ZORAN GLIGORIC*, CEDOMIR BELJIC*, BRANKO GLUSCEVIC* CEDOMIR CVIJOVIC**

UNDERGROUND LEAD-ZINC MINE PRODUCTION PLANNING USING FUZZY STOCHASTIC INVENTORY POLICY

PLANOWANIE WYDOBYCIA CYNKU I OŁOWIU W KOPALNIACH PODZIEMNYCH Z WYKORZYSTANIEM PODEJŚCIA STOCHASTYCZNEGO Z ELEMENTAMI LOGIKI ROZMYTEJ DO OKREŚLANIA NIEZBĘDNEGO POZIOMU ZAPASÓW

Methodology for long-term underground lead-zinc mine planning based on fuzzy inventory theory is presented in this paper. We developed a fuzzy stochastic model of inventory control problem for planning lead-zinc ore production under uncertainty. The final purpose of this article is to find the optimal quantity of mined ore that should be stockpiled, in order to enable "feeding" of mineral processing plant in cases when the production in underground mine is interrupted, by using Possibilistic mean value of fuzzy number for defuzzing the fuzzy total annual inventory costs, and by using Extension of the Lagrangean method for solving inequality constrain problem. The different types of costs involved in mined ore inventory problems affect the efficiency of production scheduling. Dynamic nature of lead and zinc metal price is described by Ornstein-Uhlenbeck stochastic mean reverting process. The model is illustrated with a numerical example.

Keywords: underground mine, production planning, ore inventory policy, uncertainties, fuzzy-stochastic modelling

W pracy przedstawiono metodologię długoterminowego wydobycia cynku i ołowiu w kopalniach podziemnych z wykorzystaniem podejścia stochastycznego z elementami logiki rozmytej do określania wymaganego poziomu zapasów. Opracowaliśmy model stochastyczny z wykorzystaniem elementów logiki rozmytej do kontroli zapasów w planowaniu wydobycia cynku i ołowiu w warunkach niepewności. Celem końcowym pracy jest określenie optymalnej ilości wydobywanej rudy, którą należy zachować jako zapas tak aby zapewnić odpowiednie jej dostawy do zakładu przeróbczego nawet w przypadku przerwania wydobycia w kopalni podziemnej, opierając się na posybilistycznej wartości średniej liczby rozmytej i wyostrzeniu całkowitych rocznych kosztów zapasów. Wykorzystano także rozszerzenie metody Lagrange'a do rozwiązywania problemu więzów w nierówności. Różnorakie koszty związane ze składowaniem

* UNIVERSITY OF BELGRADE, FACULTY OF MINING AND GEOLOGY, DJUSINA 7, 11000 BELGRADE, REPUBLIC OF SERBIA. E-MAIL: zoran.gligoric@rgf.bg.ac.rs; cedomir.beljic@rgf.bg.ac.rs; branko.gluscevic@rgf.bg.ac.rs
** BELGRADE UNIVERSITY COLLEGE OF APPLIED STUDIES IN CIVIL ENGINEERING AND GEODESY, DEPARTMENT OF GEODESY; HAJDUK STANKA 2, 11000 BELGRADE; REPUBLIC OF SERBIA. E-MAIL: cedomircvijovic@vggs.rs

zapasów wydobywanej rudy mają wpływ na wydajność planowanej produkcji. Dynamiczne zmiany cen cynku i ołowiu zostały określone z wykorzystaniem rewersji średniej stochastycznej, w pracy Ornsteina-Uhlenbecka. Zaprezentowano przykład numeryczny jako ilustrację modelu.

Słowa kluczowe: kopalnia podziemna, planowanie produkcji, określanie niezbednego poziomu zapasów rudy, warunki niepewności, modelowania stochastyczne z elementami logiki rozmytej

1. Introduction

A mining production system is commonly composed of two main producing units. The first unit is related to mine and the second to mineral processing plant. The mine produces the ore which is further processed by the mineral processing plant. The final product is the metal concentrate which is sold on the market. The concentrate is produced on a continuous production line while the ore is produced in discontinuous way, i.e., in production cycles. Therefore, ore is stockpiled until it is needed for processing. This paper treats only mine producing unit, i.e. underground mine.

Underground mining is used when the ore deposit is located deep under the earth's surface. Underground mine represents a very complex technological system composed of many subsystems such as: drilling and blasting, supporting, transport and haulage, ventilation, dewatering, etc. (see Fig. 1).

Interruption in any of these subsystems directly causes interruption of production. For example, in the underground metallic ores mines there is a presence of very harmful gases produced
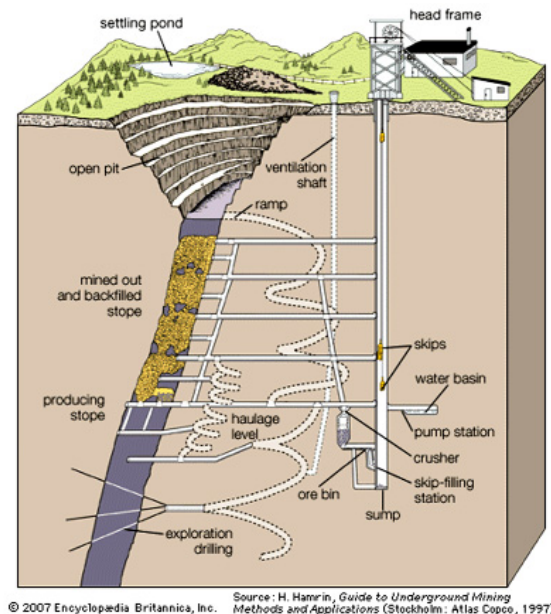

Fig. 1. Schematic view of underground mine
(source: H. Hamrin, *Guide to Underground Mining Methods and Applications*, Stockholm; Atlas Copco, 1997)

by diesel powered machines and if the main ventilation system is broken down all miners must put on self rescue apparatus and take leave of the mine immediately. The situation is even more serious in the underground coal mines where there is a presence of methane (highly explosive gas). Another example is related to the stability of main transport roadways which directly depends on the efficiency of the supporting systems. High values of the stress around the roadway can cause the deformation and even close of roadway. In such situations the efficiency of the transportation system of personnel and materials as well as the haulage of mined ore is considerably reduced. Repairment of damaged roadway can last days even weeks, entailing interruption of production because there is no alternative direction of transport. Such accidents are classified into group of unplanned production interruptions. Certainly the environment of production associated with the mining company is unique when compared with environment encountered by typical manufacturing companies. As in any manufacturing company, there are planned interruptions of production so there are planned interruptions of production in every underground mine and they are classified into group of planned production interruptions. Both groups are characterized by loss of production time.

