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ON CHARACTERISTICS OF THE M^θ/G/1/m AND M^θ/G/1 QUEUES WITH QUEUE-SIZE BASED PACKET DROPPING

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Abstract. We study the M^θ/G/1/m and M^θ/G/1 queuing systems with the function of the random dropping of customers used to ensure the required characteristics of the system. Each arriving packet of customers can be rejected with a probability defined depending on the queue length at the service beginning of each customer. The Laplace transform for the distribution of the number of customers in the system on the busy period is found, the mean duration of the busy period is determined, and formulas for the stationary distribution of the number of customers in the system are derived via the approach based on the idea of Korolyuk's potential method. The obtained results are verified with the help of a simulation model constructed with the assistance of GPSS World tools.

Keywords: *queuing systems, packet arrival of customers, active queue management, random dropping of customers, busy period, distribution of the number of customers*

Introduction

For the purpose of preventing overloads in the nodes of packet-switched networks (ATM, TCP/IP, etc.) the *active queue management* (AQM) algorithms are used. In the queuing system simulating a network node, each arriving packet can be dropped with a certain probability dependent on the queue length, even if a buffer is not completely filled. Dependence of the probability of packet dropping on the queue length is called a dropping function [1].

In AQM algorithms different dropping functions are used, for example in a known algorithm RED (Random Early Detection) [2], this function is linear. Thanks to preventive random packet dropping, AQM algorithm indirectly informs the sender that uses the TCP protocol on the oncoming overload. Application of this algorithm in the router can bring a lot of beneficial effects, including reduction of queue and network delay time (more details can be found in [3]).

Studies show [1] that the mechanism of the dropping function is a powerful tool for parameter control of a queuing system. This mechanism can not only regulate the queue length, loss probability of customers, waiting time, and queue length variance, but also regulates several of these parameters simultaneously. The models with the dropping also have deep universal meaning. If we can not regulate the parameters of a queuing system through changes in the input flow or service process, application of dropping function is a simple and effective way to provide the required parameters of the queuing system.

The use of the queuing theory to analyze the active queue management algorithms started in recent years [1, 4-8]. As a rule, the authors restrict the study of systems with exponential distribution of the service time and in the case of consideration of the general law of the service time distribution, the assumption of ordinary input flow is used.

In this paper we consider the M^{θ}/G/1/m and M^{θ}/G/1 queuing system with the dropping function of general form. Each arriving packet of customers can be accepted for service with a probability dependent on the queue length. We assign this probability according to the rule: if at the beginning of a customer service n customers are in the system then each packet of customers arriving in the course of the service of this customer is accepted for service with probability β_n and leaves the system (is discarded) with probability $1-\beta_n$.

In paper [8], we also studied the queuing systems with the dropping function of general form. In contrast to this article, in [8] we consider the systems of M^{θ}/M/n and M^{θ}/M/1/m types and the probability β_n is assigned depending on the queue length at the time of the arrival of each customer packet.

We use an approach based on the idea of Korolyuk's potential method [9], in particular, its modification developed in [10] for the queuing systems with operating parameters dependent on the queue length.

1. Description of the model and basic notation

Let λ be a parameter of the exponential distribution of the time intervals between moments of arrival of customer packets, β_n is the probability of acceptance for the service of an arriving customer packet appointed at the start of customer service according to the algorithm described above. If at the beginning of a customer service n customers are in the system, then in the course of its service the time intervals between moments of arrival of customer packets received for service are distributed exponentially with parameter $\lambda_n = \lambda\beta_n$. Therefore, the functioning of the queuing system with a dropping function of customer packets can be represented as a sequence of modes which differ values of the parameters λ_n $(n \geq 1)$ of the input flow. Introducing the notation $\lambda_0 = \lambda$ we obtain the sequence λ_n $(n \geq 0)$.

Consider an M^{θ}/G/1/m queuing system, which can formally be described as follows. Assume that sequences of independent and identically distributed random variables $\{\alpha_{ni}\}$, $\{\theta_i\}$, $\{\delta_i\}$ $(i \geq 1, n \geq 0)$ are specified for the n-th functioning mode. Here α_{ni} is the time between arrivals of the $(i-1)$th and i-th customer packets, θ_i is the number of customers in the i-th packets and δ_i is the service time of i-th customer. We assume that $\mathbf{P}\{\alpha_{ni} < x\} = 1 - e^{-\lambda_n x}$ $(\lambda_n > 0)$, $\mathbf{P}\{\delta_i < x\} = F(x)$ $(x \ge 0)$, $F(0) = 0$; $\mathbf{P}\{\theta_i = j\} = a_j$, $\sum_{j=1}^{\infty} a_j = 1$. If $\mathbf{P}\{\theta_i = 1\} = a_1 = 1$, then customers arrive at the system one by one.

