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Regularized Han-type Algorithms for Inconsistent Maritime Container Transportation Problems

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ABSTRACT: In this paper we analyse several ways to compute the weights from the Regularized Han (**RH**) algorithm, the regularized version of Han's algorithm for approximating the least squares solutions of inconsistent (incompatible) systems of linear inequalities. We tested our approaches on a classical transportation problem, aiming to provide a cost optimized solution to real world transportation problems, which often are unbalanced and inconsistent.

1 INTRODUCTION

Many practical problems give rise to systems of linear inequalities as

$$Ax^* \le b \tag{1}$$

in which the inequalities are componentwise, i.e.

$$\sum_{j=1}^{n} A_{ij} x_j^* \le b_i, \forall i = 1, \dots, m \tag{2}$$

where A_{ij} is the (i, j) element of $A \in \mathbb{R}^{m x n}$ and $b = (b_i)_{1 \le i \le m} \in \mathbb{R}^m$. Also, let A^T, A^+ denote the transpose and the Moore-Penrose pseudoinverse of A, respectively. $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ will stand for the Euclidean scalar product and norm.

If (1) is consistent, many classes of efficient solvers, mostly iterative ones, have been designed for its numerical solution (see for a good overview the monograph [3]). In the inconsistent case of (1), for any $x^* \in \mathbb{R}^n$ it exists at least one index $i \in \{1, \dots, m\}$ such that the i-th inequality in (2) is violated, i.e. the set $I(x) \subset \{1, \dots, m\}$, defined by

$$I(x) = \{ i \in \{1, \dots, m\}, \langle A_i, x \rangle \ge b_i \} \tag{3}$$

is non-empty for any $x \in \mathbb{R}^n$. Moreover, let us suppose that the set $I(x) = \{i_1, \dots, i_p\}$ is ordered such that $i_1 < i_2 < \dots < i_p$, then, $A_{I(x)}, b_{I(x)}$ will denote the submatrix of A with the rows $A_{i_1}, \dots, A_{i_p}$ and the subvector of b with components $b_{i_1}, \dots, b_{i_p}$, respectively. For any vector $y \in \mathbb{R}^m$ we define $y_+ \in \mathbb{R}^m$ by $(y_+)_i = \max\{y_i, 0\}$, and the convex sets by $C_i = \{ x \in \mathbb{R}^n, \langle A_i, x \rangle \le b_i \}$, $i = 1, \dots, m$. The inconsistency of the system (1) is equivalent to $\bigcap_{i=1}^{m} C_i = 0$, and we reformulate it in a "least squares sense" as (see e.g. [5]): find $x^* \in \mathbb{R}^n$ such that

$$f\left(x^*\right) = \min\left\{f\left(x\right), x \in \mathbb{R}^n\right\} \tag{4}$$

with

$$f\left(x\right) = \frac{1}{2}\left\|\left(Ax - b\right)_+\right\|^2$$

The following results are proved also in [5].

Proposition 1.

(i) The objective function f from (4) is continuously differentiable, convex and (see also (3))

$$f'\left(x\right) = A^T\left(Ax - b\right)_+ = A^T_{I(x)}\left(A_{I(x)}x - b_{I(x)}\right) \tag{5}$$

(ii) There exists (at least) a least squares solution of (1).

(iii) A vector $x^* \in \mathbb{R}^n$ of (1) if and only if

$$f'\left(x^*\right) = A^T\left(Ax^* - b\right)_+ = 0 \tag{6}$$

The paper is organized as follows: in sections 2 and 3 we present Han's original algorithm for inconsistent systems of linear inequalities and the Regularized version of it, respectively. Section 4 presents the characteristics of a classical transportation problem and the way it can be modelled so that the Han-type algorithms can be applied on. Section 5 is dedicated to the methods we used for choosing the best weights of the Regularized Han algorithm, the numerical experiments being done over an unbalanced and inconsistent transportation problem.

