dr n. tech. Andrzej Antoni CZAJKOWSKI[a,b], dr inż. Wojciech Kazimierz OLESZAK[a]

[a] Higher School of Humanities of Common Knowledge Society in Szczecin, Faculty of Applied Sciences
Wyższa Szkoła Humanistyczna Towarzystwa Wiedzy Powszechnej w Szczecinie, Wydział Nauk Stosowanych

[b] Higher School of Technology and Economics in Szczecin, Faculty of Automotive Systems
Wyższa Szkoła Techniczno-Ekonomiczna w Szczecinie, Wydział Systemów Automotive

PARTIAL LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND *MATHEAMTICA*

Abstract

Introduction and aims: The paper describes the method of solving first order linear differential homogeneous differential equations using *Mathematica* program. The purpose of the work is to provide algorithms for analytical and symbolic solutions in *Mathematica* for three selected examples.
Material and methods: The work uses selected literature from first order linear partial differential equations. The method of characteristics was used in analytical solutions, and the *Mathematica 5* program in numerical solutions.
Results: The characteristics method was used in analytical solutions of selected examples of first order linear partial differential equations. In addition to numerical solutions, graphic interpretation was given using spatial and contour charts.
Conclusion: *Mathematica* program solves the first order linear partial differential equations with given boundary conditions using the *pde* and *DSolve* procedures. *Mathematica* program also allows for first order linear partial differential equations with boundary conditions to show some geometric interpretation of their solutions using the *Plot3D* and *ContourPlot* commands.

Keywords: Partial differential equations, linear, homogeneous, the first order, solutions, *Mathematica.*

(Received: 01.03.2019; Reviewed: 15.03.2019; Accepted: 20.03.2019)

RÓWNANIA RÓŻNICZKOWE CZĄSTKOWE LINIOWE JEDNORODNE RZĘDU PIERWSZEGO I PROGRAM MATHEMATICA

Streszczenie

Wstęp i cele: *W pracy opisano metodę rozwiązywania równań różniczkowych cząstkowych liniowych jednorodnych pierwszego rzędu z wykorzystaniem programu Mathematica. Celem pracy jest podanie algorytmów rozwiązań analitycznych i symbolicznych w programie Mathematica dla wybranych trzech różnych przykładów.*
Materiał i metody: *W pracy wykorzystano wybraną literaturę z równań różniczkowych cząstkowych liniowych rzędu pierwszego. W rozwiązaniach analitycznych zastosowano metodę charakterystyk, a w rozwiązaniach numerycznych program Mathematica 5.*
Wyniki: *Metodę charakterystyk zastosowano w rozwiązaniach analitycznych wybranych przykładów równań różniczkowych cząstkowych liniowych rzędu pierwszego. Oprócz rozwiązań numerycznych podano interpretację graficzną stosując wykresy przestrzenne i konturowe.*
Wnioski: *Program Mathematica rozwiązuje liniowe jednorodne równania różniczkowe cząstkowe pierwszego rzędu z zadanymi warunkami brzegowymi stosując procedury pde i DSolve. Program Mathematica umożliwia również dla równań różniczkowych cząstkowych liniowych rzędu pierwszego z warunkami brzegowymi pokazanie geometrycznej interpretacji ich rozwiązań za pomocą poleceń Plot3D i ContourPlot.*

Słowa kluczowe: *Równania różniczkowe cząstkowe, liniowe, jednorodne, rząd pierwszy, rozwiązania, Mathematica.*

(Otrzymano: 01.03.2019; Zrecenzowano: 15.03.2019; Zaakceptowano: 20.03.2019)

1. Introduction

1.1. First order linear partial homogeneous differential equations

Homogeneous partial linear differential equation with an unknown function $u = u(x_1, x_2, \ldots, x_n)$ has the form [1], [3]-[5]:

$$A_1 \frac{\partial u}{\partial x_1} + A_2 \frac{\partial u}{\partial x_2} + \ldots + A_n \frac{\partial u}{\partial x_n} = 0, \tag{1}$$

where functions $A_1, A_2, \ldots, A_n$, are functions which are continuously differentiable in a certain area of n - dimensional space and depend only on variables $x_1, x_2, \ldots, x_n$.