Customers are served one by one, a served customer leaves the system, and the server immediately starts serving a customer from the queue, if one exists, or waits for the arrival of the next customer packet. The first-in first-out (FIFO) service discipline is used. A queue inside one customer batch can be arbitrarily organized, since the characteristics under study are independent of the way in which the queue is organized. Let m be the maximum number of customers that can simultaneously be in the queue.

Denote by $\mathbf{P}_n$ the conditional probability, provided that at the initial time the number of customers in the queueing system is $n \ge 0$, and by $\mathbf{E}(\mathbf{P})$ the conditional expectation (the conditional probability) if the system starts to work at the time of arrival of the first batch of customers. We denote the described system by $M^{\theta}_{\beta_n}$/G/1/m.

We introduce the following notations: $\eta(x, \lambda_n)$ is the number of customers arriving in the system during the time interval $[0;x)$ under the condition that the time intervals between moments of arrival of the customer packets are exponentially distributed with parameter λ_n; a_j^{k*} is the k-fold convolution of the sequence a_j;

$$f(s) = \int_0^{\infty} e^{-sx} dF(x), \quad M = \int_0^{\infty} x dF(x) < \infty, \quad \overline{F}(x) = 1 - F(x);$$

$$\alpha(z) = \sum_{k=1}^{\infty} z^k a_k; \quad \overline{a}_j = \sum_{k=j}^{\infty} a_k; \quad \overline{p}_j = \sum_{k=j}^{\infty} p_k; \quad \overline{q}_j = \sum_{k=j}^{\infty} q_k; \quad e_a = \sum_{k=1}^{\infty} k a_k < \infty;$$

$$p_{ni}(s) = \frac{1}{f(s)} \int_0^{\infty} e^{-sx} \mathbf{P}\{\eta(x, \lambda_n) = i + 1\} dF(x) =$$

$$= \frac{1}{f(s)} \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^{\infty} e^{-(\lambda_n + s)x} \frac{(\lambda_n x)^k}{k!} dF(x) \quad (n \ge 1, \ i \ge -1); \tag{1}$$

$$q_{ni}(s) = \int_0^{\infty} e^{-sx} \mathbf{P}\{\eta(x, \lambda_n) = i\} \overline{F}(x) dx =$$

$$= \sum_{k=0}^{i} a_i^{k*} \int_0^{\infty} e^{-(\lambda_n + s)x} \frac{(\lambda_n x)^k}{k!} \overline{F}(x) dx \quad (n \ge 1, \ i \ge 0);$$

$$R_{n1}(s) = \frac{1}{f(s) p_{n,-1}(s)},$$

$$R_{n,k+1}(s)=\frac{R_{nk}(s)-f(s)\sum_{i=0}^{k-1}p_{ni}(s)R_{n,k-i}(s)}{f(s)p_{n,-1}(s)} \quad (n\geq 1,\ k\geq 1);$$

$$p_{ni}=\lim_{s\to+0}p_{ni}(s),\quad q_{ni}=\lim_{s\to+0}q_{ni}(s),\quad R_{ni}=\lim_{s\to+0}R_{ni}(s). \tag{2}$$

All dependences on s we consider for values $\operatorname{Re} s\geq 0$ of the argument.

Note that for all $n\in\{1,2,\dots,m+1\}$ the equality $\sum_{i=-1}^{\infty}p_{ni}=1$ holds, therefore sequence p_{ni} can be interpreted as the distribution of jumps in a certain semi-continuous from below random walk.

