Opuscula Math. 41, no. 6 (2021), 881–898
https://doi.org/10.7494/OpMath.2021.41.6.881 Opuscula Mathematica

μ-HANKEL OPERATORS ON HILBERT SPACES

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Communicated by P.A. Cojuhari

Abstract. A class of operators is introduced (μ-Hankel operators, μ is a complex parameter), which generalizes the class of Hankel operators. Criteria for boundedness, compactness, nuclearity, and finite dimensionality are obtained for operators of this class, and for the case $|\mu| = 1$ their description in the Hardy space is given. Integral representations of μ-Hankel operators on the unit disk and on the Semi-Axis are also considered.

Keywords: Hankel operator, μ-Hankel operator, Hardy space, integral representation, nuclear operator, integral operator.

Mathematics Subject Classification: 47B25, 47B35.

1. INTRODUCTION

As is well known, classical Hankel operators form one of the most significant classes of operators in spaces of analytic functions. This class has a lot of applications to various parts of Analysis, Probability, Control Theory, etc. (see [11–13]). Therefore, it is not surprising that there are a large number of generalizations and analogues of operators of this class (see ibid, and also, e.g., [8], and the bibliography therein).

Hankel operators are significant, in particular, for an important class of Toeplitz operators (see [11,13]). On the other hand, an interesting generalization of Toeplitz operators (the "λ-Toeplitz operators") was given in [5] (see also [7]). In this paper, we consider a new class of operators in Hilbert spaces (the "μ-Hankel operators"; μ is a complex parameter), which are related to λ-Toeplitz operators as well as Hankel operators with Toeplitz ones. We give criteria for boundedness, compactness, nuclearity, and finite dimensionality for μ-Hankel operators. Operators of this class turned out to be associated with the Hankel ones in the case $|\mu| = 1$, but a new feature is the nuclearity of these operators for $|\mu| \neq 1$.

The last part of the paper can be considered as a contribution to the theory of integral operators. We shaw that some natural classes of integral operators are μ-Hankel and apply our results obtained in an abstract setting to these operators. In particular, μ-Hankel operators are closely related to the complex moment problem.

2. BOUNDEDNESS AND NUCLEARITY OF μ-HANKEL OPERATORS

Definition 2.1. Let μ, and ν be complex numbers, $\alpha = \{\alpha_j\}_{j\geq 0}$ be a sequence of complex numbers, and let $\mathcal{H}, \mathcal{H}'$ be separable Hilbert spaces. We call the operator $A_{(\mu,\nu),\alpha} : \mathcal{H} \to \mathcal{H}'$ (μ,ν)-*Hankel* if for some orthonormal bases $(e_k)_{k\geq 0} \subset \mathcal{H}$ and $(e'_j)_{j\geq 0} \subset \mathcal{H}'$ the matrix $(a_{jk})_{k,j\geq 0}$, of this operator (recall that $a_{jk} = \langle A_{(\mu,\nu),\alpha} e_k, e'_j\rangle$; here and below, the angle brackets denote the dot product) consists of elements of the form

$$a_{jk} = \mu^k \nu^j \alpha_{k+j}.$$

In particular, $A_{(1,1),\alpha}$ is a Hankel operator (for the latter see, e.g., [11, 13]).

Remark 2.2. For accuracy one can assume that the operator $A_{(\mu,\nu),\alpha}$ is initially defined on the linear span of the set $\{e_k : k \in \mathbb{Z}_+\}$.

Instead of $A_{(\mu,1),\alpha}$, we will further write $A_{\mu,\alpha}$ (or A_μ) and call such an operator μ-*Hankel*. To avoid the trivial case, for these operators we will assume that $\mu \neq 0$.

Thus, the matrix of a μ-Hankel operator has the form $(\mu^k \alpha_{k+j})_{k,j\geq 0}$, i. e.

$$(a_{jk})_{k,j\geq 0} = \begin{pmatrix} \alpha_0 & \mu\alpha_1 & \mu^2\alpha_2 & \mu^3\alpha_3 & \mu^4\alpha_4 & \dots \\ \alpha_1 & \mu\alpha_2 & \mu^2\alpha_3 & \mu^3\alpha_4 & \dots & \\ \alpha_2 & \mu\alpha_3 & \mu^2\alpha_4 & \dots & & \\ \alpha_3 & \mu\alpha_4 & \dots & & & \\ \alpha_4 & \dots & & & & \\ \vdots & & & & & \end{pmatrix}. \tag{2.1}$$

For what follows, it is useful to note that the adjoint operator $A^*_{(\mu,\nu),\alpha}$ has a matrix $\overline{\mu}^j \overline{\nu}^k \overline{\alpha_{k+j}}$ (the bar denotes complex conjugation), and, therefore, $A^*_{(\mu,\nu),\alpha} = A_{(\overline{\nu},\overline{\mu}),\overline{\alpha}}$. In particular, $A^*_{\mu,\alpha} = A_{(1,\overline{\mu}),\overline{\alpha}}$.

Since $\mu^k \nu^j \alpha_{k+j} = (\mu/\nu)^k (\nu^{k+j}\alpha_{k+j})$ for $\nu \neq 0$, we have $A_{(\mu,\nu),\alpha} = A_{\mu/\nu,\alpha'}$, where $\alpha'_k = \nu^k \alpha_k$, and thus the consideration of (μ,ν)-Hankel operators is reduced to the consideration of μ-Hankel operators, what we are going to do in this paper.

Like for Hankel operators, μ-Hankel operators can be characterized as operators satisfying some commuting relation.

Theorem 2.3. *A bounded operator A in the space $\ell^2(\mathbb{Z}_+)$ is μ-Hankel if and only if the following commuting relation is true:*

$$AS = \mu S^* A, \tag{2.2}$$

where S is a shift in $\ell^2(\mathbb{Z}_+)$.

Proof. Let the operator A be μ-Hankel. For all $k, j \in \mathbb{Z}_+$ we have

$$\begin{aligned} \langle ASe_k, e_j\rangle &= \langle Ae_{k+1}, e_j\rangle = \mu^{k+1}\alpha_{k+j+1} \\ &= \mu\langle Ae_k, e_{j+1}\rangle = \mu\langle Ae_k, Se_j\rangle \\ &= \langle \mu S^* Ae_k, e_j\rangle. \end{aligned}$$

Due to the boundedness of the operator A, this implies (2.2).

Conversely, let (2.2) be valid. Then for $k, j \in \mathbb{Z}_+$, $k > 0$ we have

$$\begin{aligned} a_{jk} &:= \langle Ae_k, e_j\rangle = \langle ASe_{k-1}, e_j\rangle \\ &= \langle \mu S^* Ae_{k-1}, e_j\rangle = \mu\langle Ae_{k-1}, Se_j\rangle \\ &= \mu\langle Ae_{k-1}, e_{j+1}\rangle = \mu a_{(j+1),(k-1)}. \end{aligned}$$

Iterating this equality, we have $a_{jk} = \mu^k a_{(k+j),0}$. Since the last equality is obvious for $k = 0$, the operator A is μ-Hankel, and the theorem is proved. □

Corollary 2.4. *If a bounded operator A is μ-Hankel, then its kernel* Ker$A$ *is an invariant subspace of the shift operator. Therefore, if A is defined in the Hardy space $H^2 = H^2(\mathbb{T})$ ($\mathbb{T}$ is a unit circle), then* Ker$A$ *has the form θH^2, where θ is an inner function.*

The next theorem gives, in particular, criteria for boundedness of operators of the class under consideration. Below $\widehat{\psi}(n)$ denotes the nth Fourier coefficient of the function ψ.

