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UPPER BOUNDS FOR THE EXTENDED ENERGY OF GRAPHS AND SOME EXTENDED EQUIENERGETIC GRAPHS

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Abstract. In this paper, we give two upper bounds for the extended energy of a graph one in terms of ordinary energy, maximum degree and minimum degree of a graph, and another bound in terms of forgotten index, inverse degree sum, order of a graph and minimum degree of a graph which improves an upper bound of Das *et al.* from [*On spectral radius and energy of extended adjacency matrix of graphs*, Appl. Math. Comput. 296 (2017), 116–123]. We present a pair of extended equienergetic graphs on n vertices for $n \equiv 0 (\mathrm{mod}\ 8)$ starting with a pair of extended equienergetic non regular graphs on 8 vertices and also we construct a pair of extended equienergetic graphs on n vertices for all $n \geq 9$ starting with a pair of equienergetic regular graphs on 9 vertices.

Keywords: energy of a graph, extended energy of a graph, extended equienergetic graphs.

Mathematics Subject Classification: 05C50.

1. INTRODUCTION

All graphs considered in this paper are simple and finite. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Two vertices v_i and v_j in $V(G)$ are said to be adjacent in G if $v_i v_j \in E(G)$. The degree of a vertex v_i in G is the number of vertices that are adjacent with v_i and we denote it by d_i. Also, we denote by Δ and δ, the maximum degree and the minimum degree of G, respectively. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix $[a_{ij}]$, where a_{ij} is 1 if the vertices v_i and v_j are adjacent in G, 0 otherwise. Since $A(G)$ is a real symmetric matrix, all its eigenvalues are real. The spectrum of G is the collection of all eigenvalues of $A(G)$.

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Throughout the paper, we denote the eigenvalues of $A(G)$ by $\lambda_i(G), i = 1, 2, \ldots, n$, where $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. Studies on graph spectrum can be found in [4,5]. The energy $\varepsilon(G)$ of a graph G is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

In 1978, Gutman [10] introduced the concept of graph energy. In recent years, the concept of graph energy has been extensively studied by many researchers. Results on graph energy can be found in a book [12] by Li *et al.* and references cited therein. Two graphs of same order are said to be equienergetic if their energies are same. In [11], Indulal and Vijayakumar have constructed a pair of equienergetic graphs on n vertices for $n = 6, 14, 18$ and for all $n \geq 20$. Later, Jianping Liu, Bolian Liu [14] and Ramane, Walikar [17] have independently proved that there exists a pair of equienergetic graphs on n vertices for all $n \geq 9$. Studies on equienergetic graphs can be found in [1–3, 9, 13, 18, 19] and references therein.

In [20], Yang *et al.* introduced a new matrix called the extended adjacency matrix, denoted by $A_{ex}(G)$ and is defined as the $n \times n$ matrix whose (i, j)-entry is equal to $\frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)$ if $v_i v_j \in E(G)$ and 0 otherwise. Since $A_{ex}(G)$ is real symmetric matrix, all its eigenvalues are real. We denote the eigenvalues of $A_{ex}(G)$ by $\eta_i(G)$ $i = 1, 2, \ldots, n$, where $\eta_1(G) \geq \eta_2(G) \geq \ldots \geq \eta_n(G)$. It can be noted that if G is a regular graph, then $A_{ex}(G) = A(G)$. The extended energy $\varepsilon_{ex}(G)$ of a graph G (cf. [6, 20]) is defined as

$$\varepsilon_{ex}(G) = \sum_{i=1}^{n} |\eta_i(G)|.$$

In analogous to equieneregtic graphs, two graphs are said to be extended equienergetic graphs if their extended graph energies are same. The forgotten topological index $F(G)$ [8] and the inverse degree sum $r(G)$ [16] of a graph G are two degree based topological indices. These are defined as

$$F(G) = \sum_{v_i \in V(G)} d_i^3 \quad \text{and} \quad r(G) = \sum_{v_i \in V(G)} \frac{1}{d_i}.$$

In [6], Das *et al.* presented various upper and lower bounds for $\eta_1(G)$ and $\varepsilon_{ex}(G)$. Motivated by this, in this paper, we give two upper bounds for the extended energy of graphs one in terms of $\varepsilon(G)$, Δ and δ, and another in terms of $F(G)$, $r(G)$, n and δ. We present a pair of extended equienergetic graphs on n vertices for $n \equiv 0(mod\ 8)$ starting with a pair of extended equienergetic non regular graphs on 8 vertices and also we construct a pair of extended equienergetic graphs on n vertices for all $n \geq 9$ starting with a pair of equienergetic regular graphs on 9 vertices.