Taking into account that

$$\sum_{k=0}^{\infty}z^k q_{nk}=\frac{1-f(\lambda_n(1-\alpha(z)))}{\lambda_n(1-\alpha(z))},$$

we obtain the equalities

$$\sum_{k=0}^{\infty}q_{nk}=M. \tag{3}$$

The sequences q_{nk} and R_{nk} can be computed using recurrence relations:

$$q_{n0}=\frac{1-f(\lambda_n)}{\lambda_n},\quad q_{nk}=\sum_{i=1}^{k}a_i q_{n,k-i}-\frac{p_{n,k-1}}{\lambda_n}\quad (n,k\geq 1); \tag{4}$$

$$R_{n1}=\frac{1}{p_{n,-1}},\quad R_{n,k+1}=\frac{R_{nk}-\sum_{i=0}^{k-1}p_{ni}R_{n,k-i}}{p_{n,-1}}\quad (n,k\geq 1). \tag{5}$$

2. Distribution of the number of customers in the system during the busy period

Let $\tau(m)=\inf\{t\geq 0:\xi(t)=0\}$ denote the first busy period for the system $M^{\theta}_{\beta_n}/G/1/m$ and

$$\varphi_n^{(m)}(t,k)=\mathbf{P}_n\{\xi(t)=k,\ \tau(m)>t\},$$
$$\Phi_n^{(m)}(s,k)=\int_0^{\infty}e^{-st}\varphi_n^{(m)}(t,k)dt\quad (1\leq n,k\leq m+1).$$

It is obvious that $\varphi_0^{(m)}(t,k) = 0.$ The total probability formula implies

$$\varphi_n^{(m)}(t,k) = \sum_{j=0}^{m-n}\int_0^t \mathbf{P}_n\{\eta(x)=j\}\varphi_{n+j-1}^{(m)}(t-x,k)dF(x) + \\ + \int_0^t \mathbf{P}_n\{\eta(x)\ge m+1-n\}\varphi_m^{(m)}(t-x,k)dF(x) + \big(\mathbf{P}_n\{\eta(t)=k-n\} + \\ + I\{k=m+1\}\mathbf{P}_n\{\eta(t)\ge m+2-n\}\big)\overline{F}(t) \quad (1\le n\le m). \tag{6}$$

Here $I\{A\}$ is the indicator of a random event A; it equals 1 or 0 depending on whether the event A occurs or not.

Let $q_n(s,k,m) = q_{n,k-n}(s) + I\{k=m+1\}\overline{q}_{n,m+2-n}(s).$ Then, with allowance for relations (1), functions $\Phi_n^{(m)}(s,k)$ are defined from the system of equations derived from (6):

$$\varPhi_n^{(m)}(s,k) = f(s)\sum_{j=0}^{m-n} p_{n,j-1}(s)\varPhi_{n+j-1}^{(m)}(s,k) + \\ + f(s)\overline{p}_{n,m-n}(s)\varPhi_m^{(m)}(s,k) + q_n(s,k,m) \qquad (1\le n\le m). \tag{7}$$

In this case, the boundary condition is written as

$$\varPhi_0^{(m)}(s,k) = 0. \tag{8}$$

The functions $\varPhi_n^{(m)}(s,k)$ can be found by solving the system of equations (7) and (8).

We will use the functions $\mathcal{R}_{ni}(s)$ defined by recurrence relations:

$$\mathcal{R}_{n,1}(s) = R_{n+1,1}(s); \quad \mathcal{R}_{n,j+1}(s) = R_{n+1,1}(s)\Big(\mathcal{R}_{n+1,j}(s) - \\ - f(s)\sum_{i=0}^{j-1} p_{n+1,i}(s)\mathcal{R}_{n+1+i,j-i}(s)\Big) \quad (1\le j\le m-n-1,\ 0\le n\le m-1). \tag{9}$$

Theorem 1 of [10] implies the following statement.

Theorem 1. For $1\le k\le m+1$ and $\operatorname{Re} s>0$ functions $\varPhi_n^{(m)}(s,k)$ are defined as

$$\varPhi_n^{(m)}(s,k) = \left(\mathcal{R}_{n,m-n}(s) - f(s)\sum_{i=1}^{m-n}\mathcal{R}_{ni}(s)\overline{p}_{n+i,m-n-i}(s)\right)\varPhi_m^{(m)}(s,k) - \\ - \sum_{i=1}^{m-n}\mathcal{R}_{ni}(s)q_{n+i}(s,k,m) \quad (1\le n\le m-1), \tag{10}$$

where

$$\Phi_m^{(m)}(s,k) = \frac{\sum_{i=1}^{m} \mathcal{R}_{0i}(s) q_i(s,k,m)}{\mathcal{R}_{0m}(s) - f(s)\sum_{i=1}^{m} \mathcal{R}_{0i}(s)\overline{p}_{i,m-i}(s)}.$$

3. Busy period and stationary distribution

For a more compact notation of the obtained formulas below, we agree that $\mathcal{R}_{m0}(s) \equiv 1$.

