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# Uncertainty evaluation of the minimal value measurements

**Abstract**

The non-standard statistical method for evaluating uncertainty for a minimal value of observations as the measurement result, is proposed. This method is based on properties of minimal order statistic. As example from the practice this method is used to evaluate the uncertainty of a percent elongation and tensile strength in testing plastic products.

**Keywords**: minimal value of observations, uncertainty type A.

## 1. Introduction

The products' quality control and testing is the necessary element of manufacturing and economy. In some cases a minimal value of observations is the base for the measurement result, and the uncertainty of this value should be found. Recommendation of its estimation is not given in GUM [1]. So some method has to be chosen, developed and tested. In this paper a general theoretical approach of processing the uncertainty based on a minimal number of measurement results is given. Then as example of applying of these theoretical backgrounds in practice of the testing of the quality control of plastic tubes is considered. In this case two parameters are measured - percent elongation and tensile strength of the plastic tube in the process of their break.

## 2. Theory of minimal value uncertainties

The uncertainty of the minimum value of percent elongation and tensile strength has two components:

- instrumental component caused by uncertainty indications of the measuring machine, additional equipment (such as caliper and micrometer) and measurement conditions different from normal,
- statistical component caused by the dispersion of observations.

Therefore the processing uncertainty of obtained results is estimated by two GUM methods [1]:

- Method type B is based on analysis and calculation of the uncertainty of metrological properties of used measuring instruments, measurement conditions and equations describing the dependence of desired parameters from directly obtained measurements results. This method is used to evaluate the standard uncertainty $u_{cB}(\varepsilon_{p1})$.
- Method type A - component $u_A(\varepsilon_{p1})$ of uncertainty of a minimal value is evaluated for $n$ independent observations taken from the normal distribution.

The next stage is the estimation of uncertainty component of the minimal value by statistical method (type A). Problem of the evaluation type A uncertainty in testing of the percent elongation and tensile strength is, as noted above, that have to be found the *minimum values* of parameters of test specimens. Therefore, it is impossible to apply directly the GUM method of uncertainty evaluation of measurements with multiple observations [1].

As the example the estimation of the uncertainty of minimum values of controlled parameters from the sample of five elements is performed. The minimal observation $x_{min}=x_{(1)}=\min(x_1, x_2, \cdots x_n)$ is the first one from the set of ordered observations: $x_{(1)}\le x_{(2)}\le x_{(3)}\le \cdots \le x_{(n)}$. The result of a test measurement is not as usual the arithmetic mean $(\bar{x})$ but the minimal (or maximal) value of observations. Then, the standard and expanded uncertainties of test results cannot be calculated according to standard GUM procedures [1]. Another procedure should be used.

It is obvious that minimal value is a random value, however its probability density function (PDF) is not equal to the PDF $p(x)$ of population.

In the next sections the minimal observation $x_{(1)}$ is denoted by $x_1$. Theoretical distribution $p(x_1)$ of minimal value $x_1$ for the normally distributed observations ($m$=0, $\sigma$=1), i.e.

$$p(x_1)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right),\qquad F(x_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_1}\exp\left(-x^2/2\right)\mathrm{d}x\ )$$

can be described [2] by formula:

$$p(x_1)=n\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)\cdot\left[1-F(x_1)\right]^{n-1}. \qquad (1)$$

This distribution for $n = 5$ is presented in Fig. 1a. From (1) the expected value $m_{01}$ of $x_1$ can be calculated as:

$$m_{0,1}=\int_{-\infty}^{\infty}x_1 p(x_1)\mathrm{d}x_1 \qquad (2)$$

and $\sigma_{0.1}$ standard deviation of the minimal observation:

$$\sigma_{0,1}=\int_{-\infty}^{\infty}x_1^2 p(x_1)\mathrm{d}x_1-m_{0,1}^2 \qquad (3)$$

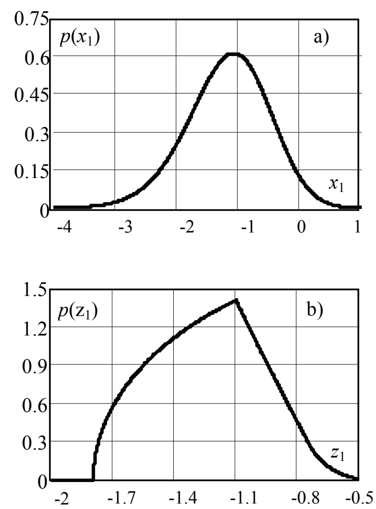


Fig. 1. Distributions: a) of the minimal observation $x_1$; b) of normalized deviation $z_1$ ($n$ = 5)