2 HAN-TYPE ALGORITHM FOR INCONSISTENT SYSTEMS OF LINEAR INEQUALITIES

S.P. Han proposed in [5] the following iterative algorithm for approximating a least squares solution of the system (1):

Algorithm H.

Let $x^0 \in \mathbb{R}^n$ be an initial datum; for $k = 0,1,...$ do:

Step 1. Find $I_k = I\left(x^k\right)$ and compute $d^k_{LS} \in \mathbb{R}^n$ as the (unique) minimal norm solution of the linear equalities least squares problem

$$\left\|A_{I_k}d - \left(b_{I_k} - A_{I_k}x^k\right)\right\| = \min! \tag{7}$$

Step 2. Compute $\lambda^k \in \mathbb{R}$ as the smallest minimizer of the function

$$\theta\left(\lambda\right) = f\left(x^k + \lambda d^k_{LS}\right),\ \lambda \in \mathbb{R} \tag{8}$$

Step 3. Set $x^{k+1} = x^k + \lambda^k d^k_{LS}$

The existence of the smaller minimizer for the convex function from (8) was proved in [5] and a procedure to find it was given in [1].

Theorem 1.

(i) Let $\left(x^k\right)_{k\geq 0}$ be the sequence generated by algorithm **H** from any $x^0 \in \mathbb{R}^n$. Then, $f'\left(x^{\bar{k}}\right) = 0$, for a $\bar{k} < \infty$ or $\lim_{k\to\infty} f'\left(x^k\right) = 0$.

(ii) For any m x n matrix A, any right-hand side $b \in \mathbb{R}^m$ and any initial datum $x^0 \in \mathbb{R}^n$, Han's algorithm **H** produces a least squares solution of the system (1) in a finite number of steps (in exact arithmetic).

3 REGULARIZED HAN (RH) ALGORITHM

The Regularized Han algorithm was introduced in [10] and thoroughly studied in [9]. In [10] the author comments on Han's algorithm by considering its major drawback in the fact that, in each iteration, initial objective function from (4) is replaced by

$$f\left(x^k\right) = \frac{1}{2}\left\|\left(Ax^k - b\right)_+\right\|^2 = \frac{1}{2}\left\|A_{I_k}x^k - b_{I_k}\right\|^2.$$

In this way, many originally satisfied constraints might be violated in the new iterative solution x^k. Regarding this aspect, he also proposed to use the complement of the set I_k, denoted by J_k and characterized by

$$J_k = J\left(x^k\right) = \left\{i \in \left\{1,...,m\right\}, A_i x^k < b_i\right\} \tag{9}$$

together with a diagonal weights matrix $W_k = \operatorname{diag}\left(w^k_1,...,w^k_{q_k}\right), w^k_i \geq 0$, where q_k is the number of elements in the set $J_k, \forall k \geq 0$. With these ideas he designed the Regularized version of Han's algorithm from below.

Algorithm RH.

Let $x^0 \in \mathbb{R}^n$ be an initial datum; for $k = 0,1,...$ do:

Step 1. Find $I_k = I\left(x^k\right)$, $J_k = J\left(x^k\right)$ as before and compute $\tilde{d}^k \in \mathbb{R}^n$ the minimal norm solution of the (regularized) linear least squares problem

$$\left\|A_{I_k}d - \left(b_{I_k} - A_{I_k}x^k\right)\right\|^2 + \left\|W_k A_{J_k} d\right\|^2 = \min! \tag{10}$$

Step 2. Compute $\lambda_k \in \mathbb{R}$ as the smallest minimizer of the function

$$\theta(\lambda) = f\left(x^k + \lambda \tilde{d}^k\right),\ \lambda \in \mathbb{R} \qquad (11)$$

Step 3. Set $x^{k+1} = x^k + \lambda^k \tilde{d}^k$.

Theorem 2.

