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ON THE DEFORMATION OF POROUS SPHERICAL BODIES UNDER RADIAL SURFACE TRACTION

Simona De Cicco
University of Naples Federico II, Department of Structures for Engineering and Architecture, Naples, Italy
e-mail: simona.decicco@unina.it

The paper deals with the linear theory of elastic materials with voids based on the concept of volume fraction. In this model, the interstitial pores are vacuous and can contract or stretch. The change in the volume fraction is measured by a scalar function, so that independent kinematical variables are four: the components of displacements and the volume fraction function. The equilibrium problem of elastic spherical bodies under radial surface traction is solved. The solution is given in closed form and applied to study three special cases. Explicit formulas of the displacement, stress distribution and volume fraction function are given.

Keywords: linear elasticity, materials with voids, equilibrium theory, spherical solids

1. Introduction

The equilibrium and motion of solid bodies bounded by a spherical surface are classical problems in the theory of elasticity. At the beginning of the nineteenth century, many scientists investigated various problems concerning spheres and spherical shells. We mention among others Poisson (1829) who, in the second section of his Mémoire, studied vibrations of an elastic sphere, and Clebsch (1862), who addressed the problem of radial vibrations. The equilibrium problem of a spherical shell subjected to a given distribution of load was solved by Lamé (1854). A complete solution of the "Lamé's problem" was obtained by Lord Kelvin (Thompson, 1863) using series involving spherical harmonics. In his famous treatise, Love (1926) introduced a method of solving the Lamé's problem in terms of series of spherical harmonics regarding these functions as functions of Cartesian coordinates and avoiding transformations to polar coordinates. In more recent years, a representation of solutions in terms of series and quadrature was presented by Kupradze (1979). The method has been applied to solve several cases of boundary value problems in three-dimensional elasticity and thermoelasticity for spheres and spherical cavities in a infinite medium.

The attention given to the topic is due to its practical applications in engineering, geotechnical sciences, geophysics and earth sciences.

In this paper, we address the equilibrium problem of spherical bodies under the action of given surface traction in the context of the linear theory of elastic materials with voids introduced by Cowin and Nunziato (1983). Differently from the well-known Biot (1941) consolidation theory, where the open pore spaces are filled with a liquid or gas, in the Cowin-Nunziato model the pores are empty, containing nothing of mechanical significance. Both theories were formulated to describe mechanical behaviour of porous materials and play an important role in engineering, soil mechanics and biomechanics.

The basis of Cowin-Nunziato theory is the concept of volume fraction. The bulk density is explicated as the product of two fields, the matrix material density field and the volume fraction field. The volume fraction corresponding to the void volume is taken as an independent kinematic parameter. In consequence, the theory of materials with voids is characterized by four independent kinematical variables, three components of the displacement and a change in the volume fraction. Consequently, for the mechanical equilibrium an additional balance equation is required. Moreover, extra boundary conditions must be added. The model is suitable to describe the behaviour of rocks, ceramics, pressed powders as well as concrete.

The theory has been subject to intensive study, and a great number of contributions regarding the fundamentals and applications has been published. Basic results and extended references may be found in the book by Ciarletta and Ieşan (1993). Theorems concerning the existence and uniqueness of the solution were established by Ieşsan (1985). Using a semi-inverse method, the Saint-Venant problem was solved by Dell'Isola and Batra (1977). Magnucki and Malinowski (2004) derived an explicit expression for the critical load of compressed porous beams. Stress concentration problems were investigated by De Cicco and De Angelis (2019). Thermoelastic deformations of porous anisotropic cylinders were studied by De Cicco and Ieşan (2013). Puri and Cowin (1985) analysed behaviour of plane harmonic waves in a medium with voids.

A contribution to the topic under consideration was given by Cowin and Nunziato (1983). They solved the problems of thick walled spherical and circular cylindrical shells under internal and external pressure. The salient feature of the solution is that the stress field is not affected by the porosity and is identical with that predicted by classical elasticity. In this problem, we generalize the pressure vessel problem proposed by Cowin and Puri (1983).

The outline of the paper is as follows. In Section 2, we present the basic equations of the equilibrium theory of elastic materials with voids. In Section 3, we derive the generalized analytical solution for a class of problems for which the kinematical variables are functions of the radial coordinate r. In Section 4, the solution is applied to study three special cases.

All the results are expressed in explicit form and generalize the solutions of analogous problems in the classical elasticity. It is worth to note that in the two cases studied, the stress distribution is not affected by a change in the volume fraction, whereas in the third case the porosity influences both the displacement and stress field.