Theorem 2.5. *Let $\mathcal{H}, \mathcal{H}'$ be a separable infinite-dimensional Hilbert spaces, and $A_\mu = A_{\mu,\alpha} : \mathcal{H} \to \mathcal{H}'$ a μ-Hankel operator. The following statements are true.*

(1) *Let $|\mu| < 1$. The operator A_μ is bounded if and only if $(\alpha_k) \in \ell^2(\mathbb{Z}_+)$. In this case, A_μ is nuclear with the Hilbert–Schmidt norm*

$$\|A_\mu\|_{\mathrm{S}_2} = \left(\sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n=k}^{\infty} |\alpha_n|^2\right)^{1/2} \tag{2.3}$$

and with a trace

$$\mathrm{tr} A_\mu = \sum_{n=0}^{\infty} \mu^n \alpha_{2n}. \tag{2.4}$$

(2) *Let $|\mu| > 1$. The operator A_μ is bounded if and only if $(\mu^k \alpha_k) \in \ell^2(\mathbb{Z}_+)$. Moreover, in this case A_μ is nuclear, and its trace is represented by the formula* (2.4)*.*

(3) *Let $|\mu| = 1$. Then $A_\mu = V_\mu \Gamma_\mu$, where $\Gamma_\mu : \mathcal{H} \to \mathcal{H}$ is Hankel with matrix $(\mu^{k+j}\alpha_{k+j})$, and $V_\mu : \mathcal{H} \to \mathcal{H}'$ is a unitary operator.*
In particular, the operator A_μ is bounded if and only if there is such a function $\psi \in L^\infty(\mathbb{T})$ that $\mu^n \alpha_n = \widehat{\psi}(n)$ for $n \in \mathbb{Z}_+$. In addition,

$$\|A_\mu\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \alpha_n = \widehat{\psi}(n) \forall n \in \mathbb{Z}_+\}.$$

Proof. (1) Let $|\mu| < 1$. Notice that

$$A_\mu e_0 = \sum_j a_{j0} e'_j = \sum_j \alpha_j e'_j.$$

Thus, if A_μ is bounded then $\sum_j |\alpha_j|^2 = \|A_\mu e_0\|^2 \le \|A_\mu\|^2$, which proves the necessity.

Now let $\alpha \in \ell^2(\mathbb{Z}_+)$. We shall show that A_μ belongs to the Hilbert–Schmidt class $\mathbf{S}_2$ (and thus it is bounded). It is known (see, e.g., [14, p.152, Th. 6.22]), that for an operator A with matrix (a_{jk}) to belong to the Hilbert–Schmidt class $\mathbf{S}_2$, it suffices that

$$C^2 := \sum_j \sum_k |a_{jk}|^2 < \infty,$$

and its Hilbert–Schmidt norm is $\|A\|_{\mathbf{S}_2} = C$. In our case, $\alpha \in \ell^2(\mathbb{Z}_+)$, and therefore

$$\begin{aligned}\sum_j \sum_k |a_{jk}|^2 &= \sum_j \sum_k |\mu|^{2k} |\alpha_{j+k}|^2 = \sum_k |\mu|^{2k} \sum_j |\alpha_{j+k}|^2 \\ &= \sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n\geq k}^{2k} |\alpha_n|^2 \leq \|\alpha\|_{l^2}^2 \sum_{k=0}^{\infty} (|\mu|^2)^k = \frac{\|\alpha\|_{l^2}^2}{1-|\mu|^2} < \infty.\end{aligned}$$

Hence, $A_\mu \in \mathbf{S}_2$ and

$$\|A_\mu\|_{\mathrm{S}_2} = \left(\sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n=k}^{\infty} |\alpha_n|^2 \right)^{1/2}.$$

Moreover, the operator A_μ is nuclear, since under our assumptions

$$\mathrm{tr} A_\mu = \sum_{n=0}^{\infty} a_{nn} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n} < \infty$$

(see, e.g., [4, Theorem 8.1]).

(2) Let $|\mu| > 1$. We denote $\alpha'_k := \mu^k \alpha_k$. As proved above the $1/\overline{\mu}$-Hankel operator $A_{1/\overline{\mu},\alpha'}$ is bounded if and only if $(\mu^k \alpha_k) = (\alpha'_n) \in \ell^2(\mathbb{Z}_+)$. The operator $A_{\mu,\alpha}$ is conjugate to the operator $A_{1/\overline{\mu},\alpha'}$ since the matrices of these operators in the bases (e_k) and (e'_j) coincide. This, in turn, means that the operator $A_{\mu,\alpha}$ is bounded if and only if $(\mu^k \alpha_k) \in \ell^2(\mathbb{Z}_+)$. Moreover, this operator is nuclear together with $A_{1/\overline{\mu},\alpha'}$ (see, e.g., [4]) and by the formula for the trace, proved in paragraph 1),

$$\mathrm{tr} A_{\mu,\alpha} = \overline{\mathrm{tr} A_{1/\overline{\mu},\alpha'}} = \overline{\sum_{n=0}^{\infty} \frac{1}{\overline{\mu}^n} \overline{\alpha}'_{2n}} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n}.$$

(3) Let $|\mu| = 1$. Consider the operator $V_\mu : \mathcal{H} \to \mathcal{H}'$, $V_\mu x := \sum_{j=0}^{\infty} \overline{\mu}^j x_j e'_j$ for $x \in \mathcal{H}$, $x = \sum_{j=0}^{\infty} x_j e_j$. Since $|\mu| = 1$, the operator V_μ is unitary. Then the operator $\Gamma_\mu := V_\mu^{-1} A_\mu$ is Hankel with the matrix $(\alpha'_{k+j})_{j,k\geq 0} = (\mu^{j+k} \alpha_{k+j})_{j,k\geq 0}$, because in view of $A_\mu e_k = \sum_{j=0}^{\infty} \mu^k \alpha_{k+j} e_j$ we have $\Gamma_\mu e_k = V_\mu^{-1} A_\mu e_k = \sum_{j=0}^{\infty} \mu^{j+k} \alpha_{k+j} e_j$. But $A_{\mu,\alpha}$ is bounded if and only if Γ_μ is bounded, and by the Nehari Theorem (see, e.g., [13, Theorem 1.1.1]) this is equivalent to the fact that there exists a function $\psi \in L^\infty(\mathbb{T})$, such that $\widehat{\psi}(n) = \alpha'_n = \mu^n \alpha_n$ for $n \in \mathbb{Z}_+$. Moreover, by virtue of the same theorem

$$\|A_{\mu,\alpha}\| = \|\Gamma_\mu\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \mu^n \alpha_n = \widehat{\psi}(n) \forall n \in \mathbb{Z}_+\},$$

which completes the proof. □

Example 2.6 (the generalized Hilbert matrix)**.** Let $\alpha_n = \frac{1}{n+1}$, $n \in \mathbb{Z}_+$. The corresponding μ-Hankel operator in $\ell^2(\mathbb{Z}_+)$ will be denoted by H_μ. (Thus, the operator H_1 is classical and has the Hilbert matrix, see, e.g., [13, p. 6]). According to Theorem 2.5 three cases are possible.

(1) $|\mu| < 1$. Then H_μ is nuclear and

$$\mathrm{tr} H_\mu = \sum_{n=0}^{\infty} \frac{\mu^n}{2n+1} = \frac{1}{2\sqrt{\mu}} \log \frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}.$$

(2) $|\mu| > 1$. In this case H_μ is unbounded in $\ell^2(\mathbb{Z}_+)$.