The system of ordinary differential equations corresponding to the partial equation (1.1) is a system with the following form:

$$\frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \ldots = \frac{dx_n}{A_n}. \tag{2}$$

The general integral of the partial differential equation (1) is found using the following theorem.

Theorem 1.

If n-1 of the compounds

$$\begin{cases} \Psi_1(x_1, x_2, \ldots, x_n) = C_1 \\ \Psi_2(x_1, x_2, \ldots, x_n) = C_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Psi_{n-1}(x_1, x_2, \ldots, x_n) = C_{n-1} \end{cases} \tag{3}$$

is the system of n-1 integrals of the first independent system of ordinary equations (2), then the composite function is given by:

$$u = \Phi(\Psi_1, \Psi_2, \ldots, \Psi_{n-1}), \tag{4}$$

where the symbol Φ is any differentiable function, and $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$, are the left sides of the first integrals (3), is the general integral of the partial differential equation (1) [7], [8].

Algorithm 1.

To find the general integral of a partial linear and homogeneous first order differential equation we must:

(1a) write the system of ordinary differentials (2) corresponding to this equation,
(1b) calculate as many first independent integrals (3) as there are equations written in the system,
(1c) create a compound function (4), where $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$ are the left sides of the first integrals determined in point (1b) [10]- [12].

1.2. Cauchy problem

Cauchy's problem is to designate such a solution $u(x_1, x_2, \ldots, x_n)$ of this equation (1), which for one of the variables $x_n = x_n^{(0)}$ is equal to function $\varphi(x_1, x_2, \ldots, x_{n-1})$ of other variables, i.e.

$$u(x_1, x_2, \ldots, x_{n-1}, x_n^{(0)}) = \varphi(x_1, x_2, \ldots, x_{n-1}). \tag{5}$$

The above-mentioned problem is solved according to the scheme that shows algorithm 1.

Algorithm 2.

(2a) We write the system of ordinary differential equations corresponding to this equation. It has the following form:

$$\frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \ldots = \frac{dx_n}{A_n}; \tag{6}$$

(2b) We find the n-1 of independent first integrals of this system:

$$\begin{cases} \Psi_1(x_1, x_2, \ldots, x_{n-1}, x_n) = C_1 \\ \Psi_2(x_1, x_2, \ldots, x_{n-1}, x_n) = C_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Psi_{n-1}(x_1, x_2, \ldots, x_{n-1}, x_n) = C_{n-1}; \end{cases} \tag{7}$$

(2c) We put in the equations (7) $x_n = x_n^{(0)}$:

$$\begin{cases} \Psi_1(x_1, x_2, \ldots, x_{n-1}, x_n^{(0)}) = C_1 \\ \Psi_2(x_1, x_2, \ldots, x_{n-1}, x_n^{(0)}) = C_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Psi_{n-1}(x_1, x_2, \ldots, x_{n-1}, x_n^{(0)}) = C_{n-1}. \end{cases} \tag{8}$$

We then get a system of n-1 equations that we solve with respect to the other variables.

(2d) Values found at point (2c) for $x_1, x_2, \ldots, x_{n-1}$ we insert into the function:

$$u = \varphi(x_1, x_2, \ldots, x_{n-1}), \tag{9}$$

which is given in the Cauchy problem.

We then obtain:

$$u = \tilde{\varphi}(C_1, C_2, \ldots, C_{n-1}). \tag{10}$$

(2e) To the function defined by the formula (10) we put in place C_1, C_2, …, C_{n-1} the left side of the corresponding first integrals (8), then we obtain the required solution [17].

2. Examples of partial linear homogeneous differential equations of the first order solved analytically and numerically in *Mathematica* program

Example 1.