2. UPPER BOUNDS FOR THE EXTENDED ENERGY

In this section, we give two upper bounds for the extended energy of a graph.

Let M be a $m \times n$ matrix. We denote the singular values of M by $s_i(M)$, $i = 1, 2, \ldots, m$, where $s_1(M) \geq s_2(M) \geq \ldots \geq s_m(M)$. It is worth to note that the sum of all singular values of $A(G)$(respectively, $A_{ex}(G)$) is the energy (respectively, extended energy) of G. We need the following lemmas (see [15]) to prove our main results.

Lemma 2.1. *If A and B are $n \times n$ complex matrices. Then*

$$\sum_{i=1}^{k} s_i(A+B) \leq \sum_{i=1}^{k} s_i(A) + \sum_{i=1}^{k} s_i(B), \quad k = 1, 2, \ldots, n.$$

Lemma 2.2. *If $A_1, A_2, \ldots, A_m$ are $n \times n$ complex matrices. Then*

$$\sum_{i=1}^{k} s_i(A_1 A_2 \cdots A_m) \leq \sum_{i=1}^{k} s_i(A_1) s_i(A_2) \cdots s_i(A_m), \quad k = 1, 2, \ldots, n.$$

In the following theorem, we give an upper for the extended energy of a graph in terms of ordinary energy.

Theorem 2.3. *Let G be a graph of order n. Then $\varepsilon_{ex}(G) \leq \frac{\Delta}{\delta}\varepsilon(G)$.*

Proof. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n$. From the definition of extended adjacency matrix of a graph, it is easy to see that

$$A_{ex}(G) = \frac{B + B^T}{2}, \quad B := D^{-1}(G)A(G)D(G). \tag{2.1}$$

Applying Lemmas 2.1 and 2.2 in (2.1), we obtain

$$\begin{aligned}
\varepsilon_{ex}(G) &\leq \sum_{i=1}^{n} s_i(B) \\
&\leq \sum_{i=1}^{n} s_i(D^{-1}(G) s_i(A(G)) s_i(D(G))) \\
&= \sum_{i=1}^{n} \frac{d_i}{d_{n-i+1}} s_i(A(G)).
\end{aligned}$$

Since $\frac{d_i}{d_{n-i+1}} \leq \frac{\Delta}{\delta}$, from the above inequality, it follows that

$$\varepsilon_{ex}(G) \leq \frac{\Delta}{\delta} \sum_{i=1}^{n} s_i(A(G)) = \frac{\Delta}{\delta}\varepsilon(G). \qquad \square$$

The following theorem gives an upper bound for $\varepsilon_{ex}(G)$ in terms of $F(G)$, $r(G)$, δ and n.

Theorem 2.4. *Let G be a graph with n vertices and m edges. We assume that G has no isolated vertices. Then*

$$\varepsilon_{ex}(G) \leq \sqrt{\frac{n}{2}\left(\frac{F(G)}{\delta^2} + \delta^2 r(G)\right)} \tag{2.2}$$

with equality holding if and only if $G \cong \frac{n}{2}K_2$.

Proof. We have

$$\begin{aligned}
2\sum_{i=1}^{n} \eta_i(G)^2 &= \sum_{v_iv_j \in E(G)} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 \\
&= 2m + \sum_{v_iv_j \in E(G)} \left(\frac{d_i^2}{d_j^2} + \frac{d_j^2}{d_i^2}\right) \\
&= 2m + \sum_{v_i \in V(G)} \frac{1}{d_i^2} \left(\sum_{v_iv_j \in E(G)} d_j^2\right) \\
&= 2m + \sum_{v_i \in V(G)} \left(\left(\frac{1}{d_i^2} \sum_{v_iv_j \in E(G)} \left(d_j^2 - \delta^2\right)\right) + \frac{\delta^2}{d_i}\right) \\
&\leq \frac{1}{\delta^2} \sum_{v_i \in V(G)} \sum_{v_iv_j \in E(G)} d_j^2 + \sum_{v_i \in V(G)} \frac{\delta^2}{d_i} \\
&= \frac{F(G)}{\delta^2} + \delta^2 r(G).
\end{aligned}$$

Thus

$$2\sum_{i=1}^{n} \eta_i(G)^2 \leq \frac{F(G)}{\delta^2} + \delta^2 r(G). \tag{2.3}$$

