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FORCED OSCILLATION AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. The paper deals with the second order nonhomogeneous linear differential equation

$$(p(t)y'(t))' + q(t)y(t) = f(t),$$

which is oscillatory under the assumption that $p(t)$ and $q(t)$ are positive, continuously differentiable and monotone functions on $[0, \infty)$. Throughout this paper we shall use pairs of quadratic forms, which obtained by different methods than Kusano and Yoshida. This form will lead to a property of qualitative behavior, including amplitudes and slopes, of oscillatory solutions of the above equation. In addition, we will discuss the existence of three types (moderately bounded, small, large) of oscillatory solutions, which are based on results due to Kusano and Yoshida.

Keywords: forced oscillation, asymptotic behavior, second order, differential equation.

Mathematics Subject Classification: 34C10, 34C11.

1. INTRODUCTION

The theory of second order differential equations about the methods used in their solutions and their wide applications have a long history, and there is many existing literature on the subject [2–4, 7, 11]. As far as oscillation theory is concerned, most articles in second order differential equations deal with whether the equations oscillate. However, to the best of our knowledge, only several researchers have attempt to establish the relations about properties of oscillatory solutions between homogeneous equations and nonhomogeneous equations. Let us consider the second order homogeneous equation

$$(p(t)y'(t))' + q(t)y(t) = 0, \tag{1.1}$$

and the nonhomogeneous equation

$$(p(t)y'(t))' + q(t)y(t) = f(t), \tag{1.2}$$

where $p(t), q(t) \in C^1([0,\infty),(0,\infty))$ and $f(t) \in C([0,\infty),(-\infty,\infty))$. Recently, Kusano and Yoshida [6] have acquired as much and detailed information as possible about the existence and the qualitative properties of oscillatory solutions of homogeneous equation (1.1) which was based on a work due to Hille [5] and Hartman [3]. So, our primary interest is to prove the existence and the qualitative properties of oscillatory solutions of nonhomogeneous equation. Therefore, while using the results of [6], we will extend their results to nonhomogeneous equation (1.2). By a solution of equation (1.1) or (1.2), we mean a function $y(t) \in C^2([T_y,\infty),(-\infty,\infty))$, $T_y \ge t_0$, which satisfies equation (1.1) or (1.2) on $[T_y,\infty)$. We restrict our attention to the nontrivial solution $y(t)$ of equations (1.1) or (1.2) only, i.e., to solutions satisfying $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. A nontrivial solution (1.1) or (1.2) is oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory.

Thus, let $y(t)$ be an oscillatory solution on $[0,\infty)$ of (1.1) or (1.2), and we adopt definition as follows.

Definition 1.1. The sequences $\{\sigma_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ are said to be distribution of zeros points and extrema points of oscillatory solution, if there exist $\sigma_k < \sigma_{k+1}$ and $\tau_k < \tau_{k+1}$ $(k = 1, 2, \ldots)$ such that $y(\sigma_k) = 0$ and $y'(\tau_k) = 0$ for any k.

Definition 1.2. The value $|y(\tau_k)|$ and $|y'(\sigma_k)|$ are referenced to as the *amplitude* $\mathcal{A}[y]$ and the *slope* $\mathcal{S}[y]$, respectively, of the k-th wave of $y(t)$.

In this paper we use the following notation:

$$\mathcal{A}^*[y] = \sup_k |y(\tau_k)|, \quad \mathcal{A}_*[y] = \inf_k |y(\tau_k)|,$$
$$\mathcal{S}^*[y] = \sup_k |y'(\sigma_k)|, \quad \mathcal{S}_*[y] = \inf_k |y'(\sigma_k)|.$$

Definition 1.3.

(i) A solution $y(t)$ of equations is said to be *large oscillatory*, if $y(t)$ satisfies $\mathcal{A}^*[y] = \infty$, i.e. $\limsup_{t\to\infty} |y(t)| = \infty$.

(ii) A bounded solution $y(t)$ of equations is said to be *small oscillatory*, if $y(t)$ satisfies $\lim_{k\to\infty} |y(\tau_k)| = 0$, i.e. $\lim_{t\to\infty} |y(t)| = 0$.

(iii) A bounded solution $y(t)$ of equations is said to be *moderately bounded oscillatory*, if $y(t)$ satisfies $\mathcal{A}_*[y] > 0$, i.e. $\liminf_{k\to\infty} |y(\tau_k)| > 0$.

The starting results for qualitative properties, including amplitude and slope, of oscillatory solutions are Hille [5] and Hartman [3], to the best of the author's knowledge, who utilized a pair of quadratic forms. Inspired by these results, Kusano and Yoshida showed that for more general equation (1.1), there always exists a pair of quadratic forms of the type

$$\mathcal{V}[y](t) = P(t)y'(t)^2 + Q(t)y(t)^2, \quad \mathcal{W}[y](t) = R(t)y'(t)^2 + S(t)y(t)^2,$$

so that established existence of three types of solutions of (1.1) referred to as moderately bound, small and large oscillatory solutions. We concern exclusively with the case where equation (1.1) is oscillatory also in this paper. It is known [7] that (1.1) is oscillatory if

$$\int_0^\infty \frac{dt}{p(t)} = \infty \quad \text{and} \quad \int_0^\infty P(t)^\lambda q(t)dt = \infty \quad \text{for some } \lambda \in [0,1), \tag{1.3}$$

where $P(t) = \int_0^t ds/p(s)$, or if

$$\int_0^\infty \frac{dt}{p(t)} < \infty \quad \text{and} \quad \int_0^\infty \pi(t)^\mu q(t)dt = \infty \quad \text{for some } \mu \in (1,2], \tag{1.4}$$

where $\pi(t) = \int_t^\infty ds/p(s)$.

2. PRELIMINARIES

In this section we introduce a pair of positive quadratic forms $\{\mathcal{V}[y], \mathcal{W}[y]\}$ and $\{\mathcal{F}[y], \mathcal{G}[y]\}$ which are an important tool in acquiring as much and detailed information as possible about the existence and qualitative properties of oscillatory solutions of equations (1.1) and (1.2). At first, we apply different method from the results [6] to derive the same following lemma as Kusano and Yoshida. The difference with the result of [6] is that it is derived directly from the equation (1.1).