In such environment it is necessary to create the plan of production in order to satisfy market demands with minimum costs. This problem was treated in different ways. Gillenwater et al. (1995) developed the mine production scheduling model based on the aggregate production planning. The model presented in their research use the coal industry as an example. Martinez and Newman (2011) have created a mixed-integer program to schedule long- and short-term production at LKAB's Kiruna mine, an underground sublevel caving mine located in northern Sweden. The model minimizes deviations from monthly preplanned production quantities while adhering to operational constraints. Optimization is based on the decomposition heuristic that, on average, obtains better solutions faster than solving the model directly. Smith et al. (2003) incorporated a variety of features into their lead and zinc underground mine model, including sequencing relationships, capacities and minimum production requirements. However, they significantly reduced the resolution of the model by aggregating stopes into larger blocks. The resulting model, with time periods of one year length, maximizes net present value over the life of the mine (here, 13 years). Carlyle and Eaves (2001) presented a model that maximizes revenue from Stillwater's platinum and palladium mine which uses the sublevel stoping mining method. The problem focuses on strategic mine expansion planning, so the integer decision variables schedule the timing of various mining activities: development and drilling and stope preparation. Sarin and West-Hansen (2005) schedule a coal mining operation to maximize net present value without penalties for irregular schedules. They expedite the solution time for their model with Bender's decomposition-based methodology. Little and Topal (2011) review optimization studies, focusing on model reduction approaches, which employ Mixed Integer Programming techniques for simultaneous optimization of stope layouts and underground production scheduling. Four theories are presented to reduce the number of variables and complex constraints without comprising its mathematical integrity. Azimi et al. (2012) applied multi-criteria ranking to select the optimal open pit mining cut-off grade strategy under metal price uncertainty. Gligoric et al. (2011) developed a hybrid model of evaluation of mine capacity expansion project in an underground lead-zinc mine using simulation and fuzzy numbers.

Finally, the company is interested in determining when to produce a batch of ore and how many ore to produce in each batch. This organizational problem can be solved by effective inventory control system. This is essential for mining company for many reasons, such as, meeting metal concentrate supply contracts, and optimization of amount of money tied up in ore inventory.

2. Basic concepts of fuzzy sets

In order to deal with the vagueness of human thought, Zadeh (1965) first introduced the fuzzy set theory. This theory was oriented to the rationality of uncertainty due to imprecision or vagueness. A fuzzy set is a class of objects with continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. A fuzzy set is an extension of a crisp set. Crisp sets only allow full membership or non-membership at all, whereas fuzzy sets allow partial membership. In other words, an element may partially belong to a fuzzy set. The role of fuzzy sets is significant when applied to complex phenomena not easily described by traditional mathematical methods, especially when the goal is to find a good approximate solution (Bojadziev, 1998). Modelling using fuzzy sets has proven to be an effective way of formulating decision problems where the information available is subjective and imprecise (Zimmermann, 1992). A tilde "~" is placed above a symbol if the symbol represents a fuzzy set.

Suppose that $X = \{x\}$ is a universe, i.e., the set of all possible (feasible, relevant) elements to be considered. Then a fuzzy subset (or a fuzzy set, for short) A in X is defined as a set of ordered pairs $\{(x, \mu_A(x))\}$, where $x \in X$ and $\mu_A : X \rightarrow [0,1]$ is the membership function of A; $\mu_A : X \in [0,1]$ is the grade of membership of x in A, from 0 for full nonbelongingness to 1 for full belongingness trough all intermediate values (Seda, 2005). It is convenient to denote fuzzy set defined in a finite universe, say A in $X = \{x_1,...,x_n\}$ as $A = \mu_A(x_1)/x_1 + ... + \mu_A(x_n)/x_n$ where "$\mu_A(x_i)/x_i$" (called a singleton) is a "grade of membership/element" pair and "+" is used in the set-theoretical sense.

The γ-cut of fuzzy set A in X is defined as an ordinary set $A_\gamma \subseteq X$ such that

$$A_\gamma = \left\{ x \in X \mid \mu_A(x) \geq \gamma \right\}, \quad \forall \gamma \in [0,1] \tag{1}$$

Similarly, the γ-level cut of a fuzzy set A in X, denoted by A^γ, is the crisp subset of X that contains all of the elements of X with exactly the given degree of membership γ:

$$A^\gamma = \left\{ x \in X \mid \mu_A(x) = \gamma \right\}, \quad \forall \gamma \in [0,1] \tag{2}$$

The level set of A, denoted L_A, is a subset $[0,1]$ containing the values γ that determine distinct γ-cuts:

$$L_A = \left\{ \gamma \in [0,1] \mid \mu_A(x) = \gamma \quad \text{for some } x \in X \right\} \tag{3}$$

Any fuzzy subset of the real line R, whose membership function satisfies the following conditions, is a generalized fuzzy number $\widetilde{M}$, (Ritha et al., 2011)
   (1) $\mu_{\widetilde{M}}(x)$ is a continuous mapping from R to the closed interval [0,1],
   (2) $\mu_{\widetilde{M}}(x) = 0, -\infty < x \leq a$,
   (3) $\mu_{\widetilde{M}}(x)$ is strictly increasing on $[a, b]$,
   (4) $\mu_{\widetilde{M}}(x) = w_M, a \leq x \leq c$,
   (5) $\mu_{\widetilde{M}}(x)$ is strictly decreasing on $[b, c]$,
   (6) $\mu_{\widetilde{M}}(x) = 0, c \leq x < +\infty$, where $0 < w_M \leq 1$, and a, b, c are real numbers. Also this type of generalized fuzzy number is denoted as $\widetilde{M}(a,b,c)$.

There are many possibilities to use different fuzzy numbers according to the situation. Triangular fuzzy numbers (TFN) are very convenient to work with because of their computational

simplicity and they are useful in promoting representation and information processing in fuzzy environment. In this paper, we use TFNs.

Triangular fuzzy numbers can be defined as a triplet (a, b, c). The parameters a, b and c respectively, indicate the smallest possible value, the most promising value and the largest possible value that describe a fuzzy event. A triangular fuzzy number $\tilde{M}$ is shown in Figure 2.
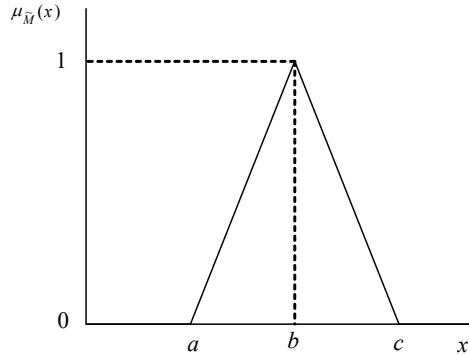

Fig. 2. Triangular fuzzy number

The membership function is defined as (Kaufmann & Gupta, 1985):

$$\mu_{\tilde{M}}(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x > c \end{cases} \tag{4}$$

Triangular fuzzy numbers can be used to perform common mathematical operations. The basic fuzzy arithmetic operations on two triangular fuzzy memberships $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are defined as follows: (1) inverse: $\tilde{A}^{-1} = (1/a_3, 1/a_2, 1/a_1)$; (2) addition: $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$; (3) subtraction: $\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$; (4) scalar multiplication: $\forall \varphi > 0, \varphi \in R, \varphi \cdot \tilde{A} = (\varphi \cdot a_1, \varphi \cdot a_3, \varphi \cdot a_3)$; (5) multiplication: $\tilde{A} \otimes \tilde{B} = (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3)$; (6) division: $\tilde{A}/\tilde{B} = (a_1/b_3, a_2/b_2, a_3/b_1)$.