Now from Cauchy-Schwarz inequality and (2.3), we have

$$\varepsilon_{ex}(G) = \sum_{i=1}^{n} |\eta_i(G)| \leq \sqrt{n \sum_{i=1}^{n} \eta_i(G)^2} \leq \sqrt{\frac{n}{2}\left(\frac{F(G)}{\delta^2} + \delta^2 r(G)\right)}.$$

Moreover, the equality holds if and only if $|\eta_1(G)| = |\eta_2(G)| = \ldots = |\eta_n(G)|$ and G is a regular graph. Let H be a regular connected component of G and $\eta_{i1}, \eta_{i2}, \ldots, \eta_{ik}$ be the extended eigenvalues of H arranged in decreasing order such that $|\eta_{i1}| = |\eta_{i2}| = \ldots = |\eta_{ik}|$. Then from Perron–Frobenius theory η_{i1} is simple and as $\sum_{j=1}^{k} \eta_{ij} = 0$, it follows that $k = 2$, i.e., $H = K_2$. This completes the proof. □

Remark 2.5. Das *et al.* [6] gave the following upper bound

$$\varepsilon_{ex}(G) \leq \sqrt{\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right)} \sqrt{\frac{nF(G)}{2\delta^2}}. \tag{2.4}$$

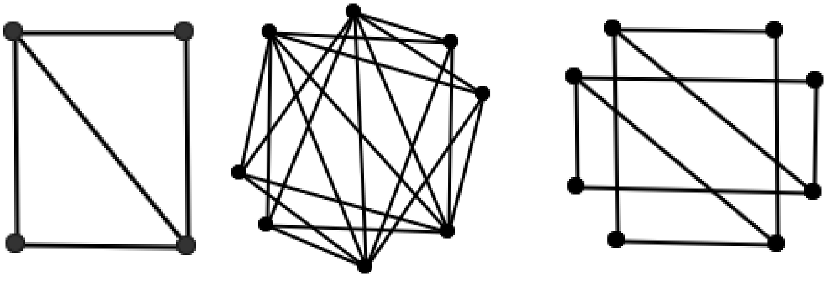
Since $\left(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\right) \geq 2$ and $\frac{F(G)}{\delta^2} \geq 2m \geq \delta^2 \sum_{i=1}^{n} \frac{1}{d_i}$, it follows that our upper bound in (2.2) is sharper than (2.4).

3. SOME FAMILIES OF EXTENDED EQUIENERGETIC GRAPHS

In this section, we describe some methods to construct extended equienergetic graphs on n vertices. We start with the following definitions (cf. [11]).

Definition 3.1. The duplication of a graph G, denoted by $Du(G)$, is the graph obtained by taking two copies of the vertex set $V(G)$ of G and then joining a vertex in the first copy of $V(G)$ to a vertex in the second copy of $V(G)$ whenever they are adjacent in G. See Figure 1.

Definition 3.2. The double graph $Du^*(G)$ is the graph obtained by taking two copies of G and then joining a vertex in the first copy of G to a vertex in the second copy of G whenever they are adjacent in G. See Figure 1.



G $\quad$ $Du^*(G)$ $\quad$ $Du(G)$

Fig. 1. Graphs $Du(G)$ and $Du^*(G)$

Let $M = [m_{ij}]$ and N be two matrices. The Kronecker product $M \otimes N$ of M and N is the matrix obtained by replacing each entry m_{ij} of M by $m_{ij}N$. If M and N are square matrices, then it is well-known that $\lambda\mu$ is an eigenvalue of $M \otimes N$ whenever λ and μ are the eigenvalues of M and N, respectively. In the following theorem, we give a method to construct a pair of extended equieneregtic graphs.

Theorem 3.3. *Let G be a graph on n vertices. Then the graphs $Du(G)$ and $Du^*(G)$ are extended equienergetic graphs.*

Proof. From the definitions of $Du(G)$ and $Du^*(G)$, and also by proper labelling of the vertices of $Du(G)$ and $Du^*(G)$, it can be easily seen that

$$A_{ex}(Du(G)) = \begin{bmatrix} 0 & A_{ex}(G) \\ A_{ex}(G) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A_{ex}(G)$$

and

$$A_{ex}(Du^*(G)) = \begin{bmatrix} A_{ex}(G) & A_{ex}(G) \\ A_{ex}(G) & A_{ex}(G) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A_{ex}(G).$$

Thus the spectrum of $A_{ex}(Du(G))$ and $A_{ex}(Du^*(G))$ are $\{\pm\eta_1(G), \pm\eta_2(G), \ldots, \pm\eta_n(G)\}$ and $\{2\eta_1(G), 2\eta_2(G), \ldots, 2\eta_n, 0, 0, \ldots, 0\}$, respectively. So $\varepsilon_{ex}(Du(G)) = \varepsilon_{ex}(Du^*(G))$. □

Let G and H be graphs with vertex sets $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. The Kronecker product of G and H, denoted by $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G \otimes H$ if and only if u_i and u_k are adjacent in G and v_j and v_l are adjacent in H. In the following theorem, we construct some extended equienergetic graphs starting with a pair of extended equienergetic non regular graphs on 8 vertices.