For $n = 5$ observations from (2) the expected value of normalized $m_{0.1} = -1.16296$ and from (3) the standard deviation $\sigma_{0.1}$=0.66898. If $m\neq 0$ and $\sigma\neq 1$ then expected value $m_1$ and standard deviation $\sigma_1$ of the minimal observation of $x_1$ are:

$$m_1=m+m_{0.1}\cdot\sigma;\qquad \sigma_1=\sigma_{0.1}\cdot\sigma\ . \qquad (4)$$

In practice the expected value $m_1$ of minimal observation of $x_1$ is unknown, but after (4) the estimate $\hat{x}_1$ of $m_1$ can be calculated as

$$\hat{x}_1=\bar{x}+m_{0,1}\cdot s_x \qquad (5)$$

where arithmetical mean and experimental standard deviation of observations are

$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i\ ,\ s_x=\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}\ . \qquad (6)$$

From (7) the experimental standard uncertainty of minimal value is calculated as

$$u_A\left(x_1\right)=\sigma_{0.1}\cdot s_x\ . \qquad (7)$$

The distribution $p_{z1}(z_1)$ of the minimal observation $x_1$ deviation from mean $\bar{x}$, normalized to $s_x$ is

$$z_1=\frac{x_1-\bar{x}}{s_x}\ . \qquad (8)$$

This distribution does not depend on $\bar{x}$ and on $s_x$. It depends only on population distribution $p(x)$ and number of observation $n$. It can be shown that the range of random value $z_1$ is independent of population PDF and equals to

$$\left(n-1\right)/\sqrt{n}\le z_1\le 1/\sqrt{n}\ . \qquad (9)$$

The distribution $p_{z1}(z_1)$ consists of $n-1$ sections, with bounds $z_{b,i}$ ($i = 1, 2,.. n-1$) that are determined by the formula:

$$z_{b,i}=-\sqrt{\left(n-1\right)\left(n-i\right)/\left(n\cdot i\right)},\quad i=1,2,\ldots,n-1\ . \qquad (10)$$

In test procedure the minimal observation $x_1$ is compared with the critical value $x_{critic}$, then after determination $x_1$, the left side of expanded uncertainty $U_{p,low}(x_1)$ must be calculated as

$$x_1-U_{p,low}\left(x_1\right)\ge x_{critic}\ . \qquad (11)$$

For the very small number of observations (for example $n = 5$) the most important is the first part (left section) with bounds

$$z_{b,1}=-\left(n-1\right)/\sqrt{n}\ ,\quad z_{b,2}=-\sqrt{\left(n-1\right)\left(n-2\right)/2n}\ . \qquad (12)$$

If $n = 5$ then $z_{b,1}=-4/\sqrt{5}\approx-1.7889$, $z_{b,2}=-\sqrt{6/5}\approx-1.0954$, because at the end of the first part the cumulative function is

$$F_{z1}\left(z_1\right)=\int_{-\left(n-1\right)/\sqrt{n}}^{-\sqrt{\left(n-1\right)\left(n-2\right)/2n}}p_{z1}\left(z_1\right)\mathrm{d}z_1>0.10\ . \qquad (13)$$

For normally distributed $n=5$ observations, the theoretical distribution $p_{z1}(z_1)$ at the left side part can be described as

$$p_{z1}\left(z_1\right)=\frac{5\sqrt{5}}{2\pi}\sqrt{1-\frac{5}{16}z_1^2}\ ,\quad -\frac{4}{\sqrt{5}}\le z_1\le-\sqrt{\frac{6}{5}}\ . \qquad (14)$$

From (17) cumulative function in this part is:

$$F_{z1}\left(z_1\right)=\int_{-\sqrt{2}}^{z_1}p_{z1}\left(z_1\right)\mathrm{d}z_1=$$
$$=\frac{5}{2}\left[\frac{1}{2\pi}\cdot z_1\sqrt{5-\left(\frac{5}{4}z_1\right)^2}+\frac{2}{\pi}\arcsin\left(\frac{\sqrt{5}}{4}z_1\right)+1\right] \qquad (15)$$