Let us assume that $\tilde{A}_i$, $i = 1, 2, ..., n$ be sample of the random variable of fuzzy number. By the extension principle of fuzzy sets (Rivera et al., 1995), (Zadeh, 1965) and definition of the triangular fuzzy number, the average operation for fuzzy number $\tilde{A}_i$ is as follows:

$$\overline{\tilde{A}} = \sum_{i=1}^{n} \tilde{A}_i / n \tag{5}$$

An important concept related to the applications of fuzzy numbers is defuzzification, which converts a fuzzy number into a crisp value. Such a transformation is not unique because different methods are possible. The most commonly used defuzzification method is the centroid defuzzi-

fication method, which is also known as center of gravity or center of area defuzzification. The centroid defuzzification method can be expressed as follows (Yager, 1981):

$$\bar{x}_0\left(\tilde{A}\right) = \int_a^c x\mu_{\tilde{A}}(x)\,dx \,/ \int_a^c \mu_{\tilde{A}}(x)\,dx \qquad (6)$$

where $\bar{x}_0(\tilde{A})$ is the defuzzified value. The defuzzification formula of triangular fuzzy numbers (a, b, c) is

$$\bar{x}_0\left(\tilde{A}\right) = (a+b+c)/3 \qquad (7)$$

and it will be used in this paper.

3. Fuzzy model

3.1. Model in crisp sense

In this chapter, we introduce the fuzzy inventory model, and find the optimal solution, i.e., optimal crisp quantity of mined ore that should be stockpiled. Because inventory polices affect profitability, the choice among policies depends upon their relative profitability. Some of the costs that determine this profitability are the producing costs, holding costs and shortage costs. First, we deal with an inventory model in crisp environment. In this model, there are some assumptions: the economic order quantity (*EOQ*) approach is used as a base of model; the way of ore production is cyclical; the demand is not changed over time of plan; time of plan is constant; shortages are not allowed; the ore production capacity of underground mine is greater than the ore demand rate, and enables the ore inventory to be replenished in continuous way.

The cost of producing an amount of ore y, per cycle, can be represented by a function $A(y)$. The simplest form of this function is one that is directly proportional to the amount of the ore mined. Another common assumption is that $A(y)$ is composed of two parts: the first term is directly proportional to the amount of the ore mined, and the second term is a constant K for y positive and is 0 for $y = 0$. For this case,

$$A\left(y\left(x,p,z\right)\right) = \begin{cases} 0 & \text{if } y = 0 \\ a \cdot y + K = a\dfrac{p \cdot x}{p-z} + K & \text{if } y > 0, p > z \end{cases} \qquad (8)$$

where
   y — the ore production (t/day),
   K — the costs involved in setting up to start a production cycle ($/cycle),
   a — the unit cost of production ($/t),
   x — the inventory level (t/day),
   p — the ore production capacity rate (t/day),
   z — the ore demand rate (t/day).

The holding cost (the storage cost) represents all the costs associated with the storage of the ore inventory until it is used. Included are the cost of stockpiling and capital tied up. The holding

cost can be assessed either continuously or on a period by period basis. In our model we use continuous approach. The form of the holding cost function per one production cycle is as follows:

$$B(x,p,z) = \left(b + \varphi \cdot \sum_{i=1}^{2} V_i^{con} \cdot \frac{G_i \cdot M_i}{m_i^{con}} \right) \cdot \frac{x^2}{2 \cdot z} \cdot \frac{p}{p-z}, \quad p > z \tag{9}$$

where

b — the unit cost of carrying inventory in stock (\$/tday),
φ — percentage of the capital tied up (%), (as a percentage of the value of the ore),
G_i — the grade of the ore mined (%),
M_i — the mill recovery rate (%),
m_i^{con} — the metal content of the concentrate (%),
V_i^{con} — the value of i-th metal concentrate (\$/t),

$$V_i^{con} = \begin{cases} P_{Pb} \cdot m_{Pb}^{con} \cdot m_{Pb}^{mr} & \text{for } Pb \\ P_{Zn} \cdot \left(m_{Zn}^{con} - m_{Zn}^{mr} \right) & \text{for } Zn \end{cases} \tag{10}$$

where

P_{Pb} — the lead metal price (\$/t),
P_{Zn} — the zinc metal price (\$/t),
m_{Pb}^{con} — the lead metal content of the concentrate (%),
m_{Zn}^{con} — the zinc metal content of the concentrate (%),
m_{Pb}^{mr} — the lead metal recovery rate (%),
m_{Zn}^{mr} — the zinc metal recovery rate (%).

The difference in equation forms for *Pb* and *Zn* arises from different recovery during the smelting process.

The ore production capacity rate, directly depends on metal concentrate demands, and can be defined as follows:

$$p(z) = q \cdot z, \quad z = \max \left[\frac{z_{Pb} \cdot m_{Pb}^{con}}{G_{Pb} \cdot M_{Pb}}, \frac{z_{Zn} \cdot m_{Zn}^{con}}{G_{Zn} \cdot M_{Zn}} \right] \tag{11}$$

where

q — coefficient enables $p > z$, ($q > 1$), in this way the inventory replenishment is continuous,
z_{Pb} — lead metal concentrate demand (t/day),
z_{Zn} — zinc metal concentrate demand (t/day).

Therefore, total costs function per cycle is defined as follows:

$$C(x,z) = A(x,z) + B(x,z) =$$
$$\left(a \cdot x + \left(b + \iota \cdot \sum_{i=1}^{2} V_i^{con} \cdot \frac{G_i \cdot M_i}{m_i^{con}} \right) \cdot \frac{x^2}{2 \cdot z} \right) \cdot \frac{q}{q-1} + K \tag{12}$$

so the total costs function per unit time is:

$$C(x,z,T) = \left[\left(a \cdot x + \left(b + \varphi \cdot \sum_{i=1}^{2} V_i^{con} \cdot \frac{G_i \cdot M_i}{m_i^{con}}\right) \cdot \frac{x^2}{2 \cdot z}\right) \cdot \frac{q}{q-1} + K\right] \cdot \frac{z \cdot T}{x} \cdot \frac{q-1}{q} \qquad (13)$$

where T — time horizon (day)

Algebraically,

$$C(x,z,T) = \left[a \cdot z + K \cdot \frac{z \cdot (q-1)}{x \cdot q} + \left(b + \varphi \cdot \sum_{i=1}^{2} V_i^{con} \cdot \frac{G_i \cdot M_i}{m_i^{con}}\right) \cdot \frac{x}{2}\right] \cdot T \qquad (14)$$

3.2. Model in fuzzy environment

In the crisp inventory models, all parameters in the total cost are known and have definite values without ambiguity. But in the reality, it is not so sure. Hence, it is necessary to consider the fuzzy inventory models (Ritha et al., 2011). Dutta et al. (2007) considered a continuous review inventory system, where the annual average demand was treated as a fuzzy random variable. (Mahata & Goswami, 2006) developed a fuzzy production-inventory model with permissible delay in payment. They assumed the demand and the production rates as fuzzy numbers and defuzzified the associated cost in the fuzzy sense using extension principle.

Uncertainty in the process of production planning is commonly evaluated with respect to internal and external parameters. Internal parameters are those that can be governed by inside considerations while external are determined by outside considerations. In our case internal parameters are: $intpars(K, a, b, \varphi, q, M_i, m_i^{con}, m_i^{mr})$. External parameters are: $extpars(z_i, P_i, G_i)$.