2. Preliminaries

We consider a regular region B of three-dimensional Euclidean space occupied by a linearly elastic material with voids. The region B is referred to a system of Cartesian coordinates $O\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We denote by ∂B the boundary of B and by $\mathbf{n}$ the outward unit normal vector of ∂B. We denote by ρ the mass density in the deformed configuration and assume that ρ has the decomposition $\rho = \sigma\widehat{\rho}$, where $\widehat{\rho}$ is the density of the matrix material and σ is the volume fraction field. In the undeformed state we have $\rho_0 = \sigma_0\widehat{\rho}_0$, where ρ_0, $\widehat{\rho}_0$ and σ_0 are the mass density, density of the matrix material and the volume fraction field in the reference configuration, respectively. We introduce the notation $\psi = \sigma - \sigma_0$ (see Appendix). The independent kinematic variables are components of the displacement u_i $(i = 1, 2, 3)$ and a change in the volume fraction ψ. The governing equations of the linear theory of elastic materials with voids are given by the geometrical equations

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{2.1}$$

where $\mathbf{u}$ is the displacement field over B, $\nabla\mathbf{u}$ is the gradient of $\mathbf{u}$, $\nabla\mathbf{u}^{\mathrm{T}}$ is the transpose of $\nabla\mathbf{u}$, and $\mathbf{E}$ is the strain tensor. The equilibrium equations are

$$\operatorname{div}\mathbf{T} + \mathbf{f} = \mathbf{0} \qquad \operatorname{div}\mathbf{h} - p + q = \mathbf{0} \tag{2.2}$$

where $\mathbf{T}$ denotes the stress tensor, $\mathbf{f}$ the body force, $\mathbf{h}$ the equilibrated stress vector, p the intrinsic equilibrated body force and q the extrinsic equilibrated body force.

In some microstructural theories as well as in specific problems of classical elasticity, there arise stress systems equivalent to two oppositely directed forces at some point, known as double force systems without moments. The terminology is justified by the fact that such systems have no net force and no resulting moment. In classical elasticity, the double force systems were identified as singularities and were discussed by Love (1926).

Now we consider the problem of an elastic sphere forced into a spherical hole of slightly smaller diameter in an infinite elastic medium. The stress distribution consists of three double force systems without moment along three mutually perpendicular axes and is called the center of dilation or center of compression. In the theory of granular materials Goodman and Cowin (1972) showed the existence of an equilibrated stress resulting in either a center of compression or center of dilation.

In equation $(2.2)_2$, $\operatorname{div}\mathbf{h}$ and p can be associated with the center of dilation. The vector $\mathbf{h}$ could be interpreted as a single double force system, whereas the term p can be considered the center of dilation, but one of them acts at a distance. Other possible interpretations were suggested by Jenkins (1975) in the context of theory of granular materials and by Mackenzie (1950) who considered a porous material with spherical voids.

The second equation of (2.2) was first suggested by Goodman and Cowin (1972) and then derived from a variational argument by Cowin and Goodman (1976) (see also Appendix). It has been a subject of detailed discussions regarding the physical meaning and specific interpretations. A comparison with analogous equations arising in microstructural theories was formulated by Nunziato and Cowin (1979) and Cowin and Nunziato (1983).

The constitutive equations are

$$\mathbf{T} = 2\mu\mathbf{E} + \lambda\operatorname{tr}\mathbf{E}\mathbf{I} + \beta\psi\mathbf{I} \qquad \mathbf{h} = \alpha\nabla\psi \qquad p = \beta\operatorname{div}\mathbf{u} + \zeta\psi \tag{2.3}$$

where $\mathbf{I}$ is the identity tensor, and μ, λ, β, α and ζ are constitutive coefficients. Assuming that the internal energy density is a positive definite form, the following inequalities hold

$$\begin{gathered}\mu > 0 \qquad \alpha > 0 \qquad \zeta > 0 \\ 2\mu + 3\lambda > 0 \qquad (2\mu + 3\lambda)\zeta > 3\beta^2\end{gathered} \tag{2.4}$$

The boundary conditions at a regular point of ∂B are expressed by

$$\mathbf{t} = \mathbf{T}\mathbf{n} \qquad h = \mathbf{h}\cdot\mathbf{n} \tag{2.5}$$

where t is the surface traction and h the equilibrated surface force. The dot denotes scalar product. With the help of equations (2.3) and (2.1), equilibrium equations (2.2) become

$$\mu\varDelta\mathbf{u} + (\mu + \lambda)\nabla\operatorname{div}\mathbf{u} + \beta\nabla\psi = 0 \qquad \alpha\nabla\psi - \zeta\psi - \beta\operatorname{div}\mathbf{u} = 0 \tag{2.6}$$

where $\varDelta$ is the Laplacian.