Theorem 3.4. *There exists a pair of extended equienergetic graphs on n vertices for all $n \equiv 0(mod\ 8)$.*

Proof. Observe that, if G is a regular graph on n vertices and H is an arbitrary graph on m vertices, then the extended adjacency matrix of $G \otimes H$, i.e., $A_{ex}(G \otimes H) = A(G) \otimes A_{ex}(H)$. Hence the spectrum of $A_{ex}(G \otimes H)$ consists of $\lambda_i(G)\eta_j(H)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Moreover, $\varepsilon_{ex}(G \otimes H) = \varepsilon(G)\varepsilon_{ex}(H)$. Thus, if H_1 and H_2 are extended equieneregtic graphs and G any regular graph, then $G \otimes H_1$ and $G \otimes H_2$ are extended equieneregtic graphs. Now from Theorem 3.3, it follows that the graphs $Du(G)$ and $Du^*(G)$ for G as depicted in Fig. 1 are extended equienergetic graphs on 8 vertices. So the graphs $K_m \otimes Du(G)$ and $K_m \otimes Du^*(G)$ are extended equienergetic graphs on $8m$ vertices for all $m > 1$. □

We denote by $J_{n_1 \times n_2}$ and $J'_{n_1 \times n_2}$, the $n_1 \times n_2$ matrix having all its entries as 1 and the matrix obtained from $J_{n_1 \times n_2}$ by replacing each entry by 0 except the first diagonal entry, respectively.

Lemma 3.5. *For $i = 1, 2$, let M_i be a normal matrix of order n_i having all its row sums equal to r_i. Suppose r_i, $\theta_{i2}, \theta_{i3}, \ldots, \theta_{in_i}$ are the eigenvalues of M_i, then for any two constants a and b, the eigenvalues of*

$$M := \begin{bmatrix} M_1 & aJ_{n_1 \times n_2} \\ bJ_{n_2 \times n_1} & M_2 \end{bmatrix},$$

are θ_{ij} for $i = 1, 2$, $j = 2, 3, \ldots, n_i$ and the two roots of the quadratic equation $(x - r_1)(x - r_2) - abn_1n_2 = 0$.

Proof. Since M_i is a normal matrix having all its row sums equal to r_i, we have $M_i = U_i D_i U_i^*$, where U_i is a unitary matirx having its first column vector as $(1,1,\ldots,1)^T/\sqrt{n_i}$ and D_i is a diagonal matrix with $r_i, \theta_{i2}, \theta_{i3}, \ldots, \theta_{in_i}$ as its diagonal entries. So

$$\begin{aligned} M &= \begin{bmatrix} U_1 D_1 U_1^* & aJ_{n_1\times n_2} \\ & \\ bJ_{n_2\times n_1} & U_2 D_2 U_2^* \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & U_1^* aJ_{n_1\times n_2} U_2 \\ & \\ U_2^* bJ_{n_2\times n_1} U_1 & D_2 \end{bmatrix} \begin{bmatrix} U_1^* & 0 \\ 0 & U_2^* \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & a\sqrt{n_1 n_2} J'_{n_1\times n_2} \\ & \\ b\sqrt{n_1 n_2} J'_{n_2\times n_1} & D_2 \end{bmatrix} \begin{bmatrix} U_1^* & 0 \\ 0 & U_2^* \end{bmatrix}. \end{aligned}$$

Thus M and

$$B := \begin{bmatrix} D_1 & a\sqrt{n_1 n_2} J'_{n_1\times n_2} \\ b\sqrt{n_1 n_2} J'_{n_2\times n_1} & D_2 \end{bmatrix}$$

are similar matrices, and hence have the same spectrum. Expanding $|xI - B|$ by Laplace's method [7] along i-th column $i = 2, 3, \ldots, n_1, n_1 + 2, \ldots, n_2$, we see that

$$|xI - B| = ((x - r_1)(x - r_2) - abn_1 n_2) \prod_{\substack{j=2 \\ i=1,2}}^{n_i} (x - \theta_{ij}).$$

This completes the proof. $\square$

Definition 3.6 ([17])**.** Let G and H be two graphs. The join $G \vee H$ of G and H is a graph obtained from G and H by joining each vertex of G to every vertex in H.