(3) $|\mu| = 1$, $\mu = e^{i\theta}$. In this case, $H_\mu = V_\mu \Gamma_\mu$, where Γ_μ is Hankel in $\ell^2(\mathbb{Z}_+)$ with matrix $(\mu^{k+j}/k+j+1)$, and V_μ is unitary in $\ell^2(\mathbb{Z}_+)$. Consider the bounded function ψ_θ on $\mathbb{T}$ defined by

$$\psi_\theta(e^{it}) = ie^{-i(t-\theta)}(\pi - (t-\theta)), t \in [0, 2\pi).$$

It is easy to see that $\widehat{\psi_\theta}(n) = e^{in\theta}\widehat{\psi_0}(n) = \frac{\mu^n}{n+1} = \mu^n \alpha_n$ for $n \in \mathbb{Z}_+$. Thus, the operator H_μ is bounded in $\ell^2(\mathbb{Z}_+)$ and $\|H_\mu\| = \|\Gamma_\mu\|$.

Corollary 2.7. *The operator $A_{\mu,\alpha}$ is not left-Fredholm provided it is bounded.*

Proof. In cases (1) and (2) of Theorem 2.5 this follows from the compactness of this operator. In case (3) the failure of left-Fredholmness for $A_{\mu,\alpha}$ follows from the failure of left-Fredholmness for Hankel operators (see, e.g., [13]). □

Recall the definition of the degree of a rational function $R = P/Q$ (P and Q are polynomials of degree $\deg P$ and $\deg Q$). If the fraction P/Q is irreducible, then the value $\deg R = \max\{\deg P, \deg Q\}$ is called the *degree of the function R*. It is equal to the sum of the multiplicities of the poles R (taking into account the possible pole at infinity).

Following [13], we associate with the sequence of complex numbers $(\alpha_k)_{k\geq 0}$ the formal power series

$$\alpha(z) := \sum_{k=0}^{\infty} \alpha_k z^k. \tag{2.5}$$

Kronecker's theorem for Hankel operators readily implies

Theorem 2.8. *The matrix* (2.1) *of the operator $A_{\mu,\alpha}$ has finite rank if and only if the series* (2.5) *defines a rational function. Moreover, the rank of the matrix* (2.1) *is* $\deg(z\alpha(z))$.

Proof. From the form of the matrix (2.1) it immediately follows that the number of its linearly independent columns is equal to the number of linearly independent columns of the matrix $(\alpha_{j+k})_{j,k\geq 0}$. It remains to apply Kronecker's theorem in the formulation proposed in [13, Theorem 1.3.1]. □

3. μ-HANKEL OPERATORS IN THE HARDY SPACE

As is known (see, e.g., [11, 13]) *Hardy space* consists of functions f analytic in the unit disk $\mathbb{D}$ for which $(\widehat{f}(n))_{n\geq 0}\in \ell^2(\mathbb{Z}_+)$, where $\widehat{f}(n)$ denotes the nth Taylor coefficient of a function f, $n\in\mathbb{Z}_+$. Moreover, $\|f\|_{H^2}=\|(\widehat{f}(n))_{n\geq 0}\|_{\ell^2}$. Thus, the mapping $f\mapsto(\widehat{f}(n))_{n\geq 0}$ is an isomorphism of the Hilbert spaces $H^2(\mathbb{D})$ and $\ell^2(\mathbb{Z}_+)$. Equivalently, $H^2(\mathbb{D})$ consists of functions f analytic in $\mathbb{D}$ and satisfying the condition

$$\|f\|_{H^2}^2 := \sup_{0<r<1}\int\limits_{-\pi}^{\pi}|f(re^{i\theta})|^2\frac{d\theta}{2\pi}<\infty.$$

The space $H^2(\mathbb{D})$ can also be identified with the following subspace of the space $L^2(\mathbb{T})$:

$$\{f\in L^2(\mathbb{T}):\widehat{f}(n)=0 \text{ for all } n<0\},$$

where $\widehat{f}(n)$ denotes the nth Fourier coefficient of the function f, $n\in\mathbb{Z}$ (see, e.g., [11]). The functions $\chi_n(z):=z^n$ $(n\in\mathbb{Z}_+)$ form a standard orthonormal basis of H^2. We also put $H^2_- := L^2(\mathbb{T})\ominus H^2$. The functions $\overline{\chi_{n+1}}$ $(n\in\mathbb{Z}_+)$ form the standard orthonormal basis of this space.

The next theorem describes bounded μ-Hankel operators in Hardy space for $|\mu|=1$.

Theorem 3.1. *Let $|\mu|=1$. For the operator $A:H^2\to H^2_-$ the following statements are equivalent:*

(1) *A has a μ-Hankel matrix in standard bases and is bounded.*
(2) *$A=V_\mu H_\varphi$, where the operator $V_\mu f(z):=f(\mu z)$ is unitary in H^2_-, operator $H_\varphi: H^2\to H^2_-$ is Hankel with a symbol $\varphi\in L^\infty(\mathbb{T})$, and $\widehat{\varphi}(-n)=\mu^n\alpha_n$ $(n\in\mathbb{Z}_+)$.*
(3) *Operator A is bounded and satisfies the generalized Hankel equation*

$$\mu P_-\mathcal{S}A = AS,$$

where $Sf(z):=zf(z)$ is the unilateral shift in H^2, $(\mathcal{S}g)(t)=tg(t)$ $(t\in\mathbb{T})$ is the bilateral shift in $L^2(\mathbb{T})$, and $P_-:L^2\to H^2_-$ is the orthogonal projection.
(4) *$A=H_\psi U_\mu$, where $(U_\mu f)(z)=f(\mu z)$ is unitary in H^2, and $H_\psi:H^2\to H^2_-$ is Hankel with a symbol $\psi\in L^\infty(\mathbb{T})$ and $\widehat{\psi}(-n)=\alpha_n$ $(n\in\mathbb{Z}_+)$.*

Proof. The proof will be carried out according to the scheme (1)⇒(2)⇒ (3)⇒(4)⇒ (1).

(1)⇒(2) Evidently, V_μ is unitary in H^2_-. We shall consider the operator $B:=V_\mu^{-1}A:H^2\to H^2_-$ and compute its matrix in standard bases. As $V_\mu\overline{\chi_{j+1}}(z)=\overline{\mu}^{j+1}\overline{\chi_{j+1}(z)}$, we have

$$\begin{aligned}\langle B\chi_k,\chi_{j+1}\rangle &= \langle V_\mu^{-1}A\chi_k,\overline{\chi_{j+1}}\rangle = \langle A\chi_k, V_\mu\overline{\chi_{j+1}}\rangle = \mu^{j+1}\langle A\chi_k,\chi_{j+1}\rangle\\ &= \mu^{j+1}\mu^k\alpha_{k+j+1}=\mu^{k+j}(\mu\alpha_{k+j+1}).\end{aligned}$$

This expression depends only on $k+j$, that is, the operator B is Hankel, $B=H_\varphi$, $\varphi\in L^\infty$.

Finally,

$$\widehat{\varphi}(-n) = \langle H_\varphi \chi_n, 1\rangle = \langle V_\mu^* A\chi_n, 1\rangle = \langle A\chi_n, V_\mu 1\rangle = \langle A\chi_n, \chi_0\rangle = \mu^n \alpha_n$$

for $n \in \mathbb{Z}_+$.