Lemma 2.1. *Let* (1.1) *be oscillatory and* $y_h(t)$ *be a solution of* (1.1) *on* $[0,\infty)$. *Suppose one of the following statements holds*:

(i) *If* $p'(t) \geq 0$ *and* $q'(t) \leq 0$, *and set*

$$\mathcal{V}[y](t) = \frac{p(t)^2}{q(t)} y_h'(t)^2 + p(t)y_h(t)^2, \quad \mathcal{W}[y](t) = p(t)y_h'(t)^2 + q(t)y_h(t)^2.$$

(ii) *If* $p'(t) \leq 0$ *and* $q'(t) \geq 0$, *and set*

$$\mathcal{V}[y](t) = p(t)y_h'(t)^2 + q(t)y_h(t)^2, \quad \mathcal{W}[y](t) = \frac{p(t)^2}{q(t)} y_h'(t)^2 + p(t)y_h(t)^2.$$

(iii) *If* $(p(t)q(t))' \geq 0$, *and set*

$$\mathcal{V}[y](t) = p(t)^2 y_h'(t)^2 + p(t)q(t)y_h(t)^2, \quad \mathcal{W}[y](t) = \frac{p(t)}{q(t)} y_h'(t)^2 + y_h(t)^2.$$

(iv) *If* $(p(t)q(t))' \leq 0$, *and set*

$$\mathcal{V}[y](t) = \frac{p(t)}{q(t)} y_h'(t)^2 + y_h(t)^2, \quad \mathcal{W}[y](t) = p(t)^2 y_h'(t)^2 + p(t)q(t)y_h(t)^2.$$

Then $(\mathcal{V}[y](t))' \geq 0$ *and* $(\mathcal{W}[y](t))' \leq 0$.

Proof. Let $y_h(t)$ be any solution of (1.1) on $[0,\infty)$. Multiplying (1.1) by $p(t)y_h'(t)$ and dividing by $q(t)$ yield

$$\left[\frac{p(t)^2}{q(t)}y_h'(t)^2 + p(t)y_h(t)^2\right]' = p'(t)y_h(t)^2 - \frac{q'(t)}{q(t)^2}\Big(p(t)y_h'(t)\Big)^2 \geq 0. \tag{2.1}$$

On the other hand, (1.1) is equivalent to the following equation

$$p(t)y_h''(t) + p'(t)y_h'(t) + q(t)y_h(t) = 0. \tag{2.2}$$

Multiplying (2.2) by $y_h'(t)$, we obtain the equation

$$\Big[p(t)y_h'(t)^2 + q(t)y_h(t)^2\Big]' = -p'(t)y_h'(t)^2 + q'(t)y_h(t)^2 \leq 0. \tag{2.3}$$

Hence, we can prove that (i) hold. From the above inequalities, if $p'(t) \leq 0$ and $q'(t) \geq 0$, then we show that (ii) hold. Clearly it is easily checked from (2.1) and (2.3) that (i) and (ii) hold. Multiplying (1.1) by $p(t)y_h'(t)$ we have

$$\Big[p(t)^2 y_h'(t)^2 + p(t)q(t)y_h(t)^2\Big]' = (p(t)q(t))'y_h(t)^2. \tag{2.4}$$

Moreover we show by multiplying (2.2) by $y_h'(t)$ and dividing by $q(t)$ that

$$\left[\frac{p(t)}{q(t)}y_h'(t)^2 + y_h(t)^2\right]' = -\frac{(p(t)q(t))'}{q(t)^2}y_h'(t)^2. \tag{2.5}$$

Similar to the proof of the cases (i) and (ii), it is obvious from these relations (2.4) and (2.5) that (iii) and (iv) hold. □

Since the equation (1.2) is linear, then we know that

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t) = y_h(t) + \int_0^t G(t,\tau)f(\tau)d\tau$$

is a solution of the nonhomogeneous equation, where $y_p(t)$ is a particular solution and

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{p(\tau)W(\tau)}$$

is the Green function, $W(\tau) = y_1y_2' - y_1'y_2$ is the Wronskian and y_1, y_2, y_h are solutions of the homogeneous equation. Also, we easy to see that particular solution satisfies the initial conditions $y_p(0) = y_p'(0) = 0$.

Lemma 2.2. *Let* (1.2) *be oscillatory and* $y(t)$ *be a solution of* (1.2) *on* $[0,\infty)$. *Suppose one of the following statements holds*:

(i) *If* $p'(t) \geq 0$ *and* $q'(t) \leq 0$, *and set*

$$\begin{aligned}\mathcal{F}[y](t) &= \frac{p(t)^2}{q(t)}\Big(y'(t)+y_p'(t)\Big)^2 + p(t)\Big(y(t)+y_p(t)\Big)^2,\\ \mathcal{G}[y](t) &= p(t)\Big(y'(t)+y_p'(t)\Big)^2 + q(t)\Big(y(t)+y_p(t)\Big)^2.\end{aligned}$$

(ii) *If* $p'(t) \leq 0$ *and* $q'(t) \geq 0$, *and set*

$$\begin{aligned}\mathcal{F}[y](t) &= p(t)\Big(y'(t)+y_p'(t)\Big)^2 + q(t)\Big(y(t)+y_p(t)\Big)^2,\\ \mathcal{G}[y](t) &= \frac{p(t)^2}{q(t)}\Big(y'(t)+y_p'(t)\Big)^2 + p(t)\Big(y(t)+y_p(t)\Big)^2.\end{aligned}$$

(iii) *If* $(p(t)q(t))' \geq 0$, *and set*

$$\begin{aligned}\mathcal{F}[y](t) &= p(t)^2\Big(y'(t)+y_p'(t)\Big)^2 + p(t)q(t)\Big(y(t)+y_p(t)\Big)^2,\\ \mathcal{G}[y](t) &= \frac{p(t)}{q(t)}\Big(y'(t)+y_p'(t)\Big)^2 + \Big(y(t)+y_p(t)\Big)^2.\end{aligned}$$

(iv) *If* $(p(t)q(t))' \leq 0$, *and set*

$$\begin{aligned}\mathcal{F}[y](t) &= \frac{p(t)}{q(t)}\Big(y'(t)+y_p'(t)\Big)^2 + \Big(y(t)+y_p(t)\Big)^2,\\ \mathcal{G}[y](t) &= p(t)^2\Big(y'(t)+y_p'(t)\Big)^2 + p(t)q(t)\Big(y(t)+y_p(t)\Big)^2.\end{aligned}$$

Then $(\mathcal{F}[y](t))' \geq 0$ *and* $(\mathcal{G}[y](t))' \leq 0$.

