delta_p - \delta_1)R"_s}{1 + (\delta_1 + \delta_2 + \delta_p)R"_s}, \quad \rho_2 := \frac{1 + (\delta_1 + \delta_p - \delta_2)R"_s}{1 + (\delta_1 + \delta_2 + \delta_p)R"_s};$$
$$\rho_p := \frac{1 + (\delta_1 + \delta_2 - \delta_p)R"_s}{1 + (\delta_1 + \delta_2 + \delta_p)R"_s}, \quad \vartheta := 2\delta_p\theta_{wp}, \quad 2\delta_i\theta_{wi} := \nu_i\vartheta \ , \ i = 1, 2, \quad \nu_1 > \nu_2 > 1$$

to rewrite the difference subsystem with $z(\tau) \equiv 0$ as

$$\begin{aligned} w_1^-(\tau) &= \rho_1 w_1^-(\tau - \nu_1\vartheta) - (1 - \rho_1)w_2^-(\tau - \nu_2\vartheta) + (1 - \rho_1)w_p^+(\tau - \vartheta), \\ w_2^-(\tau) &= -(1 - \rho_2)w_1^-(\tau - \nu_1\vartheta) + \rho_2 w_2^-(\tau - \nu_2\vartheta) + (1 - \rho_2)w_p^+(\tau - \vartheta), \\ w_p^+(\tau) &= -(1 - \rho_p))w_1^-(\tau - \nu_1\vartheta) - (1 - \rho_p)w_2^-(\tau - \nu_2\vartheta) - \rho_p w_p^+(\tau - \vartheta). \end{aligned} \tag{5.20}$$

A rather tedious calculation gives the following characteristic equation of (5.20)

$$\begin{aligned} \Delta(\lambda) = (1 - \mathrm{e}^{-\nu_1\vartheta\lambda})&(1 - \mathrm{e}^{-\nu_2\vartheta\lambda})(1 + \rho_p\mathrm{e}^{-\vartheta\lambda}) \\ &+ (1 + \mathrm{e}^{-\vartheta\lambda})[(1 - \rho_2)\mathrm{e}^{-\nu_2\vartheta\lambda}(1 - \mathrm{e}^{-\nu_1\vartheta\lambda}) \\ &+ (1 - \rho_p)(1 - \rho_1)\mathrm{e}^{-\nu_1\vartheta\lambda}(1 - \mathrm{e}^{-\nu_2\vartheta\lambda})]. \end{aligned} \tag{5.21}$$

Denoting $z := \mathrm{e}^{-\vartheta\lambda}$, equation (5.21) becomes

$$\begin{aligned} \pi(z) = (z^{\nu_1} - 1)&(z^{\nu_2} - 1)(z + \rho_p) \\ &+ (z + 1)[(1 - \rho_2)(z^{\nu_1} - 1) + (1 - \rho_p)(1 - \rho_1)(z^{\nu_2} - 1)]. \end{aligned} \tag{5.22}$$

Obviously $\pi(1) = 0$ hence stability in the sense of Theorem 9.6.1 of [24] is not fulfilled and, at this level, the invariance principle cannot be applied. If an invariant set of the basic system might be pointed out, then the root $z = 1$ ($\lambda = 0$) would be eliminated (on the invariant set) and the characteristic equation of the resulting difference operator might then display arithmetic properties as in Section 4. This analysis is outside of this paper.

6. SUMMARIZING CONCLUSIONS

We have started from the one-to-one correspondence between the solutions of two mathematical objects: an initial boundary value problem for hyperbolic partial differential equations describing lossless and distortionless propagation, with nonstandard boundary conditions, and certain functional differential equations with deviated argument, in most cases of neutral type. These equations have been illustrated by the models of engineering applications arising from various fields. There was considered the stability problem, based on “weak” Lyapunov functionals of the energy type. As it is known from several classical references of the stability theory e.g. [3, 30, 31, 48, 50], such Lyapunov function(al)s ensure only stability in the sense of Lyapunov and (possibly) global boundedness of solutions. Obtaining asymptotic stability requires application of the Barbashin–Krasovskii–LaSalle principle.

Throughout this paper we considered the use of the invariance principle to the functional differential equations associated to the considered engineering applications. These functional differential equations resulted all of neutral type. According to the well known result of [24] – Theorem 9.8.2, p. 293 – the invariance principle is proven under the assumption of the asymptotic stability of the difference operator associated to the neutral functional differential equations.

However, the applications considered in our paper are all in critical cases, i.e. the difference operator is stable but *not asymptotically stable*. Among the aforementioned applications in critical cases, we considered in some detail two applications arising from the modeling of the water hammer in hydraulic engineering. The models are highly idealized while in accordance with the engineering practice, by neglecting almost all hydraulic losses (lumped, i.e. local and distributed). As a result, the first model (the standard one - described by (4.1) associated with (4.11) -) displays a *fragile stability*: being a system with two delays, the difference operator is asymptotically stable only for rational delay ratios of the form p/q with both p and q being *odd*. If p is even or the delay ratio is irrational, the difference operator is only stable.

The other case of water hammer – with three delays, described by (5.1), associated with (5.11) – is in a more complicated situation: the equilibrium is not unique, being in fact an equilibrium set. This situation is not acceptable from the engineering point of view, while suggesting existence of an invariant set during the transients also. The criticality of the case is suggested by the fact that the characteristic equation of (5.11) has a zero root.

Obviously we enumerated the conclusions as a list of possible open problems to be considered in future research.

7. WHAT IS TO BE DONE

Here our discussion will be twofold. Firstly we shall discuss the problems of the models leading to critical cases, accepting these mathematical objects as such. We start from a comment in [49, p. 341]. It is mentioned there that the assumption on the asymptotic stability for the difference operator is necessary to obtain pre-compactness of the positive orbits whenever the solution is bounded. It is suggested there to embed the resulting semi-dynamical system in a space wherein the positive orbits are pre-compact. To illustrate this suggestion, the reader is sent to an application in Chapter V, Section 4, p. 252. Interesting enough, the application there is a boundary value problem for a hyperbolic partial differential equation. With the one-to-one correspondence between the solutions of the boundary value problem for the hyperbolic partial differential equation and those of the associated system of neutral functional differential equations, the problem becomes one of choosing the state space for the neutral functional differential equations – other than $\mathcal{C}$ – see [22].

Another aspect, induced by the fragility of the asymptotic stability, is the practical measurement (in site of a certain hydroelectric power plant) displaying some oscillatory modes. Such "real world" measurements should stimulate revival of the old studies which have been obscured by the asymptotic stability of the difference operator assumed in [8]. The book [14] and certain of its references [17, 18, 51, 52] might be a good starting point in this direction.

The second aspect of the discussion is related to the fact that all models discussed in this paper are idealized, even *strongly idealized*. For instance, the hydraulic models with distributed parameters (i.e. described by boundary value problems for hyperbolic partial differential equations) are lossless except the throttling of the surge tank. On the other hand the corresponding models with lumped parameters (i.e. described by ordinary differential equations) incorporate also losses. Therefore, considering distributed parameters and losses might eliminate certain criticalities. *Too much idealization may turn harmful!*

Summarizing, there is plenty of place for research and interesting mathematical results.

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