For $z_1=-\sqrt{6/5}$ the cumulative function is $F_{z1}\left(-\sqrt{6/5}\right)=0.6806$. Total distribution $p_{z1}(z_1)$ of $z_1$ is shown in Fig. 2b. The lower $k_{low}(p)$ coverage factor for the confidence level $p$ can be calculated from equation

$$\int_{-2}^{k_{low}\left(n,p\right)}p_{z1}\left(z_1\right)\mathrm{d}z_1=F_{z1}\left(z_1\right)=1-p\ . \qquad (16)$$

The values of $k_{low}(5, p)$ for $p = 0.90$; 0.95; 0.975 and 0.995, and for $n = 5$ are presented in Tab. 1.

Tab. 1. The numeric values of coverage factors

| $p$ | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 |
|---|---|---|---|---|---|
| $k_{low}(5, p)$ | –1.6016 | –1.6714 | –1.7156 | –1.7489 | –1.7637 |

Using the equation (11) and Tab. 1 the lower limit $U_{1-p,low}(x_1)$ of expanded uncertainties of minimal value is expressed by equation:

$$x_{1,p}=\bar{x}+k_{low}\left(n,p\right)\cdot s_x\ . \qquad (17)$$

## 3. Measurements of plastic tubes

In accordance with regulatory documents [3, 4, 5], the test procedure of plastic tube has to be carried out not less than 24 h after manufacturing the tubes. The experimental research with quality control of plastic tubes used for gas and water networks or other needs is performed in the laboratory of "Elplast-Lviv" Ltd. [6]. The test specimens of shapes, shown in Fig. 3, are prepared by cutting from segments tube, with length not less than 150 mm. From each control segment at least five specimens should be cut uniformly around the tube perimeter.

a)

b)

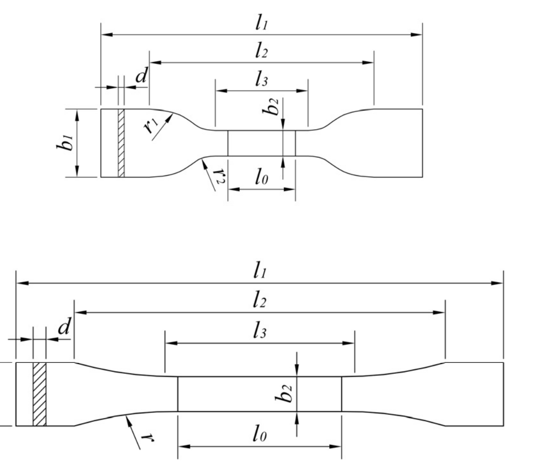


Fig. 2. Shape and dimensions of the test specimens: a) type 1, b) type 2

Before the beginning of test procedure, samples are conditioned for minimum 2 h at temperature (23±2)ºC. For the cold water tubes the testing is made on type 2 test specimens. It is carried with the speed tensile of the grips of testing machine (100±10) mm/min according to the nominal wall thickness. It is up to 5 mm for test specimens of type 1 and more than 5 mm for test specimens of type 2 [6]. Tubes for combustible gases are also tested by type 2 specimens [1, 2]. The tensile tests are carried on testing machine which provides measurement accuracy of load (from 5000 to 10000 N) with permissible error less than ±1% of measuring value. The speed of testing may be controlled in a wide range.

Calculation of the percent elongation $\varepsilon_p$ at break, given in (18), should be rounded to two significant figures [1, 3]:

$$\varepsilon_p = \frac{\Delta l_p}{L_0} \cdot 100\% , \qquad (18)$$

where: $\Delta l_P$ - elongation of the test specimen at the moment of break (in mm), as the measurement result of its smallest value, $L_0$ - initial length of test specimen (in mm).

For determination the tensile strength at yield, the thickness $D$ and width $b_2$ in working part are measured with a micrometer. It is made minimum in three cross-sections with permissible deviations of the width up to ±0.05 mm and of the thickness - up to ± 0.01 mm. A basis are measurement results with an accuracy up to ± 0.001 mm². The initial cross-sectional area of each test specimen is

$$A_0 = D \cdot b_2 . \qquad (19)$$

In calculating of the tensile strength at yield $\sigma_{PT}$, the minimum of cross-sectional area of the test specimen $A_0$ (in mm$^2$) and load $F_{PT}$ (in N) are used. Then the tensile strength at yield is achieved:

$$\sigma_{PT} = \frac{F_{PT}}{A_0} . \qquad (20)$$

For each tested specimen the testing machine records the elongation $\Delta l_P$ (in mm) at the moment of the specimen break. Results of these measurements and load $F_{PT}$ (in N), at which the tensile strength is achieved at yield, are given in [6]. Obtained results of measurements are used for calculation the percent elongation at break $\varepsilon_P$ and tensile strength at yield $\sigma_{PT}$ for each of five test specimens. According to the test requirements, as the result the minimum values of the tensile strength at yield (20) and tensile strength at yield are calculated, and can be rounded to two significant digits.