Proof. Let $y(t)$ be any solution of (1.2) on $[0,\infty)$. Case (i). Rewriting (1.2) as

$$\Big(p(t)y'(t)+p(t)y_p'(t)\Big)' + q(t)\Big(y(t)+y_p(t)\Big) = 0. \tag{2.6}$$

Multiplying (2.6) by $(p(t)y'(t)+p(t)y_p'(t))$ and dividing by $q(t)$ yield

$$\begin{aligned}&\left[\frac{p(t)^2}{q(t)}\Big(y'(t)+y_p'(t)\Big)^2 + p(t)\Big(y(t)+y_p(t)\Big)^2\right]'\\ &= p'(t)\Big(y(t)+y_p(t)\Big)^2 - \frac{q'(t)p(t)^2}{q(t)^2}\Big(y'(t)+y_p'(t)\Big)^2 \geq 0.\end{aligned}$$

Furthermore, (1.2) shows that

$$p(t)y''(t)+p'(t)y'(t)+q(t)y(t) = (p(t)y_p'(t))' + q(t)y_p(t),$$

and so

$$p(t)\Big(y''(t)+y_p''(t)\Big)+p'(t)\Big(y'(t)+y_p'(t)\Big)+q(t)\Big(y(t)+y_p(t)\Big)=0. \tag{2.7}$$

Multiplying (2.7) by $(y'(t)+y_p'(t))$, we obtain

$$\begin{aligned}&\left[p(t)\Big(y'(t)+y_p'(t)\Big)^2+q(t)\Big(y(t)+y_p(t)\Big)^2\right]'\\&=-p'(t)\Big(y'(t)+y_p'(t)\Big)^2+q'(t)\Big(y(t)+y_p(t)\Big)^2\le 0.\end{aligned}$$

Obviously, (i) hold. The proof for (ii) is similar. In order to establish the cases (iii) and (iv), multiplying (2.6) by $p(t)(y'(t)+y_p'(t))$ becomes

$$\left[p(t)^2\Big(y'(t)+y_p'(t)\Big)^2+p(t)q(t)\Big(y(t)+y_p(t)\Big)^2\right]'=(p(t)q(t))'\Big(y(t)+y_p(t)\Big)^2.$$

Multiplying (2.7) by $(y'(t)+y_p'(t))$ yields

$$\left[\frac{q(t)}{p(t)}\Big(y'(t)+y_p'(t)\Big)^2+\Big(y(t)+y_p(t)\Big)^2\right]'=-\frac{q(t)^2}{(p(t)q(t))'}\Big(y(t)+y_p'(t)\Big)^2.$$

If the condition $(p(t)q(t))'\ge 0$, then the case (iii) hold, and if the condition $(p(t)q(t))'\le 0$, then the case (iv) hold. The proof is complete. □

The problem of establishing oscillation criteria for second order nonhomogeneous linear equations has been investigated by several authors [1, 8–10, 12–14]. They studied how oscillation of the solutions of homogeneous equation (1.2) can be influenced by the forcing term $f(t)$. From one important approach pursued by Tefteller [12], if $\{y_1(t), y_2(t)\}$ is normalized, then the general solution of (1.2) can be expressed as

$$y(t)=\left[c_1-\int_0^t y_2(\tau)f(\tau)d\tau\right]y_1(t)+\left[c_2+\int_0^t y_1(\tau)f(\tau)d\tau\right]y_2(t).$$

Making use of this solution, he led to the following results.

Lemma 2.3 ([12]). *Suppose $y(t)$ is a solution of (1.2) and $y_h(t)$ is any nontrivial solution of (1.1). Then*

$$W(y_h,y)=k+\int_{t_0}^t f(\tau)y_h(\tau)d\tau$$

for some constant k.

Theorem 2.4 ([12]). *Suppose $f(t)\neq 0$ on $[0,\infty)$, $y(t)$ is a solution of (1.2), and $y_h(t)$ is a nontrivial solution of (1.1). Then $W(y_h,y)\neq 0$ on $[0,\infty)$ if and only if $y(t)$ has only simple zeros and the zeros of $y(t)$ and $y_h(t)$ separate on $[0,\infty)$.*

Theorem 2.5 ([12]). *Let $y_1(t)$ solve* (1.1) *and suppose that for a given k and $f(t) \neq 0$, the function $k + \int_0^t f(\tau)y_1(\tau)d\tau$ is nonoscillatory. Then $W(y_h, y) \neq 0$, and so,* (1.2) *is oscillatory if and only if* (1.1) *is oscillatory.*

Corollary 2.6 ([12]). *Suppose the forcing function $f(t)$ is a solution of* (1.2)*. Then $W(y_h, y) \neq 0$, and so,* (1.2) *is oscillatory if and only if* (1.1) *is oscillatory.*

Among these literature, Skidmore and Leighton [10] and Abramovich [1] proved theorem concerning the oscillatory phenomena and qualitative behaviors of solutions of (1.2) with $p(t) = 1$. Thus, we shall revise and extend this results, and state the following useful results.

Theorem 2.7. *Assume that* (1.2) *is oscillatory and $f(t)$ is nonnegative. If $(p(t)q(t))' \geq 0$ and $(p(t)f(t))' \geq 0$, then each peak of oscillatory solution of* (1.2) *is at least as high as the next and its amplitude is at least as great as that of the next pit.*

Proof. Let $t_0 < t_1 < t_2 < t_3 < t_4$ be consecutive zeros of oscillatory solution $y(t)$ of (1.2) with $y(t) > 0$ on (t_0, t_1), $y(t) < 0$ on (t_1, t_2), $y(t) > 0$ on (t_2, t_3), $y(t) < 0$ on (t_3, t_4). Let $t = b, c, d, e$ be, respectively, the points on these intervals at which $y(t)$ attains its maximum, minimum, maximum, minimum values. For this, let us denote

$$-y(c) \geq y(d) \quad \text{and} \quad -y(c) \geq -y(e). \tag{2.8}$$

Then we assume the contrary, and let $t = c_1$ be the minimum point on (t_1, t_2), and

$$-y(c_1) < y(d).$$

By changing the variables $u(t) = -y(t)$, we obtain

$$u(c_1) < -u(d)$$

and

$$(p(t)u'(t))' + q(t)u(t) = -f(t). \tag{2.9}$$

Multiplying (2.9) by $2p(t)u'(t)$ and integrating over $[c_1, d]$, we have

$$\Big[(p(t)u'(t))^2 + p(t)q(t)u(t)^2\Big]_{c_1}^{d} = \int_{c_1}^{d} \Big\{(p(t)q(t))'u(t)^2 - 2(p(t)f(t))u'(t)\Big\}dt. \tag{2.10}$$

Since $u'(t) \geq 0$ on the interval (c_1, d), this equation becomes

$$\begin{aligned} p(d)q(d)u(d)^2 - p(c_1)q(c_1)u(c_1)^2 &\leq \int_{c_1}^{d} (p(t)q(t))'u(t)^2 dt \\ &\leq u(d)^2\Big\{p(d)q(d) - p(c_1)q(c_1)\Big\}. \end{aligned}$$

Then it follows that

$$p(c_1)q(c_1)\{u(d)^2 - u(c_1)^2\} \leq 0,$$

which provides a contradiction. Accordingly, we see that $-y(c_1) \geq y(d)$.

If $y(c_1)$ is the absolute minimum $y_{\min}$ on (t_1, t_2), then $-y(c) \geq y(d)$. If it is not, then $-y_{\min} > -y(c_1) \leq y(d)$.