To decrease uncertainty related to production planning, some of these input parameters are quantified by triangular fuzzy numbers and some by a specific stochastic behaviors (see Table 1).

TABLE 1

Quantification of input parameters

Parameter	Quantification
K	TFN(min, central, max)
a	TFN(min, central, max)
b	TFN(min, central, max)
φ	TFN(min, central, max)
q	TFN(min, central, max)
G_i	Normal distribution
M_i	Uniform distribution
z_i	TFN(min, central, max)
P_i	Mean reversion process

The market risks related to metal price are modeled with a special dynamic stochastic process; a mean reversion process. The past values of the changes in this risk parameter help predict

the future. We will use a model where the metal spot price is assumed to follow an Ornstein-Uhlenbeck stochastic mean reverting process:

$$dP = k(\bar{P} - P)dt + \sigma dW \tag{15}$$

where $\bar{P}$ is the long-run equilibrium metal price, k measures the speed of mean reversion to the long run mean log price $\bar{P}$, dW is an increment to a standard Brownian motion and σ refers to the price volatility rate, for more details see (Schwartz, 1997), (Dixit and Pindyck, 1994). The solution of the equation (15) is given by the exact discrete-time equation for P_t:

$$P_t = \exp\left\{ \ln(P_{t-1})e^{-k\Delta t} + \left[\ln(\bar{P}) - \frac{\sigma^2}{2k}\right](1 - e^{-k\Delta t}) + N(0,1)\sigma\sqrt{(1 - e^{-2k\Delta t})/2k} \right\} \tag{16}$$

In order to estimate the parameters of the mean reversion process, we run the following regression:

$$dP_{t+1} = \beta_0 + \beta_1 P_t + \varepsilon \tag{17}$$

where $\beta_0 = k\bar{P}dt$ and $\beta_1 = -kdt$. Hence, if we regress observation dP against P, we can obtain estimates of β_0 and β_1; σ is the standard deviation from the regression. Figure 3 presents a sample path of the lead price simulated using the equation (16).
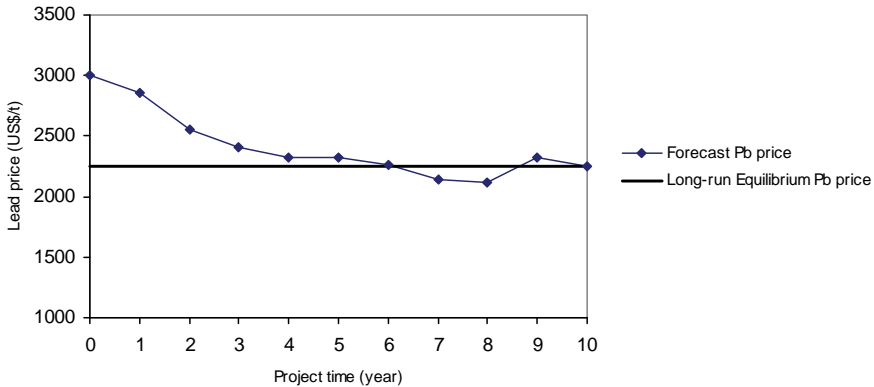

Fig. 3. One simulated lead price path on a yearly time resolution

Accordingly, the second major component of the holding cost function is the cost of capital tied up which is also modeled as dynamic stochastic process because the ore value (OV) directly depends on the metal price (see equations (10), (9)). The relation between risk variables affecting the ore value over the project time is described as follows:

$$OV(t) = \sum_{i=1}^{2} V_i^{con}(t) \cdot \frac{G_i \cdot M_i}{m_i^{con}}; \quad V_i^{con}(t) = \begin{cases} P_{Pb}(t) \cdot m_{Pb}^{con} \cdot m_{Pb}^{mr} & \text{for } Pb \\ P_{Zn}(t) \cdot \left(m_{Zn}^{con} - m_{Zn}^{mr}\right) & \text{for } Zn \end{cases} \tag{18}$$

Parameters G_i, M_i and $P_i(t)$ have its own stochastic law of performance. By simulating them, we obtain ore value for every year of the project. For each simulation, the input values and the *OV* result represent one possible path of the ore value. Simulated values of the *OV* are obtained by performing the following calculations:

$$OV^s = OV\left(P_{i,t}^s, G_i^s, M_i^s, m_i^{con}, m_i^{mr}\right), \quad s = 1, 2, ..., S; \quad t = 1, 2, ..., N; \quad i = 1, 2 \qquad (19)$$

where *S* denotes the number of simulations and *N* number of project years. The main objective of using simulation is to determine the distribution of the *OV* for every year of the project. In this way we obtain the sequence of probability density functions of *OV*; $OV_t \sim (pdf_t, \eta_t, \sigma_t)$, $t = 1, 2, ..., N$, where *N* is a total project time. Sequence of obtained pdf of OV_t can be transformed into a sequence of triangular fuzzy numbers of OV_t; $OV_t \sim TFN_t$, $t = 1, 2, ..., N$, i.e, $OV_1 \sim pdf_1 \rightarrow OV_1 \sim TFN_1$; $OV_2 \sim pdf_2 \rightarrow OV_2 \sim TFN_2$; ... ; $OV_N \sim pdf_N \rightarrow OV_N \sim TFN_N$. The way of transformation is based on the following facts: the support of the membership function and the *pdf* are the same, and the point with higher probability (likelihood) has the higher possibility (see Figure 4). For more details see (Swishchuk, 2008). The uncertainty in the *OV* parameter is modeled by triangular fuzzy number with the membership function which has the support of $\eta_t - 2\sigma_t < OV_t < \eta_t + 2\sigma_t$, $t = 1, 2, ..., N$, set up for around 95% confidence interval of distribution function. If we take into consideration that the triangular fuzzy number is defined as a triplet (a_t, b_t, c_t), then a_t and c_t are lower bound and upper bound obtained from lower and upper bound of 5% of the distribution, and the most promising value b_t is equal to mean value of the distribution. For more details see (Do et al., 2005).
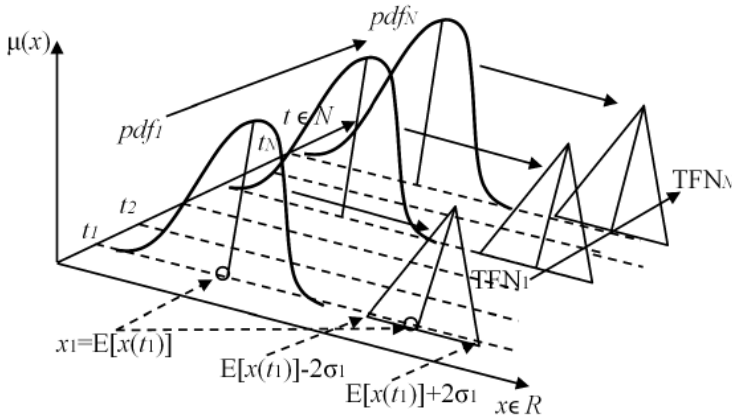