In the following theorem, we give the extended spectrum of $G \vee H$ when both G and H are regular graphs.

Theorem 3.7. *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices. Then the extended spectrum of $G_1 \vee G_2$ consists of $\lambda_j(G_i)$ for $i = 1, 2$, $j = 2, 3, \ldots, n_i$ and the two roots of the quadratic equation*

$$(x - r_1)(x - r_2) - \frac{n_1 n_2}{4} \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2.$$

Proof. Since G_i is a r_i regular graph on n_i vertices, we have

$$A_{ex}(G_1 \vee G_2) = \begin{pmatrix} A(G_1) & \frac{1}{2}\left(\frac{r_1+n_2}{r_2+n_1} + \frac{r_2+n_1}{r_1+n_2}\right) J_{n_1\times n_2} \\ \frac{1}{2}\left(\frac{r_1+n_2}{r_2+n_1} + \frac{r_2+n_1}{r_1+n_2}\right) J_{n_2\times n_1} & A(G_2) \end{pmatrix}.$$

Letting $a = b = \frac{1}{2}\left(\frac{r_1+n_2}{r_2+n_1} + \frac{r_2+n_1}{r_1+n_2}\right)$ in Lemma 3.5 we arrive at the result. $\square$

Theorem 3.8. *There exists a pair of extended equienergetic graphs on n vertices for all $n \geq 9$.*

Proof. Let H_1 and H_2 be graphs as shown in Figure 2.

Fig. 2. Graphs H_1 and H_2

It can be seen that the line graphs $L(H_1)$ and $L(H_2)$ are equienergetic 4-regular graphs [17] on 9 vertices.

Thus from the above theorem, the graphs $K_m \vee L(H_1)$ and $K_m \vee L(H_2)$ are extended equienergetic graphs on $9 + m$ vertices for $m = 1, 2, \ldots$. □

REFERENCES

[1] C. Adiga, B.R. Rakshith, *On spectra of variants of the corona of two graphs and some new equienergetic graphs*, Discuss. Math. Graph Theory **36** (2016), 127–140.

[2] R. Balakrishnan, *The energy of a graph*, Linear Algebra Appl. **387** (2004), 287–295.

[3] V. Brankov, D. Stevanović, I. Gutman, *Equienergetic chemical trees*, J. Serb. Chem. Soc. **69** (2004), 549–553.

[4] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, Berlin, 2012.

[5] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Application*, Academic Press, New York, 1980.

[6] K.Ch. Das, I. Gutman, B. Furtula, *On spectral radius and energy of extended adjacency matrix of graphs*, Appl. Math. Comput. **296** (2017), 116–123.

[7] W.L. Ferrar, *A Text-Book of Determinants, Matrices and Algebraic Forms*, Oxford University Press, 1953.

[8] B. Furtula, I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015) 4, 1184–1190.

[9] S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, *Borderenergetic graphs*, MATCH Commun. Math. Comput. Chem. **74** (2015), 321–332.

[10] I. Gutman, *The energy of a graph*, Ber. Math. Statist. sekt. Forschungsz. Graz. **103** (1978), 1–22.

[11] G. Indulal, A. Vijayakumar, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. **55** (2006), 83–90.

[12] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[13] X. Li, M. Wei, S. Gong, *A computer search for the borderenergetic graphs of order 10*, MATCH Commun. Math. Comput. Chem. **74** (2015), 333–342.

[14] J. Liu, B. Liu, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. **59** (2008), 275–278.

[15] A.W. Marshall, I. Olkin, B.C. Arnold, *Inequalities: Theory of Majorization and its Applications*, Springer, New York, 2011.

[16] S. Mukwembi, *On diameter and inverse degree of a graph*, Discrete Math. **310** (2010), 940–946.

[17] H.S. Ramane, H.B. Walikar, *Construction of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. **57** (2007), 203–210.

[18] H.S. Ramane, I. Gutman, H.B. Walikar, S.B. Halkarni, *Equienergetic complement graphs*, Kragujevac J. Sci. **27** (2005), 67–74.

[19] L. Xu, Y. Hou, *Equienergetic bipartite graphs*, MATCH Commun. Math. Comput. Chem. **57** (2007), 363–370.

[20] Y.Q. Yang, L. Xu, C.Y. Hu, *Extended adjacency matrix indices and their applications*, J. Chem. Inf. Comput. Sci. **34** (1994), 1140–1145.

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