Solve the equation [16]:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \tag{11}$$

with condition

$$u(1,y) = y^2. \tag{12}$$

• **Analytical method *(a method of characteristics)***

The ordinary differential equation for equation (11) has the form:

$$\frac{dx}{x} = \frac{dy}{y}. \tag{13}$$

We determine the first integral of the equation (13). After separating the variables we have:

$$\frac{dy}{y} = \frac{dx}{x}. \tag{14}$$

Equation (14) is integrated on both sides with respect to the variables x and y:

$$\int \frac{dy}{y} = \int \frac{dx}{x}. \tag{15}$$

$$\ln|y| = \ln|x| + \ln|C|. \tag{16}$$

where we get:

$$y = Cx, \quad C \in \Re. \tag{17}$$

After calculating the constant from equation (17), it takes the form:

$$C = \frac{y}{x}. \tag{18}$$

The general solution of the partial equation (11) has the following form:

$$u(x, y) = F\left(\frac{y}{x}\right). \tag{19}$$

Let us look for a particular solution of the partial equation (11) with the condition (12). For this purpose we put in the equation (17) x = 1.

We then receive:

$$y = C. \tag{20}$$

After inserting (18) to equation (20) with the condition (21) we get the particular solution of equation (11) in the following form:

$$u(x, y) = \frac{y^2}{x^2}. \tag{21}$$

• **Numerical method *(Mathematica)*** [2], [8], [9], [13]-[16]

```
In[1]:=pde=x*D[u[x,y],x]+y*D[u[x,y],y]==0
  solution=DSolve[pde,u[x,y],{x,y}]
  f[x_,y_]=u[x,y]/.solution[[1]]
  result=DSolve[{pde,u[1,y]==y^2},u[x,y],{x,y}]
  Plot3D[u[x,y]/.result,{x,0.5,2},{y,-2,2},ColorFunction->Hue]
  ContourPlot[u[x,y]/.result,{x,0.5,2},{y,-2,2},
  ColorFunction->Hue]
Out[1]= yu^(0,1)[x,y]+xu^(1,0)[x,y]==0
Out[2]= {{u[x,y]->C[1][y/x]}}
```

```
Out[3]= C[1][y/x]
Out[4]= {{u[x,y]->y²/x²}}
```

Out[5]=

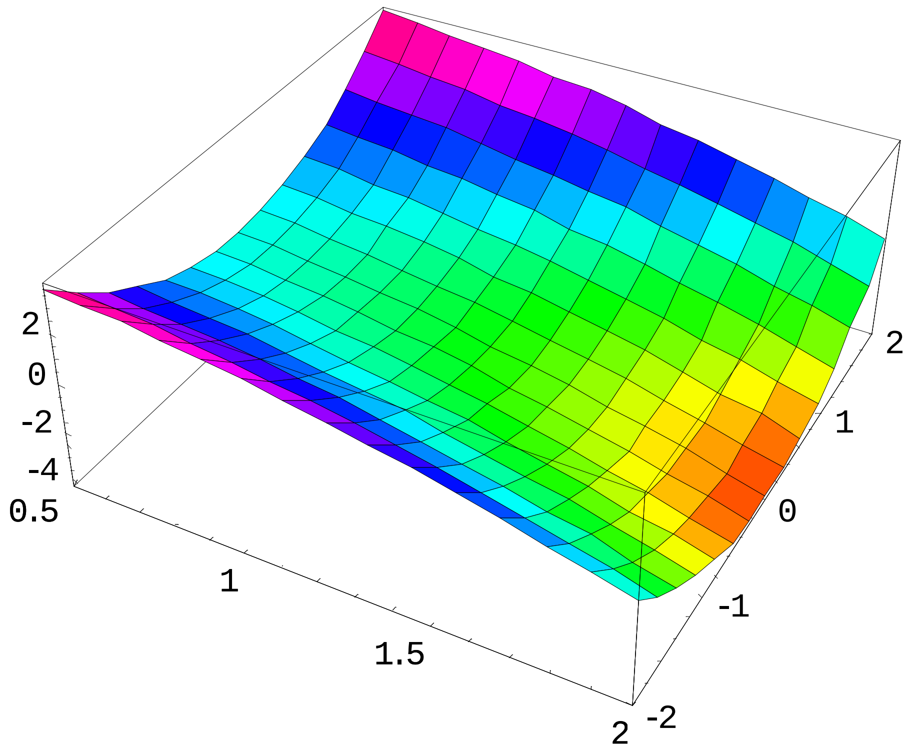

Fig. 1. Graphical interpretation of the solution (21) of differential equation (11) for 0.5 < x < 2, -–2 < y < 2 using *Plot3D* command in *Mathematica*
Source: Elaboration of the Author