If the system starts functioning at the moment when the first packet of customers arrives, then for all $1 \le k \le m+1$ using the formula of total probability we obtain the equalities

$$\int_0^\infty e^{-st}\,\mathbf{P}\{\xi(t)=k,\ \tau(m)>t\}dt = \sum_{n=1}^{m} a_n \Phi_n^{(m)}(s,k) + \overline{a}_{m+1}\Phi_{m+1}^{(m)}(s,k). \tag{11}$$

Taking into account that

$$\varphi_{m+1}^{(m)}(t,k) = \int_0^t \varphi_m^{(m)}(t-x,k)dF(x) + I\{k=m+1\}\overline{F}(t),$$

$$\Phi_{m+1}^{(m)}(s,k) = \Phi_m^{(m)}(s,k) + I\{k=m+1\}\frac{1-f(s)}{s},$$

and using the relations (10), we can rewrite the equality (11) in the form

$$\begin{aligned}\int_0^\infty e^{-st}\,\mathbf{P}\{\xi(t)=k,\ \tau(m)>t\}dt = &\left(\sum_{n=1}^{m} a_n\left(\mathcal{R}_{n,m-n}(s) - f(s)\sum_{j=1}^{m-n}\mathcal{R}_{nj}(s)\overline{p}_{n+j,m-n-j}(s)\right)+\right.\\ &\left.+\ \overline{a}_{m+1}\right)\Phi_m^{(m)}(s,k) - \sum_{n=1}^{m-1} a_n \sum_{j=1}^{m-n}\mathcal{R}_{nj}(s) q_{n+j}(s,k,m) + \overline{a}_{m+1} I\{k=m+1\}\frac{1-f(s)}{s}.\end{aligned} \tag{12}$$

To obtain a representation for $\int_0^\infty e^{-st}\,\mathbf{P}\{\tau(m)>t\}dt$ we sum up equalities (12) for k running from 1 to $m+1$. Given the definitions of $q_n(s,k,m)$ and $q_{ni}(s)$, it is not difficult to ascertain that

$$\sum_{k=1}^{m+1} q_n(s,k,m) = \sum_{k=0}^{\infty} q_{nk}(s) = \frac{1-f(s)}{s} \quad (1 \le n \le m). \tag{13}$$

Thus, (12) confirms the following statement.

Theorem 2. For the system $M^{\theta}_{\beta_n}$/G/1/m the Laplace transform of the distribution function of the busy period is defined as

$$\int_0^{\infty} e^{-st}\mathbf{P}\{\tau(m)>t\}dt = \left(\sum_{n=1}^{m} a_n\left(\mathcal{R}_{n,m-n}(s)-f(s)\sum_{j=1}^{m-n}\mathcal{R}_{nj}(s)\overline{p}_{n+j,m-n-j}(s)\right)+\overline{a}_{m+1}\right)\times$$
$$\times\frac{1-f(s)}{s}\frac{\sum_{n=1}^{m}\mathcal{R}_{0n}(s)}{\mathcal{R}_{0m}(s)-f(s)\sum_{n=1}^{m}\mathcal{R}_{0n}(s)\overline{p}_{n,m-n}(s)}-\frac{1-f(s)}{s}\left(\sum_{n=1}^{m-1}a_n\sum_{j=1}^{m-n}\mathcal{R}_{nj}(s)-\overline{a}_{m+1}\right). \quad (14)$$

To find $\int_0^{\infty}\mathbf{P}\{\tau(m)>t\}dt=\mathbf{E}\tau(m)$ we need to pass to the limit in (14) as $s\to+0$. We use the sequences $\{p_{ni}\}$, $\{q_{ni}\}$ and $\{R_{ni}\}$, defined by (2), as well as sequences $\{\mathcal{R}_{ni}\}$, obtained by limit passage $\mathcal{R}_{ni}=\lim_{s\to+0}\mathcal{R}_{ni}(s)$. For $\mathcal{R}_{ni}$ (9) implies the recurrence relations

$$\mathcal{R}_{n,1}=R_{n+1,1};$$
$$\mathcal{R}_{n,j+1}=R_{n+1,1}\left(\mathcal{R}_{n+1,j}-\sum_{i=0}^{j-1}p_{n+1,i}\mathcal{R}_{n+1+i,j-i}\right)\quad(1\le j\le m-n-1,\ 0\le n\le m-1). \quad (15)$$

It follows from (5) and (15) that

$$\mathcal{R}_{nk}-\sum_{i=1}^{k}\mathcal{R}_{ni}\overline{p}_{n+i,k-i}=1\quad(n\ge 0,\ k\ge 1). \quad (16)$$

Given (3), (13) and (16), using (14) we obtained the following statement.