Fig. 4. The way of transformation over time

According to above way of transformation and equation (5), average value of the cost of capital tied up over the project time is defined as follows:

$$\bar{\bar{C}}_{tup} = \tilde{\varphi} \cdot \frac{\sum_{t=1}^{N} O\tilde{V}_t}{N} = \tilde{\varphi} \cdot \frac{\sum_{t=1}^{N} (a_t, b_t, c_t)}{N} \qquad (20)$$

The market uncertainties related to metal concentrate demand (z_i) is expressed by fuzzy triangular number. If we take into consideration that the grade of the ore mined (G_i) and mill recovery rate (M_i) are defined by normal and uniform stochastic law respectively, then the ore demand rate (z) can be defined by following hybrid model:

$$z^s = z\left(\tilde{z}_i, G_i^s, M_i^s, m_i^{con}\right), \quad s = 1, 2, ..., S; \quad i = 1, 2 \tag{21}$$

Since we use fuzzy number to describe vagueness of metal concentrate demand, then after calculations (see equation (11)) we obtain a sample of the random variable of the fuzzy number ($\tilde{z}_i^s$, $s = 1, 2, ..., S$, $i = 1, 2$). By applying equation (5), the average fuzzy number is obtained $\left(\bar{\tilde{z}}_{Pb} = \sum_{s=1}^{S} \tilde{z}_{Pb}^s / S; \bar{\tilde{z}}_{Zn} = \sum_{s=1}^{S} \tilde{z}_{Zn}^s / S\right)$. According to equation (11), the ore demand rate is defined as follows:

$$\tilde{z} = \max\left(\bar{\tilde{z}}_{Pb}, \bar{\tilde{z}}_{Zn}\right) \tag{22}$$

Finally, the total cost function in fuzzy environment can be represented as follows:

$$\tilde{C}\left(\tilde{x}, \tilde{z}, T\right) = \left[\tilde{a} \cdot \tilde{z} + \tilde{K} \cdot \frac{\tilde{z} \cdot (\tilde{q} - 1)}{\tilde{x} \cdot \tilde{q}} + \left(\tilde{b} + \bar{\tilde{C}}_{tup}\right) \cdot \frac{\tilde{x}}{2}\right] \cdot T =$$

$$\left[(a_1, a_2, a_3) \cdot (z_1, z_2, z_3)\right] \cdot T + \left[(K_1, K_2, K_3) \cdot \frac{(z_1, z_2, z_3) \cdot ((q_1, q_2, q_3) - 1)}{(x_1, x_2, x_3) \cdot (q_1, q_2, q_3)}\right] \cdot T + \tag{23}$$

$$\left[\left((b_1, b_2, b_3) + (C_{tup1}, C_{tup2}, C_{tup3})\right) \cdot \frac{(x_1, x_2, x_3)}{2}\right] \cdot T$$

Using the fuzzy arithmetical operations we obtain the total cost function:

$$\tilde{C}\left(\tilde{x}, \tilde{z}, T\right) = \left(C_1 \cdot T, C_2 \cdot T, C_3 \cdot T\right) \tag{24}$$

where

$$C_1 = a_1 z_1 + \frac{K_1 z_1 (q_1 - 1)}{q_3 x_3} + \frac{\left(b_1 + C_{tup1}\right) x_1}{2} \tag{25}$$

$$C_2 = a_2 z_2 + \frac{K_2 z_2 (q_2 - 1)}{q_2 x_2} + \frac{\left(b_2 + C_{tup2}\right) x_2}{2} \tag{26}$$

$$C_3 = a_3 z_3 + \frac{K_3 z_3 (q_3 - 1)}{q_1 x_1} + \frac{\left(b_3 + C_{tup3}\right) x_3}{2} \tag{27}$$

By equation 7, we defuzzify the total cost function and obtain planning model as follows: Minimize

$$C\left(x, z, T\right) = \frac{1}{3} \cdot \left(C_1 + C_2 + C_3\right) \cdot T \tag{28}$$

subject to $0 < x_1 \leq x_2 \leq x_3$.

In order to find the minimization of $C(x,z,T)$ we applied Extension of the Lagrangian Method, for more details see (Ritha et al., 2011). The derivatives of $C(x,z,T)$ with respect to x_1, x_2, and x_3 are:

$$\frac{\partial C(x,z,T)}{\partial x_1} = \frac{T}{3} \cdot \left[\frac{b_1 + C_{tup1}}{2} - \frac{K_3 \cdot z_3 \cdot (q_3 - 1)}{q_1 \cdot x_1^2} \right] \tag{29}$$

$$\frac{\partial C(x,z,T)}{\partial x_2} = \frac{T}{3} \cdot \left[\frac{b_2 + C_{tup2}}{2} - \frac{K_2 z_2 (q_2 - 1)}{q_2 x_2^2} \right] \tag{30}$$

$$\frac{\partial C(x,z,T)}{\partial x_3} = \frac{T}{3} \cdot \left[\frac{b_3 + C_{tup3}}{2} - \frac{K_1 z_1 (q_1 - 1)}{q_3 x_3^2} \right] \tag{31}$$

Now, let all the above partial derivatives equal to zero, and solve x_1, x_2, and x_3; then we get:

$$\frac{\partial C(x,z,T)}{\partial x_1} = 0 \quad \Rightarrow \quad x_1 = \sqrt{\frac{2K_3 z_3 (q_3 - 1)}{q_1 (b_1 + C_{tup1})}} \tag{32}$$

$$\frac{\partial C(x,z,T)}{\partial x_2} = 0 \quad \Rightarrow \quad x_2 = \sqrt{\frac{2K_2 z_2 (q_2 - 1)}{q_2 (b_2 + C_{tup2})}} \tag{33}$$

$$\frac{\partial C(x,z,T)}{\partial x_3} = 0 \quad \Rightarrow \quad x_3 = \sqrt{\frac{2K_1 z_1 (q_1 - 1)}{q_3 (b_3 + C_{tup3})}} \tag{34}$$

Equations (32)-(34) show that $x_1 > x_2 > x_3$; it means that constraint $0 < x_1 \leq x_2 \leq x_3$ is not satisfied. According to Extension of the Lagrangian Method we convert the first inequality constraint into equality constraint $x_2 - x_1 = 0$ and minimize $C(x,z,T)$ subject to $x_2 - x_1 = 0$. Lagrangian function is $L_1(x_1,x_2,x_3,\lambda_1) = C(x,z,T) - \lambda_1(x_2 - x_1)$. Let all the partial derivatives equal to zero and solve x_1, x_2, and x_3; then we get:

$$x_1 = x_2 = \sqrt{\frac{2K_2 z_2 q_1 (q_2 - 1) + 2K_3 z_3 q_2 (q_3 - 1)}{q_1 q_2 (b_1 + C_{tup1} + b_2 + C_{tup2})}} \tag{35}$$

$$x_3 = \sqrt{\frac{2K_1 z_1 (q_1 - 1)}{q_3 (b_3 + C_{tup3})}} \tag{36}$$

Because the above equations (35), (36) show that $x_1 > x_3$, it means that constraint $0 < x_1 \leq x_2 \leq x_3$ is not satisfied, therefore it is not a local optimum and it is necessary to convert the second inequality constraint into equality constraint $x_3 - x_2 = 0$ and minimize $C(x,z,T)$ subject to $x_2 - x_1 = 0$, $x_3 - x_2 = 0$. Lagrangian function is $L_2(x_1,x_2,x_3,\lambda_1,\lambda_2) = C(x,z,T) - \lambda_1(x_2 - x_1) - \lambda_2(x_3 - x_2)$. Let all the partial derivatives equal to zero and solve x_1, x_2, and x_3; then we get:

$$x_1 = x_2 = x_3 = \sqrt{\frac{2K_1 z_1 q_1 q_2 (q_1 - 1) + 2K_2 z_2 q_1 q_3 (q_2 - 1) + 2K_3 z_3 q_2 q_3 (q_3 - 1)}{q_1 q_2 q_3 \left(b_1 + C_{tup1} + b_2 + C_{tup2} + b_3 + C_{tup3} \right)}} \tag{37}$$

Since the equation (37) shows that all inequality constraints are satisfied, the procedure terminates with $\tilde{x} = (x_1, x_2, x_3)$ as a local optimum solution to the problem i.e., the optimal ore inventory level.