Next we assume the contrary that $-y(c) < -y(e)$, and so, $-u(d) < u(c) < u(e)$. We apply (2.10) on the interval (c, e), then

$$\begin{aligned}\Big[(p(t)u'(t))^2 + p(t)q(t)u(t)^2\Big]_c^e &= \int_c^e \Big\{(p(t)q(t))'u(t)^2 - 2(p(t)f(t))u'(t)\Big\}dt \\ &= \Big[-2p(t)f(t)u(t)\Big]_c^e \\ &\quad + \int_c^e \Big\{(p(t)q(t))'u(t)^2 + 2(p(t)f(t))'u(t)\Big\}dt.\end{aligned}$$

It is obvious that

$$\begin{aligned}&p(e)q(e)u(e)^2 - p(c)q(c)u(c)^2 \\ &\leq \Big\{-2p(e)f(e)u(e) + 2p(c)f(c)u(c)\Big\} + u(e)^2\Big\{p(e)q(e) - p(c)q(c)\Big\} \\ &\quad + u(e)\Big\{2p(e)f(e) - 2p(c)f(c)\Big\}.\end{aligned}$$

Thus, we see that

$$p(c)q(c)\Big\{u(e)^2 - u(c)^2\Big\} \leq -2p(c)f(c)\Big\{u(e) - u(c)\Big\},$$

which implies to

$$q(c)\Big\{u(e) + u(c)\Big\} \leq -2f(c).$$

From $u(c) < u(e)$ it follows that

$$-f(c) - q(c)u(c) > 0.$$

However, $t = c$ is relative maximum point of $u(t)$, and

$$u''(t) = \frac{1}{p(c)}\Big\{-f(c) - q(c)u(c)\Big\} \leq 0.$$

From this contradiction we can lead to the inequalities (2.8), and the conclusion of the theorem follows. □

Analogously, we can prove the following corollary.

Corollary 2.8. *Assume that* (1.2) *is oscillatory and* $f(t)$ *is nonnegative. If* $(p(t)q(t))' \geq 0$ *and* $(p(t)f(t))' \leq 0$, *then each peak of oscillatory solution of* (1.2) *is at least as high as the next and its amplitude is at least as great as that of the next pit.*

Proof. The same as in Theorem 2.3, it is sufficient to show that

$$y(b) \geq -y(c) \quad \text{and} \quad y(d) \geq y(d). \tag{2.11}$$

Then we suppose to the contrary that there exists $t = b_1$ such that

$$y(b_1) < -y(c).$$

Multiplying (1.2) by $2p(t)y'(t)$ and integrating over (b_1, c), we obtain

$$p(b_1)q(b_1)\Big\{y(c)^2 - y(b_1)^2\Big\} \leq 0,$$

which provides a contradiction. Next we turn our attention to the second inequality (2.11) and suppose that $y(b) < y(d)$. It is obvious that $-y(c) \geq y(b) < y(d)$. As in the same proof of Theorem 2.3, it is easy to see that

$$f(b) - q(b)y(b) > 0.$$

The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. □

On the other hand, the following theorem is based on the work [1]. This theorem will be used to prove the main results of Sections 3 and 4 in this paper.

Theorem 2.9. *Assume that* (1.1) *has bounded oscillatory solutions and* (1.2) *is oscillatory. If* $(\frac{f(t)}{q(t)})' \leq 0$, *then the maxima of a solution of* (1.2) *are nonincreasing. In particular, if* $(\frac{f(t)}{q(t)})' < 0$, *then the maxima of a solution of* (1.2) *are decreasing.*

Proof. By changing the variables $u_1(t) = -y_h'(t)$ or $u(t) = -y'(t)$, respectively, equation (1.1) or (1.2) are transformed into

$$\left(\frac{(p(t)u_1(t))'}{q(t)}\right)' + u_1(t) = 0 \tag{2.12}$$

or

$$\left(\frac{(p(t)u(t))'}{q(t)}\right)' + u(t) = -\left(\frac{f(t)}{q(t)}\right)', \tag{2.13}$$

respectively. Let $t = d$, $t = e$ and $t = h$ be, respectively, a consecutive maximum, minimum and maximum of a solution $y(t)$ of (1.2). They are consecutive zeros of a solution (2.14). Let $t = \bar{d}$, $t = \bar{e}$ and $t = \bar{h}$ be, respectively, a consecutive maximum, minimum and maximum of a solution $y_h(t)$ of (1.1). They are consecutive zeros of a solution (2.12). Let $u_1(t)$ be the solution of (2.12) that satisfies

$$u_1(\bar{e}) = 0, \quad u_1'(\bar{e}) < 0.$$

Multiplication of (2.12) by $u(t)$, (2.14) by $u_1(t)$, subtraction of the resulting equations, and integration over $(t, \bar{e})$ and $(\bar{e}, t)$ yield

$$\begin{aligned}
-\int_t^{\bar{e}} \left(\frac{f(t)}{q(t)}\right)' p(t)u_1(t)dt &= \left[\frac{(p(t)u(t))'}{q(t)}p(t)u_1(t) - \frac{(p(t)u_1(t))'}{q(t)}p(t)u(t)\right]_t^{\bar{e}} \\
&= -\frac{p(\bar{e})u_1'(\bar{e})}{q(\bar{e})}p(\bar{e})u(\bar{e}) \\
&\quad - \frac{1}{q(t)}\Big\{(p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)\Big\}
\end{aligned}$$

for $t \in (\bar{d}, \bar{e})$, and

$$\begin{aligned}
-\int_{\bar{e}}^{t} \left(\frac{f(t)}{q(t)}\right)' p(t)u_1(t)dt &= \frac{p(\bar{e})u_1'(\bar{e})}{q(\bar{e})}p(\bar{e})u(\bar{e}) \\
&\quad + \frac{1}{q(t)}\Big\{(p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)\Big\}
\end{aligned}$$

for $t \in (\bar{e}, \bar{h})$. Since $u_1(\bar{e}) = 0$, $u_1'(\bar{e}) < 0$ and $u(\bar{e}) < 0$, we obtain

$$\frac{1}{q(t)}\Big\{(p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)\Big\} \le \int_t^{\bar{e}} \left(\frac{f(t)}{q(t)}\right)' p(t)u_1(t)dt$$

and

$$\frac{1}{q(t)}\Big\{(p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)\Big\} \le -\int_{\bar{e}}^{t} \left(\frac{f(t)}{q(t)}\right)' p(t)u_1(t)dt.$$

In the interval $(\bar{d}, \bar{h})$, we shall prove that

$$\Big((p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)\Big) \le 0,$$

because $-\left(\frac{f(t)}{q(t)}\right)'$ is nonnegative. Here it is easy to show that

$$\left(\frac{u_1(t)}{u(t)}\right)' = \left(\frac{p(t)u_1(t)}{p(t)u(t)}\right)' = \frac{(p(t)u(t))'p(t)u_1(t) - (p(t)u_1(t))'p(t)u(t)}{(p(t)u(t))^2} \ge 0.$$