(2)⇒(3) If $A = V_\mu H_\varphi$, then A is bounded and, since $S\chi_k = \chi_{k+1}$ and $V_\mu^{-1}\overline{\chi_{j+1}} = \mu^{j+1}\overline{\chi_{j+1}}$, we have

$$\begin{aligned}\langle AS\chi_k, \overline{\chi_{j+1}}\rangle &= \langle V_\mu H_\varphi \chi_{k+1}, \overline{\chi_{j+1}}\rangle = \langle H_\varphi \chi_{k+1}, V_\mu^{-1}\overline{\chi_{j+1}}\rangle \\ &= \langle H_\varphi \chi_{k+1}, \mu^{j+1}\overline{\chi_{j+1}}\rangle = \overline{\mu}^{j+1}\langle H_\varphi \chi_{k+1}, \overline{\chi_{j+1}}\rangle = \overline{\mu}^{j+1}\widehat{\varphi}(-k-j-2).\end{aligned}$$

On the other hand, since $P_-\overline{\chi_{j+1}} = \overline{\chi_{j+1}}$ and $S^*\overline{\chi_{j+1}} = \overline{\chi_{j+2}}$, we have

$$\begin{aligned}\langle \mu P_- \mathcal{S} A\chi_k, \overline{\chi_{j+1}}\rangle &= \mu\langle \mathcal{S} A\chi_k, \overline{\chi_{j+1}}\rangle = \mu\langle \mathcal{S} V_\mu H_\varphi \chi_k, \overline{\chi_{j+1}}\rangle \\ &= \mu\langle V_\mu H_\varphi \chi_k, \overline{\chi_{j+2}}\rangle = \mu\langle H_\varphi \chi_k, V_\mu^{-1}\overline{\chi_{j+2}}\rangle \\ &= \mu\overline{\mu}^{j+2}\langle H_\varphi \chi_k, \overline{\chi_{j+2}}\rangle = \overline{\mu}^{j+1}\widehat{\varphi}(-k-j-2).\end{aligned}$$

Since both sides of the generalized Hankel equation contain bounded operators, they coincide on the entire space H^2.

(3)⇒(4) Let the operator A be bounded and satisfy the generalized Hankel equation. We consider the operator $H := AU_\mu^{-1} = AU_{\overline{\mu}}$ and show that it satisfies the Hankel equation $P_-\mathcal{S}H = HS$ (for the latter see, e.g., [13, Theorem 1.1.8]).

Indeed, by virtue of the generalized Hankel equation $P_-\mathcal{S}AU_{\overline{\mu}} = \overline{\mu}ASU_{\overline{\mu}}$ and thus

$$\langle P_-\mathcal{S}H\chi_k, \overline{\chi_{j+1}}\rangle = \overline{\mu}\langle ASU_{\overline{\mu}}\chi_k, \overline{\chi_{j+1}}\rangle. \tag{3.1}$$

But $SU_{\overline{\mu}}\chi_k = \overline{\mu}^k S\chi_k = \overline{\mu}^k\chi_{k+1}$, and $V_{\overline{\mu}}S\chi_k = V_{\overline{\mu}}\chi_{k+1} = \overline{\mu}^{k+1}\chi_{k+1}$. Therefore, $SU_{\overline{\mu}}\chi_k = \mu U_{\overline{\mu}}S\chi_k$. Substituting the last expression in (3.1), we get that for all $k, j \in \mathbb{Z}_+$

$$\langle P_-\mathcal{S}H\chi_k, \overline{\chi_{j+1}}\rangle = \overline{\mu}\mu\langle AU_{\overline{\mu}}S\chi_k, \overline{\chi_{j+1}}\rangle = \langle HS\chi_k, \overline{\chi_{j+1}}\rangle.$$

It follows by the continuity that $P_-\mathcal{S}H = HS$, and therefore (see, e.g., [13, Theorem 1.1.8]) H is Hankel, $H = H_\psi$, $\psi \in L^\infty$.

Moreover,

$$\widehat{\psi}(-n) = \langle H_\psi \chi_n, 1\rangle = \langle AU_{\overline{\mu}}\chi_n, 1\rangle = \overline{\mu}^n\langle A\chi_n, \chi_0\rangle = \alpha_n$$

for $n \in \mathbb{Z}_+$.

(4)⇒(1) If $A = H_\psi U_\mu$, then it is obvious that the operator A is bounded. Let us find its matrix in standard bases. Taking into account that $U_\mu\chi_k = \mu^k\chi_k$, we have

$$\langle H_\psi U_\mu \chi_k, \overline{\chi_{j+1}}\rangle = \mu^k\langle H_\psi \chi_k, \overline{\chi_{j+1}}\rangle = \mu^k\widehat{\psi}(-k-j-1)$$

and (1) follows. □

The previous theorem allows, in the case $|\mu| = 1$, to derive a number of properties of μ-Hankel operators from the corresponding properties of Hankel ones. Here are some examples.

Corollary 3.2. *Let $|\mu| = 1$, and A is μ-Hankel, $A : H^2 \to H^2_-$. Operator A is compact if and only if it can be represented in the form $A = H_\psi U_\mu$, where $\psi \in C(\mathbb{T}) + H^\infty(\mathbb{T})$.*

Proof. By Theorem 3.1, $A = H_\psi U_\mu$. Since U_μ is unitary, operator A is compact if and only if H_ψ is compact. So, the corollary follows from the Hartman's Theorem (see, e.g., [13, Theorem 1.5.5]). □

Corollary 3.3. *Let $|\mu| = 1$, and A is μ-Hankel, $A : H^2 \to H^2_-$. Then A is an operator of finite rank if and only if $A = H_\psi U_\mu$ and $P_-\psi$ is a rational function. Moreover,* $\operatorname{rank} A = \deg P_-\psi$.

Proof. In the notation of Theorem 3.1 operator A is an operator of finite rank if and only if $H_\psi = AU_\mu^{-1}$ is of finite rank. This is equivalent to the fact that $P_-\psi$ is a rational function (see, e.g., [13, Corollary 1.3.2]) and $\operatorname{rank} A = \operatorname{rank} H_\psi = \deg P_-\psi$, as required. □

Corollary 3.4. *Let $|\mu| = 1$, and operators $A, B : H^2 \to H^2_-$ are μ-Hankel and bounded, $A = H_\varphi U_\mu$, $B = H_\psi U_\mu$, $\varphi, \psi \in L^\infty$. Then the operator A^*B is unitarily equivalent to the semi-commutator $[T_{\overline{\varphi}}, T_\psi) := T_{\overline{\varphi}\psi} - T_\varphi T_\psi$ of Toeplitz operators. In particular, the operator A^*B is compact if φ or ψ belongs to $H^\infty(\mathbb{T}) + C(\mathbb{T})$.*

Proof. We have

$$A^*B = U_\mu^* H_\varphi^* H_\psi U_\mu = U_\mu^{-1} H_\varphi^* H_\psi U_\mu,\ \varphi, \psi \in L^\infty.$$

It remains to use the fact that the operator $H_\varphi^* H_\psi$ is equal to $[T_{\overline{\varphi}}, T_\psi)$ (see, e.g., [13, p. 89, (1.5)]). The compactness statement now follows from the corresponding property of Toeplitz operators (see, e.g.,[11, p. 253]). □

Corollary 3.5. *Let $|\mu| = 1$, $\varphi \in L^\infty$. The following statements are equivalent for the operator $A = V_\mu H_\varphi$:*

(1) *A has a non-trivial kernel,*
(2) *the image* $\operatorname{Im} A$ *of the operator A is not dense in H^2_-,*
(3) *$\varphi = \theta\varphi_1$ for some inner function θ and function φ_1 from H^∞.*