3. Elastic porous spheres

In this Section, we shall consider the equilibrium of a porous body bounded by a spherical surface under the action of given surface tractions. We assume that the region B is referred to the interior of a sphere of radius a, i.e. $B = \{\mathbf{x}|\ x_1^2 + x_2^2 + x_3^2 < a^2\}$. The system of Cartesian coordinates $O\{x_1, x_2, x_3\}$ is chosen so that the origin O is in the center of the sphere. In the following, for our convenience, the displacement vector and the volume fraction function will be expressed in spherical coordinates (r, φ, ϑ) related to the Cartesian coordinates by the expressions

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \varphi = \arccos\frac{x_3}{r} \qquad vt = \arctan\frac{x_2}{x_1}$$

We denote by $(u_r, u_\varphi, u_\vartheta)$ the components of the displacement vector $\mathbf{u}$. We have

$$
\begin{aligned}
&u_r = u_r(r,\varphi,\vartheta) \qquad\qquad u_\varphi = u_\varphi(r,\varphi,\vartheta) \\
&u_\vartheta = u_\vartheta(r,\varphi,\vartheta) \qquad\qquad \psi = \psi(r,\varphi,\vartheta) \\
&0 \leqslant r < a \qquad\qquad 0 \leqslant \varphi < \pi \qquad\qquad 0 \leqslant \vartheta < 2\pi
\end{aligned}
\tag{3.1}
$$

Now, we restrict our attention to the class of problems with spherical symmetry. In consequence, all the quantities are independent of φ and ϑ and depend only upon the variable r. Precisely, we suppose that

$$
u_r = u(r) \qquad\qquad u_\varphi = u_\vartheta = 0 \qquad\qquad \psi = \psi(r)
\tag{3.2}
$$

The components of the strain tensor in spherical coordinates are given by

$$
e_{rr} = u' \qquad\qquad e_{\varphi\varphi} = e_{\vartheta\vartheta} = \frac{1}{r}u \qquad\qquad e_{r\varphi} = e_{r\vartheta} = e_{\varphi\vartheta} = 0
\tag{3.3}
$$

where the prime stands for derivation with respect to r.

The constitutive equations become

$$
\begin{aligned}
&\sigma_{rr} = (2\mu+\lambda)u' + 2\lambda\frac{1}{r}u + \beta\psi \qquad\qquad \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = 2(\mu+\lambda)\frac{1}{r}u + \lambda u' + \beta\psi \\
&\sigma_{r\varphi} = \sigma_{r\vartheta} = \sigma_{\varphi\vartheta} = 0 \qquad\qquad h_r = \alpha\psi' \qquad\qquad h_\varphi = h_\vartheta = 0 \\
&p = \beta\Big(u' + \frac{2}{r}u\Big) + \zeta\psi
\end{aligned}
\tag{3.4}
$$

From (3.2) and (3.3), equilibrium equation (2.6) can be written in the form

$$
u'' + \Big(\frac{2}{r}u\Big)' + \nu\psi' = 0 \qquad\qquad \alpha\Big(\psi'' + \frac{2}{r}\psi'\Big) - \zeta\psi - \beta\Big(u' + \frac{2}{r}u\Big) = 0
\tag{3.5}
$$

where

$$
\nu = \frac{\beta}{2\mu+\lambda'}
$$

The boundary conditions reduce to

$$
\sigma_{rr} = t_r \qquad\qquad \sigma_{\varphi r} = t_\varphi \qquad\qquad \sigma_{\vartheta r} = t_\vartheta \qquad\qquad h = h_r
\tag{3.6}
$$

where t_r and h_r are prescribed constants.

From the first equation of (3.5) we get

$$
u' + \frac{2}{r}u + \nu\psi = B_1
\tag{3.7}
$$

in which B_1 is an arbitrary constant. By considering equation (3.7), the second equation of (3.5) become

$$
\psi'' + \frac{2}{r}\psi' - \xi^2\psi = \frac{\beta}{\alpha}B_1
\tag{3.8}
$$

where

$$
\xi^2 = \frac{1}{\alpha}\Big(\zeta - \frac{\beta^2}{2\mu+\lambda}\Big)
$$

We note that $\xi^2 > 0$.

Equation (3.8) has solution

$$\psi = C_1 i_0(\xi_r) + C_2 k_0(\xi_r) - \frac{\beta}{\alpha\xi^2} B_1 \tag{3.9}$$

where C_1 and C_2 are arbitrary constants and i_n and k_n are spherical modified Bessel functions of the first and second kind, respectively.