Integrating (2.12) over (α, β) yields

$$\frac{u_1(\beta)}{u(\beta)} \ge \frac{u_1(\alpha)}{u(\alpha)}, \tag{2.14}$$

where $\alpha \in (\bar{d}, e)$ and $\beta \in (\bar{e}, h)$. Set $\frac{u_1(\alpha)}{u(\alpha)} \equiv k > 0$ for some α, then (2.14) reduces to

$$u_1(\beta) - ku(\beta) \leq 0$$

in view of $u(\beta) \leq 0$. On the other hand, choose $\frac{u_1(\beta)}{u(\beta)} \equiv k > 0$ for some β so that

$$u_1(\alpha) - ku(\alpha) \leq 0$$

for $u(\alpha) \geq 0$. Consequently, we can establish the following

$$u_1(t) - ku(t) \leq 0, \quad t \in (\bar{d}, e) \cup (\bar{e}, h).$$

From the relations $u_1(t) = -\bar{y}(t)$ and $u(t) = -y'(t)$, we obtain

$$\Big(ky(t) - y_h(t)\Big)' \leq 0, \quad t \in (\bar{d}, e) \cup (\bar{e}, h),$$

which implies that

$$k\Big(y(\bar{d}) - y(h)\Big) \geq y_h(\bar{d}) - y_h(h).$$

From this fact, using the results of Kusano and Yoshida [6], we can lead to

$$k\Big(y(d) - y(h)\Big) \geq k\Big(y(\bar{d}) - y(h)\Big) \geq y_h(\bar{d}) - y_h(\bar{h}) \geq 0.$$

As in the other theorem, it is easy to see that $y(d) > y(h)$ if $\left(\frac{f(t)}{p(t)}\right)' \neq 0$, $d \leq t \leq h$. Thus, the conclusion of the theorem follows. □

3. MODERATELY BOUNDED OSCILLATORY SOLUTIONS

Our aim in this section is to establish explicit upper bounds for $\mathcal{A}^*$ and $\mathcal{S}^*$ as well as explicit lower bounds for $\mathcal{A}_*$ and $\mathcal{S}_*$ for all oscillatory solutions of (1.1) or (1.2) satisfying the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{3.1}$$

and α, β are any given constants such that $(\alpha, \beta) \neq 0$. By applying the similar proof of Theorems 3.1 and 3.2 in [6] which can be extended to the equation (1.2). Thus, we establish the following four pairs of inequalities. If $p'(t) \geq 0$ and $q'(t) \leq 0$,

$$\frac{p(0)q(0)\alpha^2 + p(0)^2\beta^2}{p(\tau_k)q(0)} - \frac{p(\tau_k)^2 y_p'(\tau_k)^2}{p(\tau_k)q(\tau_k)} \leq \Big(y(\tau_k) + y_p(\tau_k)\Big)^2,$$

$$\Big(y(\tau_k) + y_p(\tau_k)\Big)^2 \leq \frac{q(0)\alpha^2 + p(0)\beta^2 - p(\tau_k)y_p'(\tau_k)^2}{q(\tau_k)}.$$

If $p'(t) \le 0$ and $q'(t) \ge 0$,

$$\frac{q(0)\alpha^2 + p(0)\beta^2 - p(\tau_k)y_p'(\tau_k)^2}{q(\tau_k)} \le \Big(y(\tau_k) + y_p(\tau_k)\Big)^2,$$

$$\Big(y(\tau_k) + y_p(\tau_k)\Big)^2 \le \frac{p(0)q(0)\alpha^2 + p(0)^2\beta^2}{p(\tau_k)q(0)} - \frac{p(\tau_k)^2 y_p'(\tau_k)^2}{p(\tau_k)q(\tau_k)}.$$

If $(p(t)q(t))' \ge 0$,

$$\frac{p(0)q(0)\alpha^2 + p(0)^2\beta^2 - p(\tau_k)^2 y_p'(\tau_k)^2}{p(\tau_k)q(\tau_k)} \le \Big(y(\tau_k) + y_p(\tau_k)\Big)^2,$$

$$\Big(y(\tau_k) + y_p(\tau_k)\Big)^2 \le \frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)} - \frac{p(\tau_k)y_p'(\tau_k)^2}{q(\tau_k)}.$$

If $(p(t)q(t))' \le 0$,

$$\frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)} - \frac{p(\tau_k)y_p'(\tau_k)^2}{q(\tau_k)} \le \Big(y(\tau_k) + y_p(\tau_k)\Big)^2,$$

$$\Big(y(\tau_k) + y_p(\tau_k)\Big)^2 \le \frac{p(0)q(0)\alpha^2 + p(0)^2\beta^2 - p(\tau_k)^2 y_p'(\tau_k)^2}{p(\tau_k)q(\tau_k)}.$$

Taking the supremum and infimum on both sides of the above inequality as $k \to \infty$, then one can show the situations in which upper and lower amplitudes $\mathcal{A}^*[y]$ and $A_*[y]$ of the solution with consideration for $\mathcal{A}^*[y] < \infty$ and $A_*[y] > 0$. We will use the following notation:

$$\overline{\mathcal{A}}[y_p] = \limsup_{t\to\infty} |y_p(t)|, \quad \underline{\mathcal{A}}[y_p] = \liminf_{t\to\infty} |y_p(t)|,$$
$$\overline{\mathcal{S}}[y_p] = \limsup_{t\to\infty} |y_p'(t)|, \quad \overline{\mathcal{S}}[y_p] = \limsup_{t\to\infty} |y_p'(t)|,$$
$$\overline{\mathcal{S}}[y_p] = \limsup_{t\to\infty} |y_p'(t)|, \quad \underline{\mathcal{S}}[y_p] = \liminf_{t\to\infty} |y_p'(t)|.$$

Theorem B and Theorem 2.5 imply that $y_p(t)$ is oscillatory particular solution of (1.2) such that

$$\underline{\mathcal{A}}[y_p] < \infty, \quad \overline{\mathcal{A}}[y_p] < \infty,$$
$$\underline{\mathcal{S}}[y_p] < \infty, \quad \overline{\mathcal{S}}[y_p] < \infty.$$

From Lemma 2.2 we obtain the following results:

Theorem 3.1. *Let* (1.1) *be oscillatory, the Wronskian* $W(y_h, y) \ne 0$ *and* $\left(\frac{f(t)}{q(t)}\right)' \le 0$ *hold, and let* $y(t)$ *be a solution of* (1.2) *on* $[0,\infty)$ *satisfying* (3.1)*.*