Theorem 3. The mean duration of the busy period of the queueing system $M^{\theta}_{\beta_n}$/G/1/m is determined in the form

$$\mathbf{E}\tau(m)=M\left(\sum_{i=1}^{m}\mathcal{R}_{0i}-\sum_{n=1}^{m-1}a_n\sum_{i=1}^{m-n}\mathcal{R}_{ni}+\overline{a}_{m+1}\right). \quad (17)$$

We introduce the notation: $\lim_{t\to\infty}\mathbf{P}\{\xi(t)=k\}=\pi_k(m),\ 1\le k\le m+1$. Reasoning as in paper [11], we obtain formulas for the stationary distribution of the number of customers in the system $M^{\theta}_{\beta_n}$/G/1/m.

Theorem 4. The stationary distribution of the number of customers in the system $M^{\theta}_{\beta_n}$/G/1/m is given by

$$\pi_0(m)=\frac{1}{1+\lambda\,\mathbf{E}\tau(m)};$$

$$\pi_k(m)=\lambda\pi_0(m)\left(\sum_{i=1}^{k}\mathcal{R}_{0i}q_{i,k-i}-\sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}\mathcal{R}_{ni}q_{n+i,k-n-i}\right)\quad(1\le k\le m);\qquad(18)$$

$$\pi_{m+1}(m)=\lambda\pi_0(m)\left(\sum_{i=1}^{m}\mathcal{R}_{0i}\overline{q}_{i,m+1-i}-\sum_{n=1}^{m-1}a_n\sum_{i=1}^{m-n}\mathcal{R}_{ni}\overline{q}_{n+i,m+1-n-i}+M\overline{a}_{m+1}\right).$$

Using (17) we find the ratio of the mean number of customers served per unit of time to the mean number of all arriving customers per unit time and obtain the formula for the stationary service probability for the system $\mathrm{M}^{\theta}_{\beta_n}/\mathrm{G}/1/\mathrm{m}$

$$\mathbf{P}_{\rm sv}(m)=\frac{\sum_{i=1}^{m}\mathcal{R}_{0i}-\sum_{n=1}^{m-1}a_n\sum_{i=1}^{m-n}\mathcal{R}_{ni}+\overline{a}_{m+1}}{e_a\left(1+\lambda\mathbf{E}\tau(m)\right)}.$$

We find the stationary queue characteristics - mean queue length $\mathbf{E}Q(m)$ and mean waiting time $\mathbf{E}w(m)$ - by the formulas

$$\mathbf{E}Q(m)=\sum_{k=1}^{m}k\pi_{k+1}(m);\quad \mathbf{E}w(m)=\frac{\mathbf{E}Q(m)}{\lambda e_a\mathbf{P}_{\rm sv}(m)}.$$

4. The system $\mathbf{M}^{\theta}_{\beta_n}$/G/1

Fixing a natural number $h\in\{2,3,\dots,m\}$ we define a set of probabilities β_n by the equalities

$$\beta_n=\begin{cases}\beta_n, & 1\le n\le h-1;\\ \beta, & h\le n\le m.\end{cases}$$

For $n\in\{h,h+1,\dots,m\}$ we introduce the notation: $R_{ni}=\widetilde{R}_i$ ($i\ge 1$), $p_{ni}=\tilde{p}_i$ ($i\ge -1$), $q_{ni}=\tilde{q}_i$ ($i\ge 0$), $\lambda_n=\tilde{\lambda}=\lambda\beta.$

Using (15) and (5) we obtain the equalities

$$\mathcal{R}_{ni}=\widetilde{R}_i,\quad n\ge h-1,\quad i\ge 1,$$

from which and the formula (17) imply

$$\mathbf{E}\,\tau(m)=M\left(\sum_{i=1}^{m}\mathcal{R}_{0i}-\sum_{n=1}^{h-2}a_n\sum_{i=1}^{h-1-n}\mathcal{R}_{ni}-\sum_{n=1}^{h-1}a_n\sum_{i=h-n}^{m-n}\mathcal{R}_{ni}-\sum_{n=h}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i+\overline{a}_{m+1}\right).\quad(19)$$

In what follows we study the corresponding queueing system with no restrictions on the queue length $(m=\infty)$, which we denote by $M^{\theta}_{\beta_n}/G/1$.