Underground mining of lead-zinc ore can be organized according to the following parameters:
Number of production cycles per year is calculated as follows:

$$\tilde{n}^* = \sqrt{\frac{(\tilde{b} + \tilde{C}_{tup})\tilde{z}T^2(\tilde{q}-1)}{2\tilde{K}\tilde{q}}} \rightarrow n^* = \left[\overline{n}^*\left(\tilde{n}^*\right)\right] + 1 \quad \text{must be integer} \tag{38}$$

The optimal cycle time is given by equation:

$$t^* = \frac{T}{n^*} \tag{39}$$

The optimal production time, when ore is mined in production part of a cycle, is calculated as follows:

$$\tilde{t}_p^* = \frac{\tilde{x}}{\tilde{z}(\tilde{q}-1)} \rightarrow t_p^* = \overline{t}_p^*\left(\tilde{t}_p^*\right) \tag{40}$$

The projected time to inventory depletion without replenishment is calculated as follows:

$$\tilde{t}_d^* = \frac{\tilde{x}}{\tilde{z}} \rightarrow t_d^* = \overline{t}_d^*\left(\tilde{t}_d^*\right) \tag{41}$$

The optimal ore quantity produced per one cycle is equal to:

$$\tilde{y}^* = \frac{\tilde{x}\tilde{q}}{\tilde{q}-1} \rightarrow y^* = \overline{y}^*\left(\tilde{y}^*\right) \tag{42}$$

The optimal ore quantity produced per one year is equal to:

$$\tilde{Y}^* = \frac{\tilde{x}\tilde{q}}{\tilde{q}-1} \cdot n^* \rightarrow Y^* = \overline{Y}^*\left(\tilde{Y}^*\right) \tag{43}$$

4. Numerical example

The management of the small Serbian underground mine company was faced with the problem of long-term supplying of mineral processing plant by lead-zinc ore. Previous ore production planning was burdened by many difficulties, primarily related to enabling continuous way of "feeding" of mineral processing plant. In the presence of input parameter uncertainties, it looked as the problem couldn't be solved satisfactory.

For this problem, the input parameters that are required for the production planning are given in the Table 2.

Input parameters

Lead metal concentrate demand (t/day)	(14 15 17)
Zinc metal concentrate demand (t/day)	(29 30 32)
Coefficient enables $p > z$, $(q > 1)$	(1.10 1.15 1.20)
Grade of the ore mined	
Lead-Normal distribution (%)	min 3.24; medium 3.71; max 4.18; volatility 0.1567
Zinc-Normal distribution (%)	min 4.45; medium 5.07; max 5.68; volatility 0.2050
Metal content of the concentrate (%)	
Lead	70
Zinc	50
Metal recovery rate (%)	
Lead	95
Zinc	42
Mill recovery rate (%)	
Lead-Uniform distribution	min 86.20; medium 88.20; max 90.20; volatility 1.154
Zinc-Uniform distribution	min 70.40; medium 72.40; max 74.40; volatility 1.154
Metal price-mean reversion process ($/t)	
Lead (yearly time resolution)-equation (16)	spot value 3000 ; equilibrium 2253; speed of mean reversion 0.245637; price volatility rate 0.035752
Zinc (yearly time resolution)-equation (16)	spot value 2470; equilibrium 1875; speed of mean reversion 0.203958; price volatility rate 0.029147
Unit cost of carrying inventory in stock ($/tday)	(0.01 0.02 0.04)
Percentage of the capital tied up (%)	(3 5 7)
Unit cost of production ($/t)	(28 30 34)
Costs of setting up a production cycle ($/cycle)	(10 11 13)
Time horizon (day)	365
Total project time (year)	10

In this model, the ore value over the project time is simulated using the mean reversion process, normal and uniform probability density functions. The ore demand rate is simulated using normal and uniform probability density functions and fuzzy number. The act of simulation is repeated 500 times and set of 500 possible states of nature is obtained. The expected values of simulated input parameters are calculated from the simulation results. Accordingly, the obtained values are as represented in Table 3.

Simulation of input parameters

S		G_i RAND	M_i RAND		Project time (year)										
					0	1	2	3	4	5	6	7	8	9	10
	Pb	0.0355	0.8727	P_{Pb} \$/t	3000	2690	2743	2498	2485	2463	2332	2307	2290	2240	2368
	Zn	0.0516	0.7215	P_{Zn} \$/t	2470	2427	2294	2205	2276	2266	2248	2227	2128	2043	1993
1				OV_{Pb} \$/t	88.41	79.29	80.84	73.64	73.24	72.60	68.75	67.99	67.50	66.03	69.79
	z_{Pb}	316 338	384 t/day	OV_{Zn} \$/t	14.73	14.47	13.68	13.15	13.58	13.51	13.41	13.28	12.69	12.18	11.89
	z_{Zn}	203 210	224 t/day	OV \$/t	103.14	93.76	94.52	86.79	86.82	86.11	82.16	81.27	80.19	78.21	81.68
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	Pb	0.0355	0.8727	P_{Pb} \$/t	3000	2690	2743	2498	2485	2463	2332	2307	2290	2240	2368
	Zn	0.0516	0.7215	P_{Zn} \$/t	2470	2427	2294	2205	2276	2266	2248	2227	2128	2043	1993
500				OV_{Pb} \$/t	88.41	79.29	80.84	73.64	73.24	72.60	68.75	67.99	67.50	66.03	69.79
	z_{Pb}	316 338	384 t/day	OV_{Zn} \$/t	14.73	14.47	13.68	13.15	13.58	13.51	13.41	13.28	12.69	12.18	11.89
	z_{Zn}	203 210	224 t/day	OV \$/t	103.14	93.76	94.52	86.79	86.82	86.11	82.16	81.27	80.19	78.21	81.68
*E	z_{Pb}	300 322	365 t/day												
	z_{Zn}	207 214	229 t/day												
	max	300 322	365 t/day	OV \$/t	107.85	101.35	96.60	93.03	90.57	88.43	86.68	85.30	84.36	83.54	83.01
σ_t					4.059	4.745	4.892	4.882	5.015	5.061	5.005	4.857	4.890	4.882	4.653

*E – expected value

Transformation of obtained pdf_t sequence of OV_t into a sequence of TFN_t of OV_t is represented in Table 4. Transformation of pdf_t, $t = 1$ of OV_t into a TFN_t of OV_t is represented in Figure 5 and 6.