(i) *Suppose that* $p'(t) \ge 0$ *and* $q'(t) \le 0$ *for* $t \ge 0$*. If* $q(\infty) > 0$*, then*

$$\mathcal{A}^*[y] \le \sqrt{\frac{|q(0)\alpha^2 + p(0)\beta^2 - p(\infty)\underline{\mathcal{S}}[y_p]^2|}{q(\infty)}} - \underline{\mathcal{A}}[y_p],$$

if $p(\infty) < \infty$ *and* $q(\infty) > 0$*, then*

$$\mathcal{A}_*[y] \geq \sqrt{\frac{\left|q(\infty)(p(0)q(0)\alpha^2 + p(0)^2\beta^2) - q(0)(p(\infty)\overline{\mathcal{S}}[y_p])^2\right|}{p(\infty)q(0)q(\infty)}} - \overline{\mathcal{A}}[y_p].$$

(ii) *Suppose that* $p'(t) \leq 0$ *and* $q'(t) \geq 0$ *for* $t \geq 0$*. If* $p(\infty) > 0$ *and* $q(\infty) < \infty$*, then*

$$\mathcal{A}^*[y] \leq \sqrt{\frac{|q(\infty)(p(0)q(0)\alpha^2 + p(0)^2\beta^2) - q(0)(p(\infty)\underline{\mathcal{S}}[y_p])^2|}{p(\infty)q(0)q(\infty)}} - \underline{\mathcal{A}}[y_p],$$

if $q(\infty) < \infty$*, then*

$$\mathcal{A}_*[y] \geq \sqrt{\frac{\left|q(0)\alpha^2 + p(0)\beta^2 - p(\infty)\overline{\mathcal{S}}[y_p]^2\right|}{q(\infty)}} - \overline{\mathcal{A}}[y_p].$$

(iii) *Suppose that* $(p(t)q(t))' \geq 0$ *for* $t \geq 0$*. If* $p(\infty)q(\infty) < \infty$*, then*

$$\mathcal{A}^*[y] \leq \sqrt{\frac{|q(\infty)(q(0)\alpha^2 + p(0)\beta^2) - q(0)p(\infty)\underline{\mathcal{S}}[y_p]^2|}{q(0)q(\infty)}} - \underline{\mathcal{A}}[y_p],$$

$$\mathcal{A}_*[y] \geq \sqrt{\frac{\left|p(0)(q(0)\alpha^2 + p(0)^2\beta^2) - (p(\infty)\overline{\mathcal{S}}[y_p])^2\right|}{p(\infty)q(\infty)}} - \overline{\mathcal{A}}[y_p].$$

(iv) *Suppose that* $(p(t)q(t))' \leq 0$ *for* $t \geq 0$*. If* $p(\infty)q(\infty) > 0$*, then*

$$\mathcal{A}_*[y] \leq \sqrt{\frac{|p(0)(q(0)\alpha^2 + p(0)^2\beta^2) - (p(\infty)\underline{\mathcal{S}}[y_p])^2|}{p(\infty)q(\infty)}} - \underline{\mathcal{A}}[y_p],$$

$$\mathcal{A}^*[y] \geq \sqrt{\frac{\left|q(\infty)(q(0)\alpha^2 + p(0)\beta^2) - q(0)p(\infty)\overline{\mathcal{S}}[y_p]^2\right|}{q(0)q(\infty)}} - \overline{\mathcal{A}}[y_p].$$

On the other hand, we establish the slopes $|y'(\sigma_k)|$ and the upper and lower slopes $\mathcal{S}^*[y]$, $\mathcal{S}_*[y]$ of oscillatory solutions $y(t)$ of (1.2) on $[0, \infty)$.

$$\frac{p(0)q(\sigma_k)}{p(\sigma_k)^2 q(0)}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)} \leq \Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2,$$

$$\Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2 \leq \frac{q(0)\alpha^2 + p(0)\beta^2 - q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)}$$

if $p'(t) \geq 0$ and $q'(t) \leq 0$,

$$\frac{q(0)\alpha^2 + p(0)\beta^2 - q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)} \leq \Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2,$$

$$\Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2 \leq \frac{p(0)q(\sigma_k)}{p(\sigma_k)^2 q(0)}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)}$$

if $p'(t) \le 0$ and $q'(t) \ge 0$,

$$\frac{p(0)}{p(\sigma_k)^2}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)} \le \Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2,$$

$$\Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2 \le \frac{q(\sigma_k)}{p(\sigma_k)q(0)}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)}$$

if $(p(t)q(t))' \ge 0$,

$$\frac{q(\sigma_k)}{p(\sigma_k)q(0)}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)} \le \Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2,$$

$$\Big(y'(\sigma_k) + y_p'(\sigma_k)\Big)^2 \le \frac{p(0)}{p(\sigma_k)^2}(q(0)\alpha^2 + p(0)\beta^2) - \frac{q(\sigma_k)y_p(\sigma_k)^2}{p(\sigma_k)}$$

if $(p(t)q(t))' \le 0$. Letting $k \to \infty$ in these inequalities, one can easily find sufficient conditions in which the upper and lower slope $\mathcal{S}^*[y]$ and $\mathcal{S}_*[y]$ of the solution with ensure $\mathcal{S}^*[y] < \infty$ and $\mathcal{S}_*[y] > 0$. Hence, we present the following result.

Theorem 3.2. *Let* (1.1) *be oscillatory, the Wronskian* $W(y_h, y) \ne 0$ *and* $\left(\frac{f(t)}{q(t)}\right)' \le 0$ *hold, and let* $y(t)$ *be a solution of* (1.2) *on* $[0,\infty)$ *satisfying* (3.1).

(i) *Suppose that* $p'(t) \ge 0$ *and* $q'(t) \le 0$ *for* $t \ge 0$. *If* $p(\infty) < \infty$, *then*

$$\mathcal{S}^*[y] \le \sqrt{\frac{|q(0)\alpha^2 + p(0)\beta^2 - q(\infty)\underline{\mathcal{A}}[y_p]^2|}{p(\infty)}} - \underline{\mathcal{S}}[y_p],$$

if $p(\infty) < \infty$ *and* $q(\infty) > 0$, *then*

$$\mathcal{S}_*[y] \ge \sqrt{\frac{\big|p(0)q(\infty)(q(0)\alpha^2 + p(0)\beta^2) - p(\infty)q(0)q(\infty)\overline{\mathcal{A}}[y_p]^2\big|}{p(\infty)^2 q(0)}} - \overline{\mathcal{S}}[y_p].$$

(ii) *Suppose that* $p'(t) \le 0$ *and* $q'(t) \ge 0$ *for* $t \ge 0$. *If* $p(\infty) > 0$ *and* $q(\infty) < \infty$, *then*

$$\mathcal{S}^*[y] \le \sqrt{\frac{|p(0)q(\infty)(q(0)\alpha^2 + p(0)\beta^2) - p(\infty)q(0)q(\infty)\underline{\mathcal{A}}[y_p]^2|}{p(\infty)^2 q(0)}} - \underline{\mathcal{S}}[y_p],$$

$$\mathcal{S}_*[y] \ge \sqrt{\frac{\big|q(0)\alpha^2 + p(0)\beta^2 - q(\infty)\overline{\mathcal{A}}[y_p]^2\big|}{p(\infty)}} - \overline{\mathcal{S}}[y_p].$$