Proof. By virtue of the unitarity of the operator V_μ, the fulfillment of properties (1) (or (2)) for the operator A is equivalent to the fulfillment of the corresponding property for the operator H_φ. The equivalence of statements (1)–(3) now follows from [13, Theorem 1.2.3]. □

Remark 3.6. Let $|\mu| = 1$. Consider the unitary operator $W_\mu f(z) := f(\mu z)$ in $L^2(\mathbb{T})$. The restrictions $U_\mu := W_\mu | H^2(\mathbb{T})$ and $V_\mu := W_\mu | H^2_-(\mathbb{T})$ are unitary in $H^2(\mathbb{T})$ and $H^2_-(\mathbb{T})$ respectively. Let $\varphi \in L^\infty(\mathbb{T})$ and T_φ denotes the Toeplitz operator with symbol φ. Then the operator $T_{\mu,\varphi} := U_\mu T_\varphi$ is called μ-Toeplitz ([5], [7, Theorem 2.5]). On the other hand, the operator $A_\mu = V_\mu H_\varphi$ is μ-Hankel by Theorem 4. It is easy to verify that

$$T_{\mu,\varphi} f + A_\mu f = W_\mu M_\varphi f, \quad f \in H^2(\mathbb{T}),$$

where M_φ stands for multiplication by φ on $L^2(\mathbb{T})$. This is the simplest relation between μ-Hankel and μ-Toeplitz operators.

4. INTEGRAL REPRESENTATIONS

In this section, we will consider two classes of integral operators that are μ-Hankel.

4.1. μ-HANKEL OPERATORS AS INTEGRAL OPERATORS ON THE UNIT DISK

Let $\mu \in \mathbb{C}$, $\mu \neq 0$, and σ is a bounded (generally speaking, complex) measure on the closed unit disk $\overline{\mathbb{D}}$ in complex plane. In the case $|\mu| < 1$, we will assume that σ is concentrated on the set $\{|\zeta| \leq |\mu|\}$. Consider the operator

$$\Gamma_{\mu,\sigma} f(z) := \mu \int\limits_{\overline{\mathbb{D}}} \frac{f(\zeta)}{\mu - \zeta z} d\sigma(\zeta) \quad (|z| < 1)$$

and the sequence of moments of the measure σ

$$\gamma_n := \int\limits_{\overline{\mathbb{D}}} \zeta^n d\sigma(\zeta) \quad (n \in \mathbb{Z}_+).$$

Theorem 4.1. *For the operator $\Gamma_{\mu,\sigma}$ to be bounded in the Hardy space $H^2(\mathbb{D})$, the condition*

$$\sup_{k \in \mathbb{Z}_+} \sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 < \infty \tag{4.1}$$

is necessary. Under this condition, this operator is μ-Hankel in $H^2(\mathbb{D})$, has the matrix $(\gamma_{k+j}/\mu^j)_{k,j \in \mathbb{Z}_+}$ with respect to the standard basis of this space, and the following statements are true.

(1) *Let $|\mu| < 1$. Then the operator $\Gamma_{\mu,\sigma}$ is nuclear and*

$$\mathrm{tr}\Gamma_{\mu,\sigma} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \mu \int\limits_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}. \tag{4.2}$$

(2) *Let $|\mu| > 1$. Then the operator $\Gamma_{\mu,\sigma}$ is bounded if and only if $(\gamma_n) \in \ell^2(\mathbb{Z}_+)$. Moreover, it is nuclear, and its trace is expressed by the formula (4.2).*

(3) *Let $|\mu| = 1$. Operator $\Gamma_{\mu,\sigma}$ is bounded if and only if there is such function $\psi \in L^\infty(\mathbb{T})$, that $\gamma_n = \widehat{\psi}(n)$ for $n \in \mathbb{Z}_+$. Moreover*

$$\|\Gamma_{\mu,\sigma}\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \gamma_n = \widehat{\psi}(n) \quad \forall n \in \mathbb{Z}_+\}.$$

Proof. Consider the standard orthonormal basis $\chi_n(z) = z^n$ $(n \in \mathbb{Z}_+)$ in $H^2(\mathbb{D})$. Taking into account that $|\zeta| \leq |\mu|$ and $|z| < 1$, we have

$$\begin{aligned} \Gamma_{\mu,\sigma}\chi_k(z) &= \mu \int\limits_{\overline{\mathbb{D}}} \frac{\zeta^k}{\mu - \zeta z} d\sigma(\zeta) = \int\limits_{\overline{\mathbb{D}}} \frac{\zeta^k}{1 - \frac{\zeta z}{\mu}} d\sigma(\zeta) = \int\limits_{\overline{\mathbb{D}}} \zeta^k \sum_{j=0}^{\infty} \left(\frac{\zeta z}{\mu}\right)^j d\sigma(\zeta) \\ &= \sum_{j=0}^{\infty} \frac{z^j}{\mu^j} \int\limits_{\overline{\mathbb{D}}} \zeta^{k+j} d\sigma(\zeta) = \sum_{j=0}^{\infty} z^j \frac{\gamma_{k+j}}{\mu^j} = \sum_{j=0}^{\infty} \frac{\gamma_{k+j}}{\mu^j} \chi_j(z). \end{aligned} \tag{4.3}$$

(The term-by-term integration of the series is legal, since for all $k \in \mathbb{Z}_+$

$$\sum_{j=0}^{\infty} \int\limits_{\overline{\mathbb{D}}} \left| z^j \frac{\zeta^{k+j}}{\mu^j} \right| d|\sigma|(\zeta) \le \sum_{j=0}^{\infty} |z|^j \int\limits_{\overline{\mathbb{D}}} d|\sigma|(\zeta) < \infty.)$$

Thus, in order for the operator $\Gamma_{\mu,\sigma}$ to act and to be bounded in $H^2(\mathbb{D})$, it is necessary for the sequence $(\gamma_{k+j}/\mu^j)_{j\in\mathbb{Z}_+}$ to belong to $\ell^2(\mathbb{Z}_+)$ for all $k \ge 0$ and the equality

$$\sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 = \|\Gamma_{\mu,\sigma}\chi_k\|^2 \le \|\Gamma_{\mu,\sigma}\|^2.$$

to be valid. This proves the necessity of the condition (4.1). Under this condition, the operator $\Gamma_{\mu,\sigma}$ has a matrix

$$a_{jk} = \frac{\gamma_{k+j}}{\mu^j} = \mu^k \alpha_{k+j}, \text{ where } \alpha_n = \frac{\gamma_n}{\mu^n},$$

with respect to the basis $(\chi_n)_{n\ge 0}$, which proves the first assertion of the theorem.

Now, by virtue of Theorem 2.5, we can assert the following.