Out[6]=

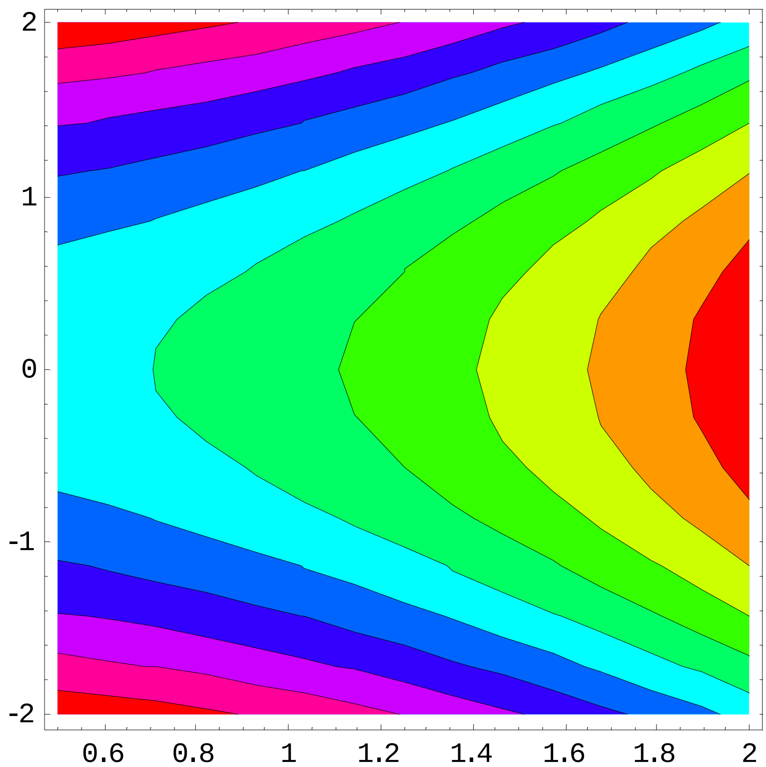

Fig. 2. Graphical interpretation of the solution (21) of differential equation (11) for 0.5 < x < 2, –2 < y < 2 using *ContourPlot* command in *Mathematica*
Source: Elaboration of the Author

Example 2.

Solve the equation [16]:

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \tag{22}$$

with condition

$$u(0,y) = \sin(y^2) \ . \tag{23}$$

• **Analytical method *(a method of characteristics)***

The ordinary differential equation for equation (22) has the form:

$$\frac{dx}{y} = \frac{dy}{x}. \tag{24}$$

We determine the first integral of the equation (2 4). After separating the variables we have:

$$ydy = xdx \ . \tag{25}$$

Equation (25) is integrated on both sides with respect to the variables x and y:

$$\int ydy = \int xdx. \tag{26}$$

where we get:

$$\frac{y^2}{2} = \frac{x^2}{2} + C, \quad C \in R. \tag{27}$$

Hence

$$y = \sqrt{2}\sqrt{\frac{x^2}{2} + C}. \tag{28}$$

After calculating the constant from equation (28), it takes the form:

$$C = \tfrac{1}{2}(y^2 - x^2). \tag{29}$$

The general solution of the partial equation (22) has the following form:

$$u(x,y) = F\left[\frac{1}{2}(y^2 - x^2)\right]. \tag{30}$$

Let us look for the particular solution of partial equation (22) with the condition (23). For this purpose we put in the equation (28) x = 0.