The test specimen of type 1 (for water pipes norm of $\varepsilon_{P,\,per}$ = 350%) has the percent elongation at break 563.38 % > 350%, and the test specimen of type 2 (for gas pipe norm of $\varepsilon_{P,\,per}$ = 500%) has 563.38% > 500%. Then, due to the percent elongation at break, these products meet the requirements of the Ukraine standards [3], which are in practice very near to EC and ASTM standards [1], [2] (cold water pipe norm: $\sigma_{PT,\,per}$ = 21.0 N/mm$^2$; gas pipe norm: $\sigma_{PT,\,per}$ = 20.0 N/mm$^2$). The actual value of the tensile strength at yield of type 1 is 22.5 N/mm$^2$ > 21.0 N/mm$^2$ and for the test specimen of a type 2 is 21.9 N/mm$^2$ > 20.0 N/mm$^2$. Then the tensile strength at yield products meet the requirements of [5].

## 4. Evaluation of uncertainties of experimental results

Below the uncertainty $u_A(x_1)$ is evaluated and analyzed. Using results of a measurements the following parameters of minimal value of percent elongation and tensile straight of each specimen were calculated (results of calculations are shown in Tab. 2): minimal value $x_{\min}$, arithmetic mean $\bar{x}$, and standard deviation $s_x$ (9); left limit $x_{1,\min}$ (8), (9) and the estimate $\hat{x}_1$ of expected value (5); estimated standard uncertainty $u_A(x_1)$ (7); relative standard uncertainty $u_{A,rel}(x_1)$; - lower $x_{1,1-p}$ limit of expanded uncertainty ($p$ = 0.95) (17), relative combined standard uncertainty $u_{c,rel}(x_1)$ (using previously assessed relative standard uncertainty $u_{c,B,rel}(x_1)$, Tab. 2); combined standard uncertainty $u_c(x_1)$. The component type B of uncertainty is previously calculated in [6]. There is: the relative combined standard uncertainty of relative elongation calculated $u_{cB,rel}(\varepsilon_{p1})$= 0.41% for the test specimen of type 1 and $u_{cB,rel}(\varepsilon_{p1})$ = 0.20% for the test specimen type 2 [6]; the relative combined standard uncertainty of the tensile strength $u_{cB,rel}(\sigma_{PT})$ = 0.614% for the test specimen type 1, and $u_{cB,\,rel}(\sigma_{PT})$ = 0.584% for the test specimen of type 2 [6].

Tab. 2. Results of uncertainty calculations of the percent elongation and the tensile strength

| Parameter | Percent elongation | | Tensile strength | |
|---|---|---|---|---|
| | Specimen 1 | Specimen 2 | Specimen 1 | Specimen 2 |
| Minimal value: $\varepsilon_{p1}$, $\sigma_{PT,1}$ | 563.38% | 563.38% | 22.49 N/mm$^2$ | 21.85 N/mm$^2$ |
| Arithmetic mean; $\bar{\varepsilon}_p$ , $\bar{\sigma}_{PT}$ | 581.89% | 573.44% | 22.57 N/mm$^2$ | 22.08 N/mm$^2$ |
| Standard deviation: $s_{\varepsilon_p}$ , $s_{\sigma_{PT}}$ | 10.87% | 5.87% | 0.081 N/mm$^2$ | 1.19 N/mm$^2$ |
| Relative combined uncertainties $u_{cB,\,rel}(\varepsilon_{p1})$, $u_{cB,\,rel}(\varepsilon_{p1})$ calculated by method type B | 0.41% | 0.20% | 0.61% | 0.58% |
| Relative standard uncertainty $u_A(\varepsilon_{p1})$, $u_{c,\,rel}(\sigma_{PT,1})$ calculated by method type A | 1.29% | 0.70% | 0.24% | 0.59% |
| Relative combined standard uncertainty of a minimal value $u_{c,\,rel}(\varepsilon_{p1})$, $u_{c,\,rel}(\sigma_{PT,1})$ | 1.36% | 0.73% | 0.66% | 0.83% |
| Combined standard uncertainty of the minimal value: $u_c(\varepsilon_{p1})$, $u_c(\sigma_{PT,1})$ | 7.6% | 3.9% | 0.15 N/mm$^2$ | 0.18 N/mm$^2$ |
| The lower limit of expanded uncertainties ($p$ = 0.95) of minimal % elongation: $\varepsilon_{pl,0.05}$ , $\sigma_{PT,1,0.05}$ | 563.72% ≈ 563.7% | 563.84% ≈ 563.8% | 22.44 N/mm$^2$ | 21.76 N/mm$^2$ |