(1) If $|\mu| < 1$, then the operator $\Gamma_{\mu,\sigma}$ is bounded if (and only if) $(\alpha_n)_{n\ge 0} = (\gamma_n/\mu^n)_{n\ge 0} \in \ell^2(\mathbb{Z}_+)$, which is true by virtue of (4.1). Moreover, this operator is nuclear, and

$$\begin{aligned} \mathrm{tr}\Gamma_{\mu,\sigma} &= \sum_{n=0}^{\infty} \mu^n \alpha_{2n} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \sum_{n=0}^{\infty} \frac{1}{\mu^n} \int\limits_{\overline{\mathbb{D}}} \zeta^{2n} d\sigma(\zeta) \\ &= \int\limits_{\{|\zeta|\le|\mu|\}} \sum_{n=0}^{\infty} \left(\frac{\zeta^2}{\mu} \right)^n d\sigma(\zeta) = \mu \int\limits_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}. \end{aligned}$$

The term-by-term integration of the series is legal, since

$$\sum_{n=0}^{\infty} \int\limits_{\{|\zeta|\le|\mu|\}} \left| \frac{\zeta^2}{\mu} \right|^n d|\sigma|(\zeta) = \sum_{n=0}^{\infty} |\mu|^n \int\limits_{\{|\zeta|\le|\mu|\}} \left| \frac{\zeta}{\mu} \right|^{2n} d|\sigma|(\zeta) \le \frac{|\sigma|(\overline{\mathbb{D}})}{1 - |\mu|} < \infty.$$

(2) If $|\mu| > 1$, then $\Gamma_{\mu,\sigma}$ is bounded if and only if $(\mu^n\alpha_n) = (\gamma_n) \in \ell^2(\mathbb{Z}_+)$. Moreover, this operator is nuclear, and

$$\mathrm{tr}\Gamma_{\mu,\sigma} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \sum_{n=0}^{\infty} \frac{1}{\mu^n} \int\limits_{\overline{\mathbb{D}}} \zeta^{2n} d\sigma(\zeta) = \mu \int\limits_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}.$$

(The term-by-term integration of the series is legal, since

$$\sum_{n=0}^{\infty} \int\limits_{\overline{\mathbb{D}}} \left| \frac{\zeta^{2n}}{\mu^n} \right| d|\sigma|(\zeta) \le \sum_{n=0}^{\infty} \left| \frac{1}{\mu} \right|^n \int\limits_{\overline{\mathbb{D}}} d|\sigma|(\zeta) < \infty.)$$

(3) Since in our case $\alpha_n = \gamma_n/\mu^n$, this follows from part (3) of Theorem 2.5 with $A\mu = \Gamma_{\mu,\sigma}$, which completes the proof. □

For $|\mu| = 1$, the sufficient boundedness condition gives the next

Corollary 4.2. *Let $|\mu| = 1$. If the function*

$$\varphi_\sigma(\zeta) := \int_{\overline{\mathbb{D}}} \frac{1 - |z|^2}{|z - \zeta|^2} d\sigma(z) \quad (\zeta \in \mathbb{T})$$

belongs to $L^\infty(\mathbb{T})$, the operator $\Gamma_{\mu,\sigma}$ is bounded in $H^2(\mathbb{D})$ and

$$\|\Gamma_{\mu,\sigma}\| \leq \|\varphi_\sigma\|_{L^\infty}.$$

Proof. We check the fulfillment of the conditions of Theorem 4.1. It is shown in [11, p. 314] that $\gamma_n = \widehat{\varphi_\sigma}(-n)$ for $n \in \mathbb{Z}_+$. Therefore we have by Parseval's equality that for all $k \in \mathbb{Z}_+$

$$\sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 = \sum_{j=0}^{\infty} |\gamma_{k+j}|^2 = \sum_{j=0}^{\infty} |\widehat{\varphi_\sigma}(-k-j)|^2 \leq \sum_{n=-\infty}^{\infty} |\widehat{\varphi_\sigma}(n)|^2 = \|\varphi_\sigma\|_{L^2}^2,$$

and therefore the condition (4.1) is satisfied. Moreover, $\gamma_n = \widehat{\varphi_\sigma^\flat}(n)$ for $n \in \mathbb{Z}_+$, where $\varphi^\flat(\zeta) := \varphi(\zeta^{-1}) = \varphi(\overline{\zeta})$, and $\varphi_\sigma^\flat \in L^\infty(\mathbb{T})$. Now it remains to apply assertion (3) of Theorem 4.1. □

Corollary 4.3. *Let the condition* (4.1) *be satisfied. The operator $\Gamma_{\mu,\sigma}$ has finite rank if and only if the function $\Gamma_{\mu,\sigma}1$ is rational. In this case* $\mathrm{rank}\Gamma_{\mu,\sigma} = \deg(z(\Gamma_{\mu,\sigma}1)(z))$.

Proof. As was shown in the proof of the previous theorem, $\alpha_n = \gamma_n/\mu^n$ for the operator $\Gamma_{\mu,\sigma}$. By Theorem 2.8, this operator has finite rank if and only if the function

$$\sum_{j=0}^{\infty} \alpha_j z^j = \sum_{j=0}^{\infty} \frac{\gamma_j}{\mu^j} z^j = \int_{\overline{\mathbb{D}}} \sum_{j=0}^{\infty} \left(\frac{\zeta z}{\mu} \right)^j d\sigma(\zeta) = \Gamma_{\mu,\sigma}1$$

is rational (the validity of the term-by-term integration of the series was substantiated in the proof of Theorem 4.1). The second statement of the corollary now also follows from Theorem 2.8. □

We are going to show that the μ-Hankel operators are related to the complex moment problem (for the latter see [2, p. 117]). The main result for the unit disk states ([2, p. 117, Theorem 4.4.12]) that for a sequence $\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ there is a bounded positive measure σ on $\mathbb{D}$ such that

$$\gamma(n, m) = \int_{\overline{\mathbb{D}}} \zeta^n \overline{\zeta}^m d\sigma(\zeta), \ (n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \tag{4.4}$$

if and only if γ is bounded and positive definite on $\mathbb{Z}_+ \times \mathbb{Z}_+$ (this means that quadratic forms $\sum_{j,k=1}^{n} c_j \overline{c_k} \gamma(s_j + s_k)$ are positive definite for all $n \in \mathbb{N}$, $\{s_1, \dots, s_n\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ [2, p. 87]).

The following proposition is a partial converse of Theorem 4.1.

Theorem 4.4. *Let a sequence* $\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ *be bounded and positive definite,* $\gamma_n = \gamma(n, 0)$*, and* $|\mu| \geq 1$*. Then there is a bounded positive measure* σ *on* $\mathbb{D}$ *such that the matrix* $(\gamma_{k+j}/\mu^j)_{k,j\in\mathbb{Z}_+}$ *corresponds to the operator* $\Gamma_{\mu,\sigma}$ *in* $H^2(\mathbb{D})$ *with respect to the standard basis of this space.*

Proof. Indeed, formula (4.4) implies that there is a bounded positive measure σ on $\mathbb{D}$ such that

$$\gamma_n = \int\limits_{\overline{\mathbb{D}}} \zeta^n d\sigma(\zeta), \ n \in \mathbb{Z}_+.$$

Since $|\mu| \geq 1$, the calculations (4.3) for the corresponding operator $\Gamma_{\mu,\sigma}$ are valid. □

We finish this section with considering the adjoint operator for $\Gamma_{\mu,\sigma}$.