We then receive:

$$y = \sqrt{2C}, \tag{31}$$

After inserting (29) the equation (30) we get the particular solution of equation (22) in the following form:

$$u(x,y) = -\sin(x^2 - y^2). \tag{32}$$

• **Numerical method *(Mathematica)*** [2], [8], [9], [13]-[16]

```
In[1]:=pde=y*D[u[x,y],x]+x*D[u[x,y],y]==0
  solution=DSolve[pde,u[x,y],{x,y}]
  f[x_,y_]=u[x,y]/.solution[[1]]
  result=DSolve[{pde,u[1,y]==Sin[y^2]},u[x,y],{x,y}]
  Plot3D[u[x,y]/.result,{x,-Pi,Pi},{y,0.001,Pi},
  ColorFunction->Hue]
  ContourPlot[u[x,y]/.result,{x,-Pi,Pi},{y,0.001,Pi},
  ColorFunction->Hue]
Out[1]= xu^(0,1)[x,y]+yu^(1,0)[x,y]==0
Out[2]= {{u[x,y]->C[1][1/2(-x^2+y^2)]}}
Out[3]= C[1][1/2 (-x^2+y^2)]
Out[4]= {{u[x,y]->-Sin[x^2-y^2]}}
```

Out[5]=

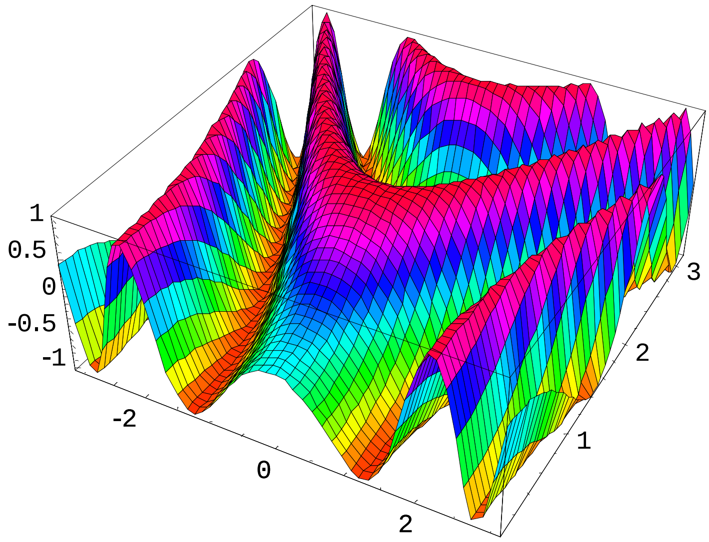

Fig. 3. Graphical interpretation of the solution (32) of differential equation (22) for –Pi < x < Pi, 0.001 < y < Pi using *Plot3D* command in *Mathematica*
Source: Elaboration of the Author

Out[6]=

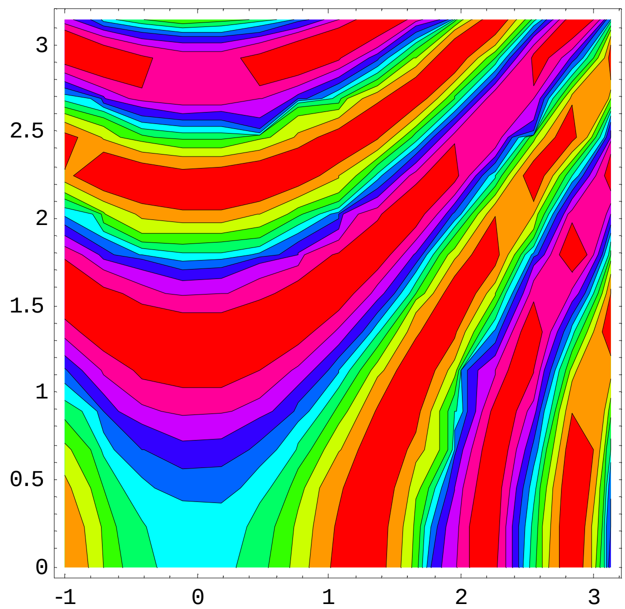

Fig. 4. Graphical interpretation of the solution (32) of differential equation (22) for –Pi < x < Pi, 0.001 < y < Pi using *ContourPlot* command in *Mathematica*
Source: Elaboration of the Author

Example 3.