For the system $M^{\theta}_{\beta_n}/G/1$ we introduce the notation: $\xi_\infty(t)$ is the number of customers in the system at time t, $\tau(\infty)=\inf\{t\ge 0:\xi_\infty(t)=0\}$ is the first busy period, $\rho_n=\lambda_n Me_a$ $(1\le n\le h-1)$, $\tilde{\rho}=\tilde{\lambda}Me_a$; $\lim_{t\to\infty}\mathbf{P}\{\xi_\infty(t)=k\}=\pi_k(\infty)$, $k\ge 0$.

Reasoning as in [11] to investigate the system $M^{\theta}/G_1,\dots,G_h/1$, after passing to the limit in (19) and (18) as $m\to\infty$ we obtain the following results.

Theorem 5. If $\tilde{\rho}<1$, then for the system $M^{\theta}_{\beta_n}/G/1$ we have

$$\mathbf{E}\tau(\infty)=M\left(\sum_{i=1}^{h-1}\mathcal{R}_{0i}-\sum_{n=1}^{h-2}a_n\sum_{i=1}^{h-1-n}\mathcal{R}_{ni}+\right.$$
$$\left.+\frac{1}{1-\tilde{\rho}}\left(e_a+\mathcal{R}_{01}(\rho_1-1)+\sum_{i=2}^{h-1}\left(\mathcal{R}_{0i}-\sum_{n=1}^{i-1}a_n\mathcal{R}_{n,i-n}\right)(\rho_i-1)\right)\right). \tag{20}$$

Theorem 6. If $\tilde{\rho}<1$, then the stationary distribution of the number of customers in the system $M^{\theta}_{\beta_n}/G/1$ is given by

$$\pi_0(\infty)=\frac{1}{1+\lambda\,\mathbf{E}\tau(\infty)};$$
$$\pi_k(\infty)=\lambda\pi_0(\infty)\left(\sum_{i=1}^{k}\mathcal{R}_{0i}q_{i,k-i}-\sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}\mathcal{R}_{ni}q_{n+i,k-n-i}\right)\quad(k\ge 1).$$

To find the stationary values of the mean queue length $\mathbf{E}Q(\infty)$, you can use the approximate formula from [11]

$$\mathbf{E}Q(\infty)=\sum_{k=1}^{\infty}k\pi_{k+1}(\infty)\approx\mathbf{E}Q_{(N)}=\sum_{k=1}^{N-1}k\pi_{k+1}(\infty)+\left(N-1+\frac{\overline{\pi}_{N+1}(\infty)}{\pi_N(\infty)}\right)\overline{\pi}_{N+1}(\infty), \tag{21}$$

where

$$\overline{\pi}_{N+1}(\infty)=1-\sum_{k=0}^{N}\pi_k(\infty).$$

The stationary value of the average waiting time $\mathbf{E}w(\infty)$ we find by the formula

$$\mathbf{E}w(\infty)=\frac{\mathbf{E}Q(\infty)}{\lambda e_a\mathbf{P}_{\text{sv}}(\infty)}. \tag{22}$$

Here, the expression for the stationary service probability of considered system

$$\mathbf{P}_{\rm sv}(\infty)=\frac{1}{e_a\left(1+\lambda\,\mathbf{E}\tau(\infty)\right)}\left(\sum_{i=1}^{h-1}\mathcal{R}_{0i}-\sum_{n=1}^{h-2}a_n\sum_{i=1}^{h-1-n}\mathcal{R}_{ni}+\right.$$
$$\left.+\frac{1}{1-\tilde\rho}\left(e_a+\mathcal{R}_{01}(\rho_1-1)+\sum_{i=2}^{h-1}\left(\mathcal{R}_{0i}-\sum_{n=1}^{i-1}a_n\mathcal{R}_{n,i-n}\right)(\rho_i-1)\right)\right)$$

is obtained by using formula (20) for $\mathbf{E}\tau(\infty)$.