Lemma 4.5. *If the operator* $\Gamma_{\mu,\sigma}$ *is bounded in* $H^2(\mathbb{D})$*, then the adjoint operator has the form*

$$\Gamma^*_{\mu,\sigma} f(z) = \int\limits_{\overline{\mathbb{D}}} \frac{f(\zeta/\mu)}{1-\zeta z} d\sigma(\zeta) \quad (|z| < 1).$$

Proof. It was shown at the beginning of the proof of Theorem 4.1 that

$$\langle \Gamma_{\mu,\sigma}\chi_k, \chi_n\rangle = \gamma_{k+n}/\mu^n \quad k, n \in \mathbb{Z}_+.$$

On the other hand, for all $n \in \mathbb{Z}_+$,

$$\Gamma^*_{\mu,\sigma}\chi_n(z) = \int\limits_{\overline{\mathbb{D}}} \frac{(\zeta/\mu)^n}{1-\zeta z} d\sigma(\zeta) = \frac{1}{\mu^n} \int\limits_{\overline{\mathbb{D}}} \zeta^n \left(\sum_{j=0}^{\infty} \zeta^j z^j\right) d\sigma(\zeta)$$

$$= \sum_{j=0}^{\infty} \frac{1}{\mu^n} \left(\int\limits_{\overline{\mathbb{D}}} \zeta^{n+j} d\sigma(\zeta)\right) z^j = \sum_{j=0}^{\infty} \frac{\gamma_{n+j}}{\mu^n}\chi_j(z).$$

The validity of the term-by-term integration of the series here follows from the fact that for $|\zeta| \leq 1$, $|\zeta/\mu| \leq 1$, $|z| < 1$ we have the estimate

$$\sum_{j=0}^{\infty} \int\limits_{\overline{\mathbb{D}}} \frac{1}{|\mu|^n} |\zeta|^{n+j}|z|^j d|\sigma|(\zeta) \leq \sum_{j=0}^{\infty} |z|^j \int\limits_{\overline{\mathbb{D}}} |\sigma|(\zeta) < \infty.$$

Thus, $\langle \Gamma_{\mu,\sigma}\chi_k, \chi_n\rangle = \langle \chi_k, \Gamma^*_{\mu,\sigma}\chi_n\rangle$ for all $k, n \in \mathbb{Z}_+$. Since the operators $\Gamma_{\mu,\sigma}$ and $\Gamma^*_{\mu,\sigma}$ are bounded, the Lemma follows. □

Theorem 4.1 and the standard facts about the adjoint operator directly imply the following

Theorem 4.6. *For the adjoint operator* $\Gamma^*_{\mu,\sigma}$ *to be bounded in the Hardy space* $H^2(\mathbb{D})$*, it is necessary that condition* (4.1) *was met. Under this condition, this operator is* $1/\overline{\mu}$*-Hankel in* $H^2(\mathbb{D})$ *with matrix* $(\overline{\gamma_{k+j}}/\overline{\mu}^k)_{k,j\in\mathbb{Z}_+}$ *with respect to the basis* $(\chi_n)_{n\geq 0}$*, the standard basis of this space, and the following statements are true.*

(1) *Let* $|\mu|<1$. *Then the operator* $\Gamma^*_{\mu,\sigma}$ *is nuclear and*

$$\mathrm{tr}\Gamma^*_{\mu,\sigma} = \sum_{n=0}^{\infty} \frac{\overline{\gamma_{2n}}}{\overline{\mu}^n} = \overline{\mu} \int\limits_{\overline{\mathbb{D}}} \frac{d\overline{\sigma}(\zeta)}{\overline{\mu} - \overline{\zeta}^2}. \tag{4.5}$$

(2) *Let* $|\mu|>1$. *In this case, the operator* $\Gamma^*_{\mu,\sigma}$ *is bounded if and only if* $(\gamma_n) \in \ell^2(\mathbb{Z}_+)$. *Moreover, this operator is nuclear, and its trace is expressed by the formula* (4.5).
(3) *Let* $|\mu|=1$. *The operator* $\Gamma^*_{\mu,\sigma}$ *is bounded if and only if there is such a function* $\psi \in L^\infty(\mathbb{T})$ *that* $\gamma_n = \widehat{\psi}(n)$ *for* $n \in \mathbb{Z}_+$. *In addition*

$$\|\Gamma^*_{\mu,\sigma}\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \gamma_n = \widehat{\psi}(n)\, \forall n \in \mathbb{Z}_+\}.$$

Corollary 4.7. *Let condition* (4.1) *be satisfied. Operator* $\Gamma^*_{\mu,\sigma}$ *has a finite rank if and only if the function* $\Gamma^*_{\mu,\sigma}1$ *is rational. Moreover,* $\mathrm{rank}\Gamma^*_{\mu,\sigma} = \deg(z(\Gamma^*_{\mu,\sigma}1)(z))$.

The proof is similar to that of Corollary 4.3.

Remark 4.8. For the case when the measure σ is concentrated on the segment $[0,1]$, operators of the form $\Gamma_{\mu,\sigma}$ were considered in [9, 10].

4.2. μ-HANKEL OPERATORS AS INTEGRAL OPERATORS ON THE SEMI-AXIS

In this subsection, we consider a certain class of μ-Hankel integral operators in the space $L^2(\mathbb{R}_+)$.

It is known (see, e.g., [15, p. 193]) that the functions

$$l_n(t) = -\imath \mathrm{L}_n(t) e^{-t/2}, \quad n \in \mathbb{Z}_+,$$

where $\mathrm{L}_n = \mathrm{L}_n^0$ are Laguerre polynomials, form an orthonormal basis of the space $L^2(\mathbb{R}_+)$. It is also clear that the functions l_n are bounded on $\mathbb{R}_+$. In what follows, we will assume that $|\mu|<1$ and put

$$k_\mu(x,t) := \sum_{n\in\mathbb{Z}_+} \mu^n l_n(t) \overline{l_n(x)}. \tag{4.6}$$

Since the series $\sum_{n\in\mathbb{Z}_+} \mu^n l_n(t)$ absolutely converges for $|\mu|<1$ and each t (see, e.g., [3], [6, Chapter XI, B]), this definition is correct, the series (4.6) absolutely converges in the norm of $L^2(\mathbb{R}_+, dx)$ for every fixed t, and therefore the function k_μ belongs to $L^2(\mathbb{R}_+)$ for each variable separately.

For a function $a \in L^2(\mathbb{R}_+)$ (which we consider to be independent of μ) we put

$$K_\mu(x,t) := \int\limits_{\mathbb{R}_+} a(x+y) k_\mu(y,t)\, dy.$$

Then the integral operator

$$\mathbf{A}_\mu f(x) := \int\limits_{\mathbb{R}_+} K_\mu(x,t)f(t)dt$$

is μ-Hankel in $L^2(\mathbb{R}_+)$. In order to check this, consider the operator

$$\mathbf{H}f(x) := \int\limits_{\mathbb{R}_+} a(x+y)f(y)dy.$$

It is known (see, e.g., [13]), that it is Hankel in $L^2(\mathbb{R}_+)$. Moreover, from the results of the work [15] (see also [13, remark after the Theorem 1.8.9]) it follows that it has Hankel matrix with respect to the basis $(l_n)_{n\in\mathbb{Z}_+}$. Indeed, it is shown in [15, p. 200] that the operator $\mathbf{H}$ is unitarily equivalent to H in $\ell^2(\mathbb{Z}_+)$, $\mathbf{H} = \mathcal{L}H\mathcal{L}^*$, where the operator $\mathcal{L} : L^2(\mathbb{R}_+) \to \ell^2(\mathbb{Z}_+)$ is unitary, and the operator H has Hankel matrix with respect to the standard basis $(e_n)_{n\in\mathbb{Z}_+}$ of the space $\ell^2(\mathbb{Z}_+)$ [15, (1.1)]. Moreover, it follows from [15, (2.27)] that $\mathcal{L}l_n = e_n$. Hence the quantity $\langle \mathbf{H}l_m, l_n\rangle = \langle He_m, e_n\rangle$ depends on $m+n$ only (and does not depend on μ).

Further, the integral operator

$$U_\mu f(x) := \int\limits_{\mathbb{R}_+} k_\mu(x,t)f(t)dt$$

has the matrix $\mathrm{diag}(1, \mu, \mu^2, \ldots)$ with respect to the basis $(l_n)_{n\in\mathbb{Z}_+}$ of $L^2(\mathbb{R}_+)$, and thus this operator is defined correctly and bounded. Moreover, the next lemma holds.