Solve the equation [16]:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \tag{33}$$

with condition

$$u(1,y) = y^2 . \tag{34}$$

• **Analytical method *(a method of characteristics)***

The ordinary differential equation for equation (33) has the form:

$$\frac{dx}{y} = \frac{dy}{-x}. \tag{35}$$

We determine the first integral of the equation (35). After separating the variables we have:

$$ydy = -xdx. \tag{36}$$

Equation (36) is integrated on both sides with respect to the variables x and y:

$$\int ydy = -\int xdx. \tag{37}$$

where we get:

$$\frac{y^2}{2} = -\frac{x^2}{2} + C, \quad C \in \Re, \tag{38}$$

$$y = \sqrt{2}\sqrt{C - \frac{x^2}{2}}. \tag{39}$$

After calculating the constant from equation (9), it takes the form:

$$C = \tfrac{1}{2}(x^2 + y^2). \tag{40}$$

The general solution of the partial equation (33) has the following form:

$$u(x,y) = F\left[\frac{1}{2}(x^2 + y^2)\right]. \tag{41}$$

Let us look for a particular solution of the partial equation (33) with the condition (34). For this purpose we put in the equation (40) x = 1. We then receive:

$$C = \frac{y^2 + 1}{2}, \tag{42}$$

and hence, after considering (40) in (42) we have:

$$y = \sqrt{x^2 + y^2 - 1}. \tag{43}$$

When we take into account (43), the condition (34) we obtain the particular solution of the equation (33) in the following form:

$$u(x,y) = x^2 + y^2 - 1. \tag{44}$$

• **Numerical method *(Mathematica)*** [2], [8], [9], [13]-[16]

```
In[1]:=pde=y*D[u[x,y],x]-x*D[u[x,y],y]==0
  solution=DSolve[pde,u[x,y],{x,y}]
  f[x_,y_]=u[x,y]/.solution[[1]]
  result=DSolve[{pde,u[1,y]==y^2},u[x,y],{x,y}]
  Plot3D[u[x,y]/.result,{x,-1,1},{y,-1,1},
  ColorFunction->Hue]
  ContourPlot[u[x,y]/.result, {x,-1,1},{y,-1,1},
  ColorFunction->Hue]
Out[1]= -xu^(0,1)[x,y]+yu^(1,0)[x,y]==0
Out[2]= {{u[x,y]->C[1][1/2(x^2+y^2)]}}
Out[3]= C[1][1/2 (x^2+y^2)]
Out[4]= {{u[x,y]->-1+x^2+y^2]}}
```

Out[5]=

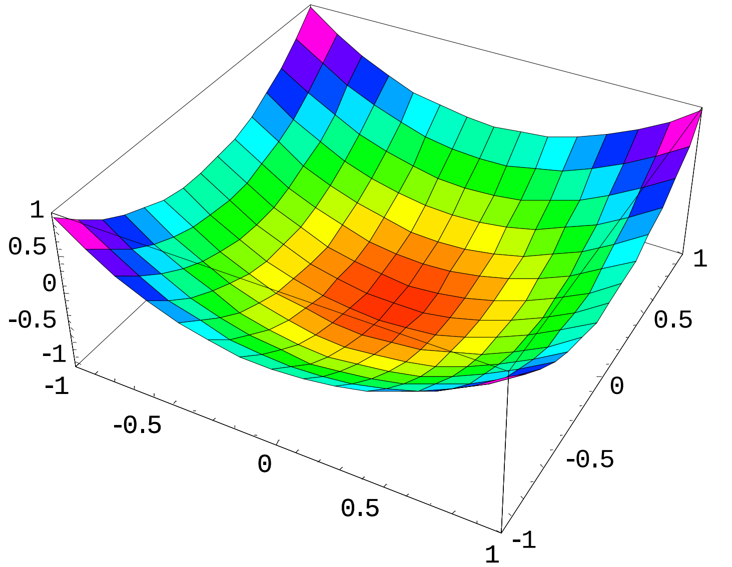

Fig. 3. Graphical interpretation of the solution (44) of differential equation (33) for $-1 < x < 1$, $-1 < y < 1$ using command *Plot3D* in *Mathematica*
Source: Elaboration of the Author