Let $x^0 \in \mathbb{R}^n$ be arbitrary fixed, and $\left(x^k\right)_{k \geq 0}$ the sequence generated by the algorithm **RH**. Then, either it exists $k_0 \geq 0$ such that $f'\left(x^{k_0}\right) = 0$, or $\lim_{k \to \infty} f'\left(x^k\right) = 0$.

4 THE TRANSPORTATION PROBLEM

A classical transportation problem implies finding the minimum cost of transporting certain quantities of a single type of commodity from a given number of loading ports (sources) to a given number of unloading ports (destinations).

At each source $\left(S_i\right)_{i \in \{1,\dots,n\}}$, supplies $\left(s_i\right)_i = 1,\dots,n$ of some goods are available, and at each destination $\left(D_j\right)_{j \in \{1,\dots,m\}}$ some demands $\left(d_j\right)_j = 1,\dots,m$ are requested. Table 1 illustrates the classical transportation problem, $\left(c_{ij}\right)_{i \in \{1,\dots,n\}, j \in \{1,\dots,m\}}$ being the costs of shipping one unit of commodity from source S_i to destination D_j.

Table 1. The classical transportation problem

	D_1	D_2	D_3	…	D_m	Supply(s)
S_1	c_{11}	c_{12}	c_{13}	…	c_{1m}	s_1
S_2	c_{21}	c_{22}	c_{23}	…	c_{2m}	s_2
…	…	…	…	…	…	…
S_n	c_{n1}	c_{n2}	c_{n3}	…	c_{nm}	s_n
Demand(d)	d_1	d_2	d_3	…	d_m	

If we denote by $x_{ij},\ i = 1,\dots,n,\ j = 1,\dots,m$ the number of units transported from source S_i to destination D_j, we get the following mathematical model of the (classical) transportation problem:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \qquad (12)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} \geq d_j,\ j = 1,\dots,m \qquad (*)$$

$$\sum_{j=1}^{m} x_{ij} = s_i,\ i = 1,\dots,n \qquad (**)$$

$$x_{ij} \geq 0,\ i = 1,\dots,n,\ j = 1,\dots,m.$$

Remark 1.
Some arguments for the relations (*)-(**) are as follows:
- for (*): at each destination, the demand has to be "at least" satisfied (e.g. the construction of a building will not be started if we do not have at least a minimal amount of materials)
- for (**): all available units must be supplied.

The problem is called **balanced** if the total supply equals the total demand (i.e. $\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j$) and **unbalanced** otherwise. In the balanced case or the unbalanced one with $\sum_{i=1}^{n} s_i \geq \sum_{j=1}^{m} d_j$, the linear program (12) is consistent and well known methods (including Simplex-type algorithms) are available (see [4]). We will consider in this paper the unbalanced case

$$\sum_{i=1}^{n} s_i < \sum_{j=1}^{m} d_j \qquad (13)$$

for which the linear program (12) becomes inconsistent.

Let us suppose that there exist 7 sources of containers $S_1,\dots,S_7$ and 7 warehouses $D_1,\dots,D_7$.

We will consider the unbalanced and inconsistent transportation problem **P** described in Table 2.

Table 2. The unbalanced transportation problem (**P**)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	Supply(s)
S_1	3	3	4	12	20	5	9	1050
S_2	7	1	5	3	6	8	4	350
S_3	5	4	7	6	5	12	3	470
S_4	4	5	14	10	9	8	7	600
S_5	8	2	12	9	8	4	2	600
S_6	6	1	8	7	2	3	1	480
S_7	9	10	6	8	7	6	5	450
Demand(d)	455	320	540	460	760	830	780	

After applying a series of refinements (see [2]), we can write the problem **P** as the linear program

$$\min\left\langle c, y \right\rangle \ \text{s.t.}\ By \geq d,\ \ y \geq 0 \qquad (14)$$

with the corresponding dual problem given by

$$\max\left\langle d, u \right\rangle \ \text{s.t.}\ B^T u \leq c,\ \ u \geq 0 \qquad (15)$$

In [2] we also proved a result that gives us the possibility to express in an equivalent way the primal-dual pair of linear programs (14-15) as the linear system of inequalities (1) where

$$A = \begin{bmatrix} c^T & -d^T \\ -B & 0 \\ 0 & B^T \\ -I & 0 \\ 0 & -I \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -d \\ c \\ 0 \\ 0 \end{bmatrix} \qquad (16)$$

So, instead of solving the linear programs (14- 15), one could solve the system (1) by applying the Han-type algorithms.