Transformation of simulated OV_t values into TFN_t values

	Project time (year)										
	0	1	2	3	4	5	6	7	8	9	10
OV \$/t	107.85	101.35	96.60	93.03	90.57	88.43	86.68	85.30	84.36	83.54	83.01
σ_t	4.059	4.745	4.892	4.882	5.015	5.061	5.005	4.857	4.890	4.882	4.653
*TFN_t	99.73	91.86	86.82	80.81	80.54	78.31	76.67	75.59	74.58	73.78	73.70
	107.85	101.35	96.60	93.03	90.57	88.43	86.68	85.30	84.36	83.54	83.01
	115.97	110.84	106.38	102.79	100.60	98.55	96.69	95.01	94.14	93.30	92.32

*TFN_t – (99.73 107.85 115.97)

Average value of the cost of capital tied up over the project time is:

$$\bar{\bar{C}}_{tup} = \frac{(0.03 \quad 0.05 \quad 0.07)}{365} \otimes (79.25 \quad 89.29 \quad 99.06) = (0.00651 \quad 0.01223 \quad 0.01899) \quad \text{\$/tday}$$

Algebraically, we obtain the total fuzzy cost function:

$$\tilde{C} = \begin{pmatrix} 0.008265x_1 + 250/x_3 + 8400, \ 0.016123x_2 + \\ 462/x_2 + 9660, \ 0.0295x_3 + 863/x_1 + 12410 \end{pmatrix} \cdot 365$$
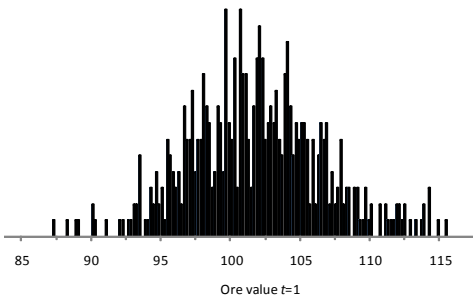
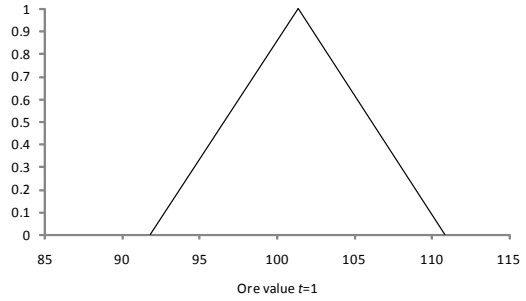Fig. 5. Probability density function of OV_t, $t = 1$

Fig. 6. Triangular fuzzy number of OV_t, $t = 1$

Using equation 7, we defuzzify the total cost function and obtain planning model as follows:

$$\min C = x_1 + 30417/x_3 + 1.96x_2 + 56210/x_2 + 3.58x_3 + 104965/x_1 + 3707183$$

subject to $0 < x_1 \leq x_2 \leq x_3$.

Results of Lagrangian Method application are represented in Table 5.

TABLE 5

Results of Langrangian method

Iteration	Ore inventory level (t/day)	Constraint
1.	$x_1 = 323$ $x_2 = 169$ $x_3 = 92$	Not satisfied
2.	$x_1 = 233$ $x_2 = 233$ $x_3 = 92$	Not satisfied
3.	$x_1 = 135$ $x_2 = 135$ $x_3 = 135$	Satisfied

Parameters of optimal production planning are represented in Table 6.

TABLE 6

Production planning parameters

Parameter	Value
1	2
Ore inventory level	135 t/day
Number of production cycles per year	100 cycle
Cycle time	3.65 day
Production time per cycle	3 day/cycle
Ore inventory depletion time per cycle	0.65 day/cycle
Ore quantity produced per cycle	1131 t/cycle
Working days per year	300 day/year

1	2
Non-working days per year	65 day/year
Annual production	113100 t/year
Daily production	377 t/day
Cost function	3709488 $

From Table 6, we can see that the company has 300 days per year to mine the ore and 65 days for planned and unplanned production interruptions, i.e. for production system maintenance. Every day 377 tonnes must be mined; ore inventory level is 135 t/day.

5. Sensitivity analysis

The sensitivity analysis of the developed model is performed by changing the values of the following fuzzy parameters z, K, a, b, φ by -30%, -20%, -10%, $+10\%$, $+20\%$ and $+30\%$, taking one parameter at a time and keeping the remaining parameters unchanged. According to $q > 1$, value of the fuzzy parameter q is changed by $+10\%$, $+20\%$ and $+30\%$. We refer to the data used in the above example as the basic data set and study sensitivity of the optimal solution to changes in the values of the different input parameters. The changes in the optimal decision values of x, n^*, t^*, Y^* and C are represented in Table 7.

TABLE 7

Sensitivity analysis

Parameter	% Change	% Change in x	% Change in n^*	% Change in t^*	% Change in Y^*	% Change in C
1	2	3	4	5	6	7
$\tilde{z}$	−30	−16.4462	−16.0000	16.0000	−16.3353	−29.9915
	−20	−10.6772	−11.0000	11.0000	−10.5587	−19.9941
	−10	−5.25885	−5.0000	5.0000	−5.13319	−9.99697
	+10	4.740286	5.0000	−5.0000	4.879199	9.996822
	+20	9.397661	9.0000	−9.0000	9.542751	19.99351
	+30	13.8647	14.0000	−14.0000	14.01571	29.99007
$\tilde{K}$	−30	−16.4462	19.0000	−19.0000	−16.3353	−0.01015
	−20	−10.6772	12.0000	−12.0000	−10.5587	−0.00656
	−10	−5.25885	5.0000	−5.0000	−5.13319	−0.00319
	+10	4.740286	−5.0000	5.0000	4.879199	0.003035
	+20	9.397661	−9.0000	9.0000	9.542751	0.005933
	+30	13.8647	−12.0000	12.0000	14.01571	0.008712
$\tilde{a}$	−30	0.0000	0.0000	0.0000	0.0000	−29.9814
	−20	0.0000	0.0000	0.0000	0.0000	−19.9876
	−10	0.0000	0.0000	0.0000	0.0000	−9.99378
	+10	0.0000	0.0000	0.0000	0.0000	9.99379
	+20	0.0000	0.0000	0.0000	0.0000	19.98758
	+30	0.0000	0.0000	0.0000	0.0000	29.98136

1	2	3	4	5	6	7
$\widetilde{b}$	−30	10.00237	−10.0000	10.0000	10.14826	−0.00622
	−20	6.233307	−7.0000	7.0000	6.3742	−0.00407
	−10	2.877447	−3.0000	3.0000	3.01389	−0.00199
	+10	−2.85505	3.0000	−3.0000	−2.72622	0.001932
	+20	−5.32836	6.0000	−6.0000	−5.2028	0.0038
	+30	−7.58848	9.0000	−9.0000	−7.46592	0.005611
$\widetilde{q}$	–	–	–	–	–	–
	–	–	–	–	–	–
	–	–	–	–	–	–
	+10	24.4375	25.0000	−25.0000	−26.523	0.015859
	+20	44.34804	42.0000	−42.0000	−36.0681	0.026561
	+30	62.00219	55.0000	−55.0000	−40.6886	0.03437
$\widetilde{\varphi}$	−30	4.905222	−6.0000	6.0000	5.044354	−0.00327
	−20	3.127228	−4.0000	4.0000	3.264001	−0.00215
	−10	1.450479	−2.0000	2.0000	1.585028	−0.00107
	+10	−1.63432	2.0000	−2.0000	−1.50386	0.001051
	+20	−3.05734	3.0000	−3.0000	−2.92877	0.00208
	+30	−4.40936	5.0000	−5.0000	−4.28258	0.003092

Here we have assumed the following sensitivity intervals (*SI*); $SI = \max - \min$:
- $SI \leq 10$ imply insensitivity,
- $10 < SI \leq 20$ imply slight sensitivity,
- $20 < SI \leq 30$ imply moderate sensitivity
- $30 < SI \leq 40$ imply high sensitivity,
- $40 < SI$ imply very high sensitivity.