Out[6]=

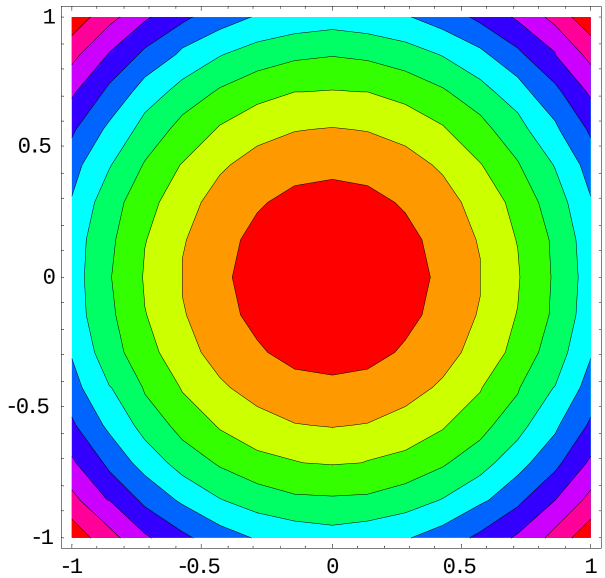

Fig. 4. Graphical interpretation of the solution (44) of differential equation (33) for $-1 < x < 1$, $-1 < y < 1$ using command *ContourPlot* in *Mathematica*
Source: Elaboration of the Author

3. Conclusions

- *Mathematica* program solves the first order linear partial differential equations with given boundary conditions using the *pde* and *DSolve* commands.
- *Mathematica* program also allows for first order linear partial differential equations with boundary conditions to show a geometric interpretation of their solutions using the *Plot3D* and *ContourPlot* commands.

Literatura

[1] Arnold W.I.: *Teoria równań różniczkowych,* PWN, Warszawa 1983.

[2] Drwal G., Grzymkowski R., Kapusta A., Słota D.: *Mathematica 5,* Wyd. Prac. Komputerowej Jacka Skalmierskiego, Gliwice 2004.

[3] Evans L.C.: *Równania różniczkowe cząstkowe,* Wyd. Naukowe PWN, Warszawa 2002.

[4] Krysicki Włodzimierz, Włodarski Lech: *Analiza matematyczna w zadaniach, Część II,* Wyd. naukowe PWN, Warszawa 1994, w. XIX.

[5] Krzyżański M.: *Równania różniczkowe cząstkowe rzędu drugiego - Część 1,* Biblioteka Matematyczna Tom 15, PWN, Warszawa 1957.

[6] Kythe P.K., Puri P., Schräferkotter M.R.: *Partial differential equations and Mathematica,* CRC Press Inc. Boca Raton, New York London Tokyo, 1997.

[7] Marcinkowska H.: *Wstęp do teorii równań różniczkowych cząstkowych,* BM Tom 43, PWN, Warszawa 1972.

[8] Pelczar A., Szarski J.: *Wstęp do teorii równań różniczkowych cząstkowych. Część 1 Wstęp do teorii równań różniczkowych zwyczajnych i równań różniczkowych cząstkowych pierwszego rzędu.* PWN, Warszawa 1987.

[9] Ross C.C.: *Differential equations, An introduction with Mathematica,* Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo Hong Kong Barcelona Budapest 1995.

[10] Sneddon I.: *Równania różniczkowe cząstkowe,* PWN, Warszawa 1962.

[11] Siewierski L. (pod red.): *Ćwiczenia z analizy matematycznej z zastosowaniami, Tom II,* PWN Warszawa 1981.

[12] Smirnow W.I.: *Matematyka wyższa. Tom 4, Część 2, Równania o pochodnych cząstkowych, Zagadnienia brzegowe.* PWN, Warszawa 1962.

[13] Trott M.: *The Mathematica guide book for graphics,* Springer, New York 2004.

[14] Trott M.: *The Mathematica guide book for numerics,* Springer, New York 2006.

[15] Trott M.: *The Mathematica guide book for symbolics,* Springer, New York 2006.

[16] Vvedensky D.: *Partial differential equations with Mathematica,* Addison-Wesley, Publishing Company, Wokingham England Reading Massachusetts Menlo Park California New York Don Mills Ontario, Amsterdam Bonn Sydney Singapore Tokyo Madrid San Juan Milan Paris Mexico City Seoul Taipei 1994.

[17] Wolska-Bochenek J., Borzymowski A., Chmaj J., Tryjarska M.: *Zarys teorii równań różniczkowych całkowych i cząstkowych,* PWN, Warszawa 1981.