5 COMPUTING THE REGULARIZATION PARAMETERS

The success of Regularized Han algorithm depends on making a good choice of the weights $w^k = \left(w_1^k,\ldots,w_{q_k}^k\right)$, $w_i^k \ge 0$. In the literature, w^k are called the regularization parameters. Hence, one question that can occur related to Regularized Han algorithm is: how can we compute the regularization parameters w^k so that the **RH** algorithm preserves its properties given by Theorem 2? In [10], the author only remarks that we can impose the appropriate penalties when x^k approaches $H_j^k = \left\{j, A_j x^k = b_j\right\}$ by assigning a larger weight w^k for j-th equation if the current iterative solution is close to the boundary H_j, and a smaller weight if is far away. Taking these remarks into consideration, we analysed several ways for choosing good regularization parameters, our goal being to compute a good estimate of the solution to 1. We tested the methods on the inconsistent problem **P** described in section 4, knowing that for this problem, the minimal cost solution is 15336, obtained with Han algorithm. Both algorithms, **H** and **RH**, were implemented in Matlab R2010a, using the built-in Matlab function pinv to compute the direction (Step 1 of the algorithms), all runs being started with the initial datum $x_0 = \left(y_0^T, 0\right)^T$ with $y_0 \ge 0$ and being terminated if at the current iterations x_k satisfy $\left\| A^T\left(Ax^k - b\right)_+ \right\|$ AT (Axk − b)+ i.

The first method considered (**M1**) tries to set each w_j^k, $j = 1,\ldots,q_k$ according to $\left\| A_{J_k} x^k - b_{J_k} \right\|$. We can implement it under the

M1

for $k = 0,1,\ldots$ do

$h^k = A_{J_k} x^k - b_{J_k}$

for $j = 1,\ldots,q_k$ do

if $\left\| h^k \right\| > 10^3$

then $w_j^k = \dfrac{1}{\left\| h^k \right\|}$

else $w_j^k = 10^{-3}$

Table 3. The cost solution of problem (**P**)

Han	RH with M1
15336	15981

Next, we took some fixed weights at each step k, by setting each w_j^k, $j = 1,\ldots,q_k$ equal to $\left\| A_{J_k} x^k - b_{J_k} \right\|$

M2

for $k = 0,1,\ldots$ do

$$w_j^k = \frac{1}{\left\| A_{J_k} x^k - b_{J_k} \right\|}, j = 1,\ldots,q_k$$

Table 4. The cost solution of problem (**P**)

Han	RH with M1
15336	16257

The third method considered uses fixed weights for all steps k

M3

$$w_j^k = 10^{-3}, \forall k, \forall j = 1,\ldots,q_k$$

Table 5. The cost solution of problem (**P**)

Han	RH with M1
15336	16113

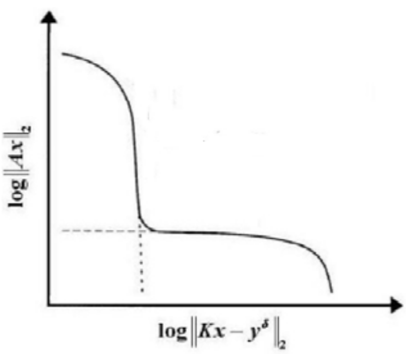
Analysing (10), we observe that we can make an analogy between the Regularized Han algorithm and Tikhonov Regularization.