5. The system $\mathrm{M}_{\beta}^{\theta}/\mathrm{G}/1$

Let's we have the system of $\mathrm{M}_{\beta_n}^{\theta}/\mathrm{G}/1$-type and $\beta_n=1,\ \lambda_n=\lambda$ for $1\le n\le h-1$ and $\beta_n=\beta<1,\ \lambda_n=\tilde\lambda=\lambda\beta$ for $n\ge h,$ then the function of the random dropping of customer packets applies only when $n\ge h.$ Such a system is denoted by $\mathrm{M}_{\beta}^{\theta}/\mathrm{G}/1.$

For $n\in\{1,2,\dots,h-1\}$ we introduce the notation: $R_{ni}=R_i$ $(i\ge1),$ $p_{ni}=p_i$ $(i\ge-1),$ $q_{ni}=q_i$ $(i\ge0),$ $\rho_n=\rho.$

Reasoning as in [11] to investigate the system $\mathrm{M}^{\theta}/\mathrm{G},\widetilde{\mathrm{G}}/1,$ we obtain the following assertion.

Theorem 7. If $\tilde\rho<1,$ then for the system $\mathrm{M}_{\beta}^{\theta}/\mathrm{G}/1$ we have

$$\mathbf{E}\,\tau(\infty)=M\left(\sum_{i=1}^{h-1}R_i\bar a_{h-i}+\frac{1}{1-\tilde\rho}\left(e_a+(\rho-1)\sum_{i=1}^{h-1}R_i\bar a_{h-i}\right)\right).\qquad(23)$$

With the help of Theorem 6 of the paper [12] we obtain formulas for the stationary distribution $\pi_k(\infty)$ $(k\ge0).$

Theorem 8. If $\tilde\rho<1,$ then the stationary distribution of the number of customers in the system $\mathrm{M}_{\beta}^{\theta}/\mathrm{G}/1$ is given by

$$\begin{aligned}
&\pi_0(\infty)=\frac{1}{1+\lambda\,\mathbf{E}\tau(\infty)};\\
&\pi_k(\infty)=\lambda\pi_0(\infty)\left(\sum_{i=1}^{k}R_iq_{k-i}-\sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}R_iq_{k-n-i}\right)\quad(1\le k\le h-1);\\
&\pi_k(\infty)=\lambda\pi_0(\infty)\left(\sum_{i=1}^{h-1}R_iq_{k-i}-\sum_{n=1}^{h-2}a_n\sum_{i=1}^{h-1-n}R_iq_{k-n-i}+\left(p_{-1}R(h)-\right.\right.\\
&\left.\left.-a_{h-1}\right)\sum_{i=1}^{k-h+1}\widetilde{R}_i\tilde q_{k-h+1-i}-\sum_{n=h}^{k-1}\left(r_n(h)+a_n\right)\sum_{i=1}^{k-n}\widetilde{R}_i\tilde q_{k-n-i}\right)\quad(k\ge h).
\end{aligned}\qquad(24)$$

With the help of (23) we obtain the formula for the stationary service probability of the system $M^\theta_\beta/G/1$

$$\mathbf{P}_{sv}(\infty) = \frac{1}{e_a\left(1+\lambda\,\mathbf{E}\tau(\infty)\right)}\left(\sum_{i=1}^{h-1} R_i\overline{a}_{h-i} + \frac{1}{1-\tilde{\rho}}\left(e_a + (\rho-1)\sum_{i=1}^{h-1} R_i\overline{a}_{h-i}\right)\right). \quad (25)$$

Consider examples of calculation of the stationary characteristics of the system $M^\theta_\beta/G/1$, using the formulas (21)-(25) and simulation system GPSS World [13, 14].

Let $\lambda = 2$, customers arrive in packets in an amount from one to five, with probabilities $a_i = 0{,}2$ $(1 \le i \le 5)$, the service time is uniformly distributed on the interval [0,2], hence $M = 1$, $e_a = 3$, $\rho = 6$. Assume that $\beta = 0{,}1$, then $\lambda_n = \tilde{\lambda} = = \lambda\beta = 0{,}2$ for $n \ge h$ and $\tilde{\rho} = 0{,}6$.

If $h = 3$, then the mean duration of the busy period $\mathbf{E}\tau(\infty)$ found by the formula (23) is equal to 265.461.