From the last line of Tab. 2 one can see that these lower limits of expanded uncertainties ($p$ = 0.95) are very closed to experimental values (Tab. 1), i.e. for percent elongation calculated minimal values are: 563.72% (specimen 1), 563.84% (specimen 2) and both experimental minimal values are equal to 563.38% (Tab. 1), for tensile strength calculated values for specimen 1 is 22.44 N/mm$^2$ - experimental values is 22.49 N/mm$^2$ and for specimen 2 calculated value is 21.76 N/mm$^2$, and experimental value is 21.85 N/mm$^2$.

## 5. Simulations by Monte Carlo method

To verify the quality of determination of statistical parameters and type A uncertainty, the Monte-Carlo method was used [7]. The number of simulations is $M = 10^5$; number of observations $n$ = 5; parameters of a test specimen 1 are from the population of normal distribution, the expected value $m = \bar{\varepsilon}_p = 581.891\%$ and standard deviation $\sigma = s_{\varepsilon p}$=10.873%. The histogram ($w_{\varepsilon p1}$) of the minimal value $\varepsilon_{p1}$ of the percent elongation is presented in Fig. 3a and histogram of standardized deviation of the minimal value from mean: $z_1 = \dfrac{\varepsilon_{p1} - \bar{\varepsilon}_p}{s_{\varepsilon_p}}$ of the percent elongation is shown in Fig. 3b.

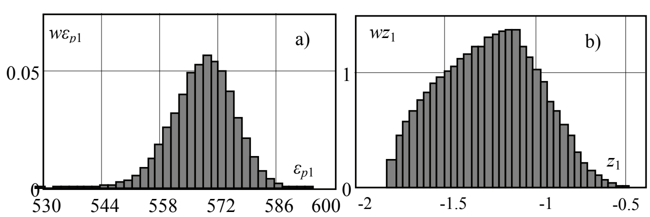


Fig. 3. Histograms a) of minimal value $\varepsilon_{p1}$, b) of normalized deviation $z_1$

Three statistical parameters of the minimal value of relative elongation $\varepsilon_{p1}$ of the test specimen 1, obtained by Monte-Carlo method, and corresponding experimental values are presented in Table 3.

Tab. 3. The experimental results and results from Monte-Carlo simulation of parameters of minimal value of relative elongation $\varepsilon_{p1}$ of a test specimen 1

| Parameters | Experimental results | Monte-Carlo simulation |
|---|---|---|
| Expected minimum value | 569.26% | 569.24% |
| Standard deviation (uncertainty) of minimum value | 7.274% | 7.275% |
| Left limit of minimal value, $p = 0.99$ | 563.72% | 563.30% |

From comparison of histogram shapes (Fig. 2a, b) and theoretical distribution (Fig. 3a, b) and also from comparison of experimental and Monte-Carlo simulation results (Tab. 3) we can see good convergence of experimental results with results of simulation.

## 6. Conclusions

A method of the uncertainty evaluation, which is based on properties of minimal order statistic, is proposed.

The standard uncertainties of the percent elongation and of the tensile strength of plastic tubes calculated from experimental measurements are very close to the results of simulations by the Monte-Carlo method.

The PDF of maximal value is symmetrical to the PDF of minimal value. Then parameters of uncertainty of maximal value can be calculated in the same way as for a minimal value [9]. Only the opposite sign of the deviation of maximal value from the expected value should be taken into account.

Proposed method can be used for result uncertainty evaluation for the test measurements of other quantities, where informative parameter is the minimal (or maximal) value of the sample of several observations.

## 7. References

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