When $n = 0$ and $\xi r > 0$, the following identities hold

$$i_0 = \sqrt{\frac{\pi}{2\xi r}} I_{1/2}(\xi r) = \frac{\sinh \xi r}{\xi r} \qquad k_0 = \sqrt{\frac{\pi}{2\xi r}} K_{1/2}(\xi r) = \frac{e^{-\xi r}}{\xi r} \tag{3.10}$$

where $I_{n+1/2}$ and $K_{n+1/2}$ are modified Bessel functions of a non-integer order and e is the Nepero number. Taking into account the relation

$$1 + \frac{\nu\beta}{\xi^2\alpha} = \frac{\zeta}{\alpha\xi^2} \tag{3.11}$$

from (3.7) and (3.9), we obtain

$$u = \frac{\zeta}{3\alpha\xi^2} B_1 r - B_2 \frac{1}{r^2} - \nu C_1 i_1(\xi r) + \nu C_2 k_1(\xi r) \tag{3.12}$$

in which B_2 is an arbitrary constant. In (3.12) the spherical modified Bessel functions i_1 and k_1 take the following expressions

$$\begin{aligned} i_1(\xi r) &= \sqrt{\frac{\pi}{2\xi r}} I_{3/2}(\xi r) = \frac{\xi r \cosh \xi r - \sinh \xi r}{\xi^2 r^2} \\ k_1(\xi r) &= \sqrt{\frac{\pi}{2\xi r}} K_{3/2}(\xi r) = \Big(\frac{1}{r\xi} + \frac{1}{\xi^2 r^2}\Big) e^{-\xi r} \end{aligned} \tag{3.13}$$

In the next Section we consider some applications of general solution (3.9) and (3.12) of the problem. We will study special cases related to the equilibrium of an elastic sphere.

4. Related problems and applications

4.1. Solid sphere loaded with purely radial pressure

In boundary conditions (3.6), let

$$t_r = t = \text{const} \qquad t_\varphi = t_\vartheta = 0 \qquad h = 0 \tag{4.1}$$

For $r = 0$, the functions u and ψ must be finite, so that

$$B_2 = 0 \qquad C_1 = 0 \qquad C_2 = 0 \tag{4.2}$$

The functions u and ψ reduce to

$$u = \frac{\zeta}{3\alpha\xi^2} B_1 r \qquad \psi = -\frac{\beta}{\alpha\xi^2} B_1 \tag{4.3}$$

From (4.3), (3.3) and (3.4) we find

$$e_{rr} = e_{\varphi\varphi} = e_{\vartheta\vartheta} = \frac{\zeta}{3\alpha\xi^2} B_1 \tag{4.4}$$

and

$$\begin{aligned}
&\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = \frac{\zeta(2\mu + 3\lambda) - 3\beta^2}{3\alpha\xi^2} B_1 \\
&h_r = h_\varphi = h_\vartheta = 0 \qquad\qquad p = 0
\end{aligned} \tag{4.5}$$

We introduce the notation

$$c = \frac{\zeta(2\mu + 3\lambda) - 3\beta^2}{3\alpha\xi^2} \tag{4.6}$$

It follows from (4.5), (4.1) and boundary conditions (3.6) that

$$B_1 = \frac{1}{c} t \tag{4.7}$$

From (4.3), we obtain a solution of an analogous problem in the classical theory of elasticity. In fact when $\beta = 0$, then

$$u = \frac{1}{2\mu + 3\lambda} \qquad\qquad \psi = 0 \tag{4.8}$$

Solution (4.3) enables us to calculate the bulk modulus of a material with voids. From (4.4) and (3.4) we get

$$\psi = -\frac{\beta}{\zeta} \operatorname{tr} \mathbf{E} \tag{4.9}$$

By constitutive equations (3.4) and equations (4.5), we obtain

$$t = K \operatorname{tr} \mathbf{E} \tag{4.10}$$

where

$$K = \frac{\zeta(2\mu + 3\lambda) - 3\beta^2}{3\zeta} \tag{4.11}$$

The constant K is the bulk modulus of an elastic material with voids. Now we introduce the notation

$$K^* = \frac{\zeta(2\mu + 3\lambda) - 3\beta^2}{3\beta} \tag{4.12}$$

From (4.3), (4.11) and (4.12), the functions u and ψ can be written in an explicit form

$$u = \frac{1}{3K} tr \qquad\qquad \psi = -\frac{1}{K^*} t \tag{4.13}$$

For $\beta = 0$, we have $1/K^* = 0$ and K reducing to

$$K^0 = \frac{2\mu + 3\lambda}{3} \tag{4.14}$$

The elastic constant K^0 is the bulk modulus in classical elasticity. We note that $K > 0$ and $K < K^0$. In fact

$$K^0 - K = \frac{\beta^2}{\zeta} > 0$$