(iii) *Suppose that* $(p(t)q(t)) \ge 0$ *for* $t \ge 0$. *If* $p(\infty) < \infty$ *and* $q(\infty) < \infty$, *then*

$$\mathcal{S}^*[y] \le \sqrt{\frac{|q(\infty)(q(0)\alpha^2 + p(0)\beta^2 - q(0)q(\infty)\underline{\mathcal{A}}[y_p]^2|}{p(\infty)q(0)}} - \underline{\mathcal{S}}[y_p],$$

$$\mathcal{S}_*[y] \ge \sqrt{\frac{\big|p(0)(q(0)\alpha^2 + p(0)\beta^2 - p(\infty)q(\infty)\overline{\mathcal{A}}[y_p]^2\big|}{p(\infty)^2}} - \overline{\mathcal{S}}[y_p].$$

(iv) *Suppose that* $(p(t)q(t))' \leq 0$ *for* $t \geq 0$*. If* $p(\infty) > 0$ *and* $q(\infty) > 0$*, then*

$$\mathcal{S}^*[y] \leq \sqrt{\frac{|p(0)(q(0)\alpha^2 + p(0)\beta^2 - p(\infty)q(\infty)\underline{\mathcal{A}}[y_p]^2|}{p(\infty)^2}} - \underline{\mathcal{S}}[y_p],$$

$$\mathcal{S}_*[y] \geq \sqrt{\frac{\left|q(\infty)(q(0)\alpha^2 + p(0)\beta^2) - q(0)q(\infty)\overline{\mathcal{A}}[y_p]^2\right|}{p(\infty)q(0)}} - \overline{\mathcal{S}}[y_p].$$

The rest of this section, we shall discuss increase/decrease behavior about the sequences of zeros and extrema points of the function. Similar to Kusano and Yoshida [6], we are interested in explicit laws or rules, if any governing the arrangement of these sequences.

Lemma 3.3. *Suppose that* (1.1) *is oscillatory. Let* $\{\sigma_k\}$ *denote the sequence of zeros of the function* $(y(t) - y_p(t))$*, where* $y(t)$ *is arbitrary solution and* $y_p(t)$ *is particular solution of* (1.2)*.*

(i) *The sequence* $\{\sigma_{k+1} - \sigma_k\}$ *is decreasing or increasing according to* $p'(t) \leq 0$ *and* $q'(t) \geq 0$*, or* $p'(t) \geq 0$ *and* $q'(t) \leq 0$*.*
(ii) *Consider the case where* $p'(t) \geq 0$ *and* $q'(t) \geq 0$ *for* $t \geq 0$*. Suppose that* $\int_0^\infty dt/p(t) < \infty$*. Put* $\pi(t) = \int_t^\infty ds/p(s)$ *and assume that* $\pi(t)^2 p(t)$ *and* $\pi(t)^4 p(t)q(t)$ *are monotone for* $t \geq 0$*. Then, the sequence* $\{\sigma_{k+1} - \sigma_k\}$ *is decreasing or increasing according to whether*

$$(\pi(t)^2 p(t))' \leq 0 \quad \text{and} \quad (\pi(t)^4 p(t)q(t))' \geq 0 \quad \text{for } t \geq 0,$$

or

$$(\pi(t)^2 p(t))' \geq 0 \quad \text{and} \quad (\pi(t)^4 p(t)q(t))' \leq 0 \quad \text{for } t \geq 0.$$

Proof. For (i), the equation (1.2) reduces to

$$\Big(p(t)\Big(y(t) - y_p(t)\Big)'\Big)' + q(t)\Big(y(t) - y_p(t)\Big) = 0.$$

Putting $y_h(t) \equiv y(t) - y_p(t)$, then $y_h(t)$ is a solution of (1.1). Applying the result of Hille [5], we can show that (i) is true.

For the case (ii), the proof here is similar to that of Kusano and Yoshida [6], and so is omitted. □

Lemma 3.4. *Suppose that* (1.1) *is oscillatory. Let* $\{\tau_k\}$ *denote the sequence of extrema points of the function* $(y(t) - y_p(t))$*, where* $y(t)$ *is arbitrary solution and* $y_p(t)$ *is particular solution of* (1.2)*.*

(i) *The sequence* $\{\tau_{k+1} - \tau_k\}$ *is decreasing or increasing according to* $p'(t) \leq 0$ *and* $q'(t) \geq 0$*, or* $p'(t) \geq 0$ *and* $q'(t) \leq 0$*.*
(ii) *Consider the case where* $p'(t) \geq 0$ *and* $q'(t) \geq 0$ *for* $t \geq 0$*. Suppose that* $\int_0^\infty dt/p(t) < \infty$*. Put* $\rho(t) = \int_t^\infty q(s)ds$ *and assume that* $\rho(t)^2/q(t)$ *and*

$\rho(t)^4/p(t)q(t)$ are monotone for $t \geq 0$. Then, the sequence $\{\tau_{k+1}-\tau_k\}$ is decreasing or increasing according to whether

$$(\rho(t)^2 p(t))' \leq 0 \quad \text{and} \quad (\rho(t)^4 p(t)q(t))' \geq 0 \quad \text{for } t \geq 0,$$

or

$$(\rho(t)^2 p(t))' \geq 0 \quad \text{and} \quad (\rho(t)^4 p(t)q(t))' \leq 0 \quad \text{for } t \geq 0.$$

Proof. The proof is similar to the proof of [6], so we omit it here. □

Combining Theorem A with Lemma 3.3 or Theorem A with Lemma 3.4, respectively, we state and prove the following results.

Theorem 3.5. *Suppose that* (1.1) *is oscillatory and the Wronskian $W(y_h, y) \neq 0$. Let $\{\sigma_k\}$ denote the sequence of zeros of the solution $y(t)$. Then $y(t)$ satisfies Lemma* 3.3 (i) *and* (ii).

Theorem 3.6. *Suppose that* (1.1) *is oscillatory and the Wronskian $W(y_h, y) \neq 0$. Let $\{\tau_k\}$ denote the sequence of extrema points of the solution $y(t)$. Then $y(t)$ satisfies Lemma* 3.4 (i) *and* (ii).

4. QUALITATIVE OSCILLATORY SOLUTIONS

Our aim in this section is to establish the existence and qualitative behavior of oscillatory solutions of nonhomogeneous equation (1.2), that is, this chapter is an attempt to obtain relations between large/small oscillatory solutions of (1.1) and qualitative oscillatory solutions of (1.2). It is known that such solutions possibly exist only if the coefficients $p(t)$ and $q(t)$ satisfy one of the following conditions:

(i) $p'(t) \geq 0$, $q'(t) \leq 0$, $p(\infty) = \infty$ and/or $q(\infty) = 0$,
(ii) $p'(t) \leq 0$, $q'(t) \geq 0$, $p(\infty) = 0$ and/or $q(\infty) = \infty$,
(iii) $(p(t)q(t))' \geq 0$, $p(\infty)q(\infty) = \infty$,
(iv) $(p(t)q(t))' \leq 0$, $p(\infty)q(\infty) = 0$.