$$\left\| A_{I_k} d - r_{I_k} \right\|^2 + \left\| \Gamma_k d \right\|^2 = \min! \qquad (17)$$

with $r_{I_k} = b_{I_k} - A_{I_k} x^k$ and $\Gamma_k = W_k A_{J_k}$. The most popular method used to find the regularization parameter of a Tikhonov Regularization is the L-curve approach. An L-curve is defined by

$$\left\{ \log\left\| Lx \right\|, \log\left\| Ax - b \right\|, \alpha \ge 0 \right\}$$

and the L-curve method selects the regularization parameter as the corner of L-curve , i.e. the point of maximum curvature (see for details [6]).



For implementation, we used Hansen's Regularization Tools package ([7]) to compute the corner of L-curve for (10).

M4

for $k = 0,1,\ldots$ do

lcorner = l_curve(U,S,rz',Tikh',L,V)

$w_j^k =$ lcorner, $\forall j = 1,\ldots,q_k$

where

1 [U,S,V] = csvd(A_{I_k}) where 'csvd' stands for Compact SVD: $A_{I_k} = U_r S_r V_r^T$

2 $rz = b_{I_k} - A_{I_k} x^k$

3 $L = IA_{J_k}$

Table 6. The cost solution of problem (**P**)

Han	RH with M1
15336	24703

After implementing **M4** on problem **P**, we noticed that an adjustment is need to be made.

M5

for k = 0,1,... do

lcorner = l_curve(U,S,rz',Tikh',L,V)

$w_j^k = \text{lcorner}, \quad \forall j = 1, ..., q_k$

if $\text{lcorner} \geq 10^{-3}$ then $\text{lcorner} = 10^{-3}$

$w_j^k = \text{lcorner}, \quad \forall j = 1, ..., q_k$

Table 7. The cost solution of problem (**P**)

Han	RH with M1
15336	15336

REFERENCES

[1] Carp, D., Popa, C. & Serban C. 2013. Iterative solution of inconsistent systems of linear inequalities, Proceedings on Applied Mathematics and Mechanics (PAMM), 13, DOI 10.1002/pamm.201310199, 407–408.

[2] Carp, D., Popa, C. & Serban C. 2014. Modifed Han algorithm for maritime containers transportation problem, ROMAI Journal, v.10, no.1, ISSN 1841-5512, ISSN Online 2065-7714, 11–23

[3] Censor, Y. & Stavros, A.Z. 1997. Parallel optimization: theory, algorithms and applications, Numer. Math. and Sci. Comp. Series, Oxford Univ. Press, New York

[4] Koopmans, T.C. & Beckmann, M. 1957. Assignment problems and location of economic activities, Econometrica, Vol.25, No.1, 53– 76.

[5] Han, S. -P. 1980. Least squares solution of linear inequalities, Tech. Rep. TR-2141, Mathematics Research Center, University of Wisconsin - Madison.

[6] Hansen, P. C. & O'Leary D. P. 1993. The use of the L-curve in the regularization of discrete ill-posed problems, SIAM J. Sci. Comput., 14, 1487-1503.

[7] Hansen, P. C. 1992. Regularization Tools, a Matlab Package for Analysis and Solution of Discrete Ill-posed Problems, Technical Report UNIC-92-03, Danish Computing Center for Research and Education, Technical University of Denmark, (Revised July 2007).

[8] Popa, C. 1998. Extensions of block- projections methods with relaxation parameters to inconsistent and rank-defficient least-squares problems, B I T, 38(1), 151–176.

[9] Popa, C. & Serban, C. 2014. Han- type algorithms for inconsistent systems of linear inequalities - a unifed approach, Applied Mathematics and Computation, Volume 246, ISSN: 0096-3003, DOI:10.1016/ j.amc.2014.08.018, 247–256

[10] Yang, K. 1990. New iterative methods for linear inequalities, Tech. report 90-6, Department of Industrial and Operations Engineering, University of Michigan, USA