Lemma 4.9. *If $a \in L^2(\mathbb{R}_+)$ and $|\mu| < 1$, then*

$$\mathbf{A}_\mu = \mathbf{H}U_\mu,$$

and this operator is μ-Hankel.

Proof. Note that the Cauchy–Schwartz–Bunyakovskii inequality implies that

$$\int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dt \le \|k_\mu(y,\cdot)\|_{L^2}\|f\|_{L^2}.$$

for all $f \in L^2(\mathbb{R}_+)$. In turn,

$$\|k_\mu(y,\cdot)\|_{L^2} \le \sum_{n\in\mathbb{Z}_+} |\mu|^n \|l_n\|_{L^2}|l_n(y)| = \sum_{n\in\mathbb{Z}_+} |\mu|^n|l_n(y)|.$$

Therefore

$$\int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dt \le \sum_{n\in\mathbb{Z}_+} |\mu|^n|l_n(y)|\|f\|_{L^2}.$$

So, again by the Cauchy–Schwartz–Bunyakovskii inequality, we have

$$\int\limits_{\mathbb{R}_+} |a(x+y)| \int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dtdy \le \int\limits_{\mathbb{R}_+} |a(x+y)| \left(\sum_{n\in\mathbb{Z}_+} |\mu|^n |l_n(y)| \right) dy \|f\|_{L^2}$$
$$= \sum_{n\in\mathbb{Z}_+} |\mu|^n \int\limits_{\mathbb{R}_+} |a(x+y)||l_n(y)|dy\|f\|_{L^2}$$
$$\le \sum_{n\in\mathbb{Z}_+} |\mu|^n \|a(x+\cdot)\|_{L^2}\|l_n\|_{L^2}\|f\|_{L^2} < \infty.$$

This justifies the application of the Fubini theorem in the following calculations:

$$\mathbf{A}_\mu f(x) = \int\limits_{\mathbb{R}_+} \left(\int\limits_{\mathbb{R}_+} a(x+y)k_\mu(y,t)dy \right) f(t)dt$$
$$= \int\limits_{\mathbb{R}_+} a(x+y) \int\limits_{\mathbb{R}_+} k_\mu(y,t)f(t)dtdy = \mathbf{H}U_\mu f(x).$$

By virtue of this equality, the matrix elements $\langle \mathbf{A}_\mu l_m, l_n\rangle = \mu^m \langle \mathbf{H} l_m, l_n\rangle$ of the operator $\mathbf{A}_\mu$ have the form $\mu^m\alpha_{m+n}$, where $\alpha_{m+n} = \langle He_m, e_n\rangle$, and thus this operator is μ-Hankel. This completes the proof. □

Theorem 4.10. *An operator $\mathbf{A}_\mu$ is nuclear in $L^2(\mathbb{R}_+)$ if, in addition to the conditions of Lemma 4.9, we require that $a = \mathcal{F}\kappa|(0,\infty)$ for some $\kappa \in L^\infty(\mathbb{R})$, where $\mathcal{F}$ denotes the Fourier transform in a sense of distributions. Moreover, its Hilbert–Schmidt norm and trace are given by the formulas*

$$\|\mathbf{A}_\mu\|_{\mathrm{S}_2} = \left(\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |K_\mu(t,s)|^2 dtds \right)^{1/2}, \ \mathrm{tr}\mathbf{A}_\mu = \int\limits_{\mathbb{R}_+} K_\mu(t,t)dt.$$

Furthermore, $\|\mathbf{A}_\mu\| \le \|\mathbf{H}\|$.

Proof. It follows from [13, Theorem 1.8.8], that the operator $\mathbf{H}$ is bounded, and thus the operator $\mathbf{A}_\mu$ is bounded by Lemma 4.9, as well. Since $|\mu| < 1$, the nuclearity of this operator immediately follows from Theorem 2.5. The formula for the Hilbert–Schmidt norm of such operators is classical (see, e.g., [1, Chapter II, item 32]).

Further,

$$\mathrm{tr}\mathbf{A}_\mu = \sum_{n\in\mathbb{Z}_+} \langle \mathbf{A}_\mu l_n, l_n\rangle = \sum_{n\in\mathbb{Z}_+} \mu^n \langle \mathbf{H} l_n, l_n\rangle$$
$$= \sum_{n\in\mathbb{Z}_+} \mu^n \int\limits_{\mathbb{R}_+} \left(\int\limits_{\mathbb{R}_+} a(t+y) l_n(y)dy \right) \overline{l_n(t)}dt. \tag{4.7}$$

Since by the Cauchy–Schwartz–Bunyakovskii inequality

$$\begin{aligned}\sum_{n\in\mathbb{Z}_+}\int\limits_{\mathbb{R}_+}|\mu|^n\left|\int\limits_{\mathbb{R}_+}a(t+y)l_n(y)dy\right||l_n(t)|dt &= \sum_{n\in\mathbb{Z}_+}|\mu|^n\int\limits_{\mathbb{R}_+}|(\mathbf{H}l_n)(t)||l_n(t)|dt\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|\mathbf{H}l_n\|_{L^2}\|l_n\|_{L^2}\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|\mathbf{H}\| < \infty,\end{aligned}$$

formula (4.7) implies that

$$\mathrm{tr}\mathbf{A}_\mu = \int\limits_{\mathbb{R}_+}\left(\sum_{n\in\mathbb{Z}_+}\mu^n\int\limits_{\mathbb{R}_+}a(t+y)l_n(y)dy\right)\overline{l_n(t)}dt. \tag{4.8}$$

In turn, since (again by the Cauchy–Schwartz–Bunyakovskii inequality)

$$\begin{aligned}\sum_{n\in\mathbb{Z}_+}|\mu|^n\int\limits_{\mathbb{R}_+}|a(t+y)||l_n(y)|dy &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|a(t+\cdot)\|_{L^2}\|l_n\|_{L^2}\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|a\|_{L^2} < \infty,\end{aligned}$$

formula (4.8) implies that

$$\begin{aligned}\mathrm{tr}\mathbf{A}_\mu &= \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}a(t+y)\sum_{n\in\mathbb{Z}_+}\mu^n l_n(y)\overline{l_n(t)}dydt\\ &= \int\limits_{\mathbb{R}_+}\left(\int\limits_{\mathbb{R}_+}a(t+y)k_\mu(y,t)dy\right)dt = \int\limits_{\mathbb{R}_+}K_\mu(t,t)dt.\end{aligned}$$

Finally, the last inequality follows from Lemma 4.9 and the obvious equality $\|U_\mu\| = 1$. □

Corollary 4.11. *Let the operator* $\mathbf{H}$ *be bounded. If the function* a *has the form*

$$a(t) = \sum_{j=1}^{n}\sum_{l=0}^{n_j-1}c_{j,l}t^l e^{\lambda_j t},$$

where $\mathrm{Re}\lambda_j < 0$*, then the operator* $\mathbf{A}_\mu$ *is finite-dimensional.*

Proof. It follows from [13, Theorem 1.8.13] that the operator $\mathbf{H}$ is finite-dimensional. To finish the proof it remains to use Lemma 4.9. □

Acknowledgements

The work of the first author was supported by the State Programm of Scientific Research of Republic of Belarus.

The work of the first author was supported by the Ministry of Education and Science of Russia, agreement No. 075-02-2021-1386.

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Received: June 23, 2021.
Accepted: August 9, 2021.