The stationary distribution of the number of customers and stationary characteristics of the system $M^\theta_\beta/G/1$, calculated by formulas (21)-(25), are shown in Tables 1 and 2. In calculating by the approximate formula (21) we use the value $N = 10$. For the sake of comparison, values of the corresponding characteristics, obtained with the help of GPSS World, are presented.

Let in the system $M^\theta_\beta/G/1$ the probability β_n be assigned depending on the number n of customers, that are in the system at the time of arrival of each customer packet. We denote such a system by $M^\theta_{\beta a}/G/1$. Let's specify that for the system $M^\theta_{\beta a}/G/1$ the dropping function of customer packets applies only for $n \ge h$, where $\beta_n = \beta < 1$, $\lambda_n = \tilde{\lambda} = \lambda\beta$ for $n \ge h$.

Table 3 contains a comparison of stationary values of the mean queue length for the systems $M^\theta_\beta/G/1$ and $M^\theta_{\beta a}/G/1$, calculated using GPSS World for $\beta = 0.1$ and different values of h. If h is increased, then the mean queue length increases, and for the same values of h we have $\mathbf{E}Q_a(\infty) < \mathbf{E}Q(\infty)$. Here we denote by $\mathbf{E}Q_a(\infty)$ the stationary value of the mean queue length for the system $M^\theta_{\beta a}/G/1$.

A comparison of the stationary values of the mean queue length for the systems $M^\theta_\beta/G/1$ and $M^\theta_{\beta a}/G/1$, computed by means of GPSS World for $h = 5$ and different values of β, is shown in Table 4. If β is increased, then the mean queue length increases, and for the same values of β we have $\mathbf{E}Q_a(\infty) < \mathbf{E}Q(\infty)$.

Table 1

Stationary distributions of the number of customers in the system $M_{\beta}^{\theta}/G/1$ for $h = 3$

Number of customers (k)	0	1	2	3	4	5
$\pi_k(\infty)$	0.00188	0.00578	0.02213	0.05977	0.06648	0.07142
$\pi_k(\infty)$ (GPSS World, $t = 10^6$)	0.00180	0.00561	0.02153	0.05809	0.06541	0.07003
Number of customers (k)	6	7	8	9	10	⋯
$\pi_k(\infty)$	0.07384	0.07368	0.07024	0.06668	0.06210	⋯
$\pi_k(\infty)$ (GPSS World, $t = 10^6$)	0.07320	0.07292	0.07043	0.06646	0.06233	⋯

Table 2

Stationary characteristics of the system $M_{\beta}^{\theta}/G/1$ for $h = 3$

Characteristic	$\mathbf{E}Q(\infty)$	$\mathbf{E}w(\infty)$	$\mathbf{P}_{sv}(\infty)$
Analytical value	9.778	9.797	0.166
Value according to GPSS World, $t = 10^5$	9.749	9.768	0.166

Table 3

Stationary values of the mean queue length for the systems $M_{\beta}^{\theta}/G/1$ and $M_{\beta a}^{\theta}/G/1$ for $\beta = 0.1$ and different values of h, computed by means of GPSS World ($t = 10^5$)

h	3	5	7	10	15	20	30	100
$\mathbf{E}Q(\infty)$	9.749	11.839	13.519	16.586	21.587	26.589	36.585	106.584
$\mathbf{E}Q_a(\infty)$	5.971	7.895	9.902	12.912	17.913	22.912	32.911	102.889

Table 4

Stationary values of the mean queue length for the systems $M_{\beta}^{\theta}/G/1$ and $M_{\beta a}^{\theta}/G/1$ for $h = 5$ and different values of β, computed by means of GPSS World ($t = 2\cdot 10^5$)

β	0	0.001	0.01	0.1	0.15	0.155
$\mathbf{E}Q(\infty)$	8.635	8.669	8.752	11.713	26.027	36.755
$\mathbf{E}Q_a(\infty)$	4.796	4.802	4.918	7.860	23.463	33.996

Conclusions

In this paper the formulas, convenient for numerical implementation, for the stationary characteristics of the queuing systems $M^{\theta}/G/1/m$ and $M^{\theta}/G/1$ with the function of a random dropping of customers have been received. The considered examples confirm that the use of the dropping function is a simple and effective way to provide the required values of the queueing system characteristics.

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