A careful analysis of Table 7 reveals the following points, see Table 8:

TABLE 8

Sensitivity between input and output parameters

Parameter	*x*	*n**	*t**	*Y**	*C*
$\widetilde{z}$	high sensitivity	moderate sensitivity	moderate sensitivity	high sensitivity	very high sensitivity
$\widetilde{K}$	high sensitivity	high sensitivity	high sensitivity	high sensitivity	insensitivity
$\widetilde{a}$	insensitivity	insensitivity	insensitivity	insensitivity	very high sensitivity
$\widetilde{b}$	slight sensitivity	slight sensitivity	slight sensitivity	slight sensitivity	insensitivity
$\widetilde{q}$	high sensitivity	moderately sensitivity	moderately sensitivity	high sensitivity	insensitivity
$\widetilde{\varphi}$	insensitivity	slight sensitivity	slight sensitivity	insensitivity	insensitivity

5. Conclusion

This paper offers a framework in which an underground lead-zinc mine production can be planned by implementing the inventory management based on the uncertainty of input data. Paying attention to the complexity and dynamic nature of the problem, it is necessary to consider factors generating uncertainty in the ore production planning. Therefore, we applied the fuzzy stochastic concept to identify the level of uncertainty in the process of planning. Inventory theory is used for obtaining optimal parameters of the ore production in order to avoid the negative influence of the production interruption and satisfy the market demands in continuous way. The developed model is very flexible and can be implemented in all mining companies because of its adjustment to the specific conditions prevailing in each mine in particular. The model is not closed, and can be extended by some constraints related to specific characteristics of production, the limitation on the available ore storage space, the limit of the total investment to increase the ore storage capacity, etc. In the future research, we plan to examine the problem of changes in demands over the project time, and find the optimal production parameters for every year of the project in particular.

Acknowledgement

References

Azimi Y., Osanloo M., Esfahanipour., 2012. *Selection of the open pit mining cut-off grade strategy under price uncertainty using a risk based multi-criteria ranking system*. Arch. Min. Sci., Vol. 57, No 3, p. 741-768.

Bellman R.E., Zadeh L.A., 1977. *Local and fuzzy logics*. [In:] Dunn J. M., Epstein G. (eds) Modern uses of multiple--valued logic. Kluwer, Boston, p. 105-151, 158-165.

Bojadziev G., Bojadziev M., 1998. *Fuzzy sets and fuzzy logic applications*. World Scientific, Singapore.

Carlyle W.M., Eaves B.C., 2001. *Underground planning at Stillwater Mining Company*. Interfaces, 31 (4), p. 50-60.

Dixit A.K., Pindyck R.S., 1994. *Investment under Uncertainty*. Princeton University Press, Princeton, N.J.

Do J., Song H., So S., Soh Y., 2005. *Comparison of deterministic calculation and fuzzy arithmetic for two prediction model equations of corrosion initiation*. Journal of Asian Architecture and Building Engineering, 4. 2., p. 447-454.

Dutta P., Chakraborty D., Roy A.R., 2007. *Continuous review inventory model in mixed fuzzy and stochastic environment*. Applied Mathematics and Computation, 188, p. 970-980.

Gillenwater E.L., Bernardo J.J., Lineberry G.T., 1995. *A Hierarchical Production Planning Approach to Work Unit Scheduling in the Coal Industry*. Mineral Resources Engineering, Vol. 4, No. 1, Imperial College Press, p. 51-69.

Gligoric Z., Beljic C., Gluscevic B., Jovanovic S., 2011. *Hybrid model of evaluation of underground lead – zinc mine capacity expansion project using Monte Carlo simulation and fuzzy numbers*. Simulation: Transactions of the Society for Modeling and Simulation International, 87, 8, 726-742.

Kaufmann A., Gupta M.M., 1985. *Introduction to Fuzzy Arithmetic: Theory and Applications*. Van Nostrand Reinhold, New York.

Little J., Topal E., 2011. *Strategies to assist in obtaining an optimal solution for an underground mine planning problem using Mixed Integer Programming*. International Journal of Mining and Mineral Engineering, Vol. 3, No. 2, p. 152-172.

Mahata G. C., Goswami A., 2006. *Production lot size model with fuzzy production rate and fuzzy demand rate for deteriorating item under permissible delay in payments*. Journal of the Operational Research Society of India, 43 (4), p. 358-375.

Martinez M.A., Newman A.M., 2011. *A solution approach for optimizing long-and short-term production scheduling at LKAB'S Kiruna mine*. European Journal of Operational Research, Vol. 211, Iss. 1, p. 184-197.

Ritha W., Rexlin Jeyakumari S., 2011. *Multi level inventory management decisions with transportation cost consideration in fuzzy environment*. Annals of Fuzzy Mathematics and Informatics, Vol. 2, No. 2, p. 171-181.

Rivera L.A.R., Hubele N.F., Lawrence F.P., 1995. C_{pk} *index estimation using data transformation*. Computers Industrial Engineering, 29 (1-4), p. 55-58.

Sarin S., West-Hansen J., 2005. *The Long-Term Mine Production Scheduling Problem*. IIE Transactions, 37(2), p. 109-121.

Schwartz E., 1997. *The stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedging*. The Journal of Finance, Vol. 52, No. 3, p. 923-973.

Seda M., 2005. *Fuzzy Shortest Paths Approximation for Solving the Fuzzy Steiner Tree Problem in Graphs*. International Journal of Mathematical and Computer Science, 1:3, p. 134-138.

Smith M.L., Shepard I., Karunatillake G., 2003. *Using MIP for strategic life-of-mine planning of the lead/zinc stream at Mount Isa Mines*. Proceedings, 31st International Symposium, Application of Computers in the Mineral Industry (APCOM), Cape town, South Africa, p. 465-474.

Swishchuk A.,Ware A., Li H., 2008. *Option pricing with stochastic volatility using fuzzy sets theory*. Northern Finance Association Conference Paper http://www.northernfinance.org/2008/papers/182.pdf

Yager R.R., 1981. *A procedure for ordering fuzzy subsets of the unit interval*. Information Sciences, 24, p. 143-161.

Zadeh L.A., 1965. *Fuzzy sets*. Information and Control, 8, p. 338-353.

Zimmermann H.J., 1992. *Fuzzy set theory and its applications*. Kluwer, Boston.