Using of the results of Kusano and Yoshida [7] in Section 2 leads to following theorems.

Lemma 4.1. *Let equation* (1.1) *be oscillatory and $y_p(t)$ be particular solution of* (1.2). *If $(p(t)q(t))' \geq 0$ [or ≤ 0] and $p(\infty)q(\infty) = \infty$ [or 0], then there exists a small [or large] oscillatory function $(y(t) - y_p(t))$.*

Lemma 4.2. *Let equation* (1.1) *be oscillatory and $y_p(t)$ be particular solution of* (1.2). *Assume that $p'(t) \geq 0$ and $q'(t) \leq 0$. If $p(\infty) = \infty$ [or $< \infty$] and $q(\infty) = \infty$ [or $< \infty$], then there exists a small [or large] oscillatory function $(y(t) - y_p(t))$.*

Lemma 4.3. *Let equation* (1.1) *be oscillatory and $y_p(t)$ be particular solution of* (1.2). *Assume that $p'(t) \leq 0$ and $q'(t) \geq 0$. If $p(\infty) > 0$ [or $= 0$] and $q(\infty) > 0$ [or $= 0$], then there exists a small [or large] oscillatory function $(y(t) - y_p(t))$.*

From the above result, the existence of such oscillatory function $(y(t)-y_p(t))$ is not in doubt, but the question is what it can be affected by force term $f(t)$. Taking into account Lemma 3.3 and Theorem 2.5, these question is answer by following results. Accordingly, it will be able to describe the oscillatory behavior of solution $y(t)$ of nonhomogeneous equation (1.2).

Theorem 4.4. *Let equation* (1.1) *be oscillatory and the Wronskian* $W(y_h, y) \neq 0$. *If* $\left(\frac{f(t)}{q(t)}\right)' < 0$, *and one of the following condition hold:*

(a) $(p(t)q(t))' \geq 0$ *and* $p(\infty)q(\infty) = \infty$,
(b) $p'(t) \geq 0$, $q'(t) \leq 0$ *and* $p(\infty) = \infty$, $q(\infty) > 0$,
(c) $p'(t) \leq 0$, $q'(t) \geq 0$ *and* $p(\infty) > 0$, $q(\infty) = \infty$,

then $y(t)$ *is a small oscillation of equation* (1.2).

Theorem 4.5. *Let equation* (1.1) *be oscillatory and the Wronskian* $W(y_h, y) \neq 0$. *If one of the following condition hold:*

(a) $(p(t)q(t))' \leq 0$ *and* $p(\infty)q(\infty) = 0$,
(b) $p'(t) \geq 0$, $q'(t) \leq 0$ *and* $p(\infty) < \infty$, $q(\infty) = 0$,
(c) $p'(t) \leq 0$, $q'(t) \geq 0$ *and* $p(\infty) = 0$, $q(\infty) < \infty$,

then $y(t)$ *is a large oscillation of equation* (1.2).

5. EXAMPLES

We illustrate the applicability and efficiency of the results via the following examples.

Example 5.1. Consider the equation

$$\left(\frac{1}{2-e^{-t}}y'(t)\right)' + (2-e^{-t})y(t) = 2-e^{-t} \tag{5.1}$$

on $[0,\infty)$. Since (1.3), $|y_p(t)| < \infty$ and the Wronskian:

$$W(y_h, y) = k + 1 - \cos(2t + e^{-t} - 1) \neq 0$$

hold, then (5.1) is clearly oscillatory. Note that $p'(t) \leq 0$, $q'(t) \geq 0$ and $(p(t)q(t))' = 0$. Let $y(t)$ be a solution of (5.1) on $[0,\infty)$ satisfying the initial condition (3.1). It is easy to check that $p(\infty) > 0$, $q(\infty) < \infty$ and $y_p(t)$ is bounded. Then (ii) of Theorem 3.1 applies and gives

$$\mathcal{A}^*[y] \leq \sqrt{2(\alpha^2+\beta^2)} + 2, \qquad \mathcal{A}_*[y] \geq \sqrt{\frac{\alpha^2+\beta^2}{2}}.$$

It follows that all solutions of (5.1) are moderately bounded, which can be shown that

$$y(t) = \sin(2t + e^{-t} - 1) - \cos(2t + e^{-t} - 1) + 1.$$

Example 5.2. Consider the equation

$$\left(e^t y'(t)\right)' + e^{3t}y(t) = e^{2t} \tag{5.2}$$

on $[0,\infty)$. It is easy to compute that (1.4), $|y_p(t)| < \infty$ and the Wronskian:

$$W(y_h, y) = k + \cos 1 - \cos e^t \neq 0$$

hold. Hence, we observe that (5.2) is oscillatory. Corollary 2.4 is not applicable to (5.2) due to satisfying $(p(t)f(t))' \geq 0$. Since $(p(t)q(t))' \geq 0$, $p(\infty)q(\infty) = \infty$, Theorem 4.4 means that there exists a small oscillatory solution

$$y(t) = e^{-t}\{(1 - \sin(1))\sin e^t - \cos(1)\cos e^t + 1\}.$$

Example 5.3. Consider the equation

$$\left(t^{-1}y'(t)\right)' + 4ty(t) = t\cos t^2 \tag{5.3}$$

on $[0,\infty)$. It easy to check that this equation is oscillatory and satisfy (1.3), $|y_p(t)| < \infty$ and the Wronskian:

$$W(y_h, y) = k + \frac{1}{4}\sin^2 t^2 \neq 0.$$

Since $p'(t) \leq 0$, $q'(t) \geq 0$ and $|y_p(\infty)| = |y_p'(\infty)| = \infty$, Theorem 4.5 implies that (5.3) has a large oscillatory solution

$$y(t) = \cos t^2 + \frac{t^2}{8}\sin t^2.$$

Example 5.4. Consider the equation

$$\left(\sqrt{t}y'(t)\right)' + \frac{1}{4\sqrt{t}}y(t) = \frac{1}{\sqrt{t}}\cos\sqrt{t} \tag{5.4}$$

on $[0,\infty)$. It easy to verify that this equation is oscillatory, which satisfy (1.3) and the Wronskian:

$$W(y_h, y) = k + \sin^2\sqrt{t} \neq 0.$$

Since $p'(t) \leq 0$, $q'(t) \geq 0$ and $|y_p(\infty)| = |y_p'(\infty)| = \infty$, Theorem 4.5 immediately implies that (5.4) has a large oscillatory solution

$$y(t) = \cos\sqrt{t} + 2\sqrt{t}\sin\sqrt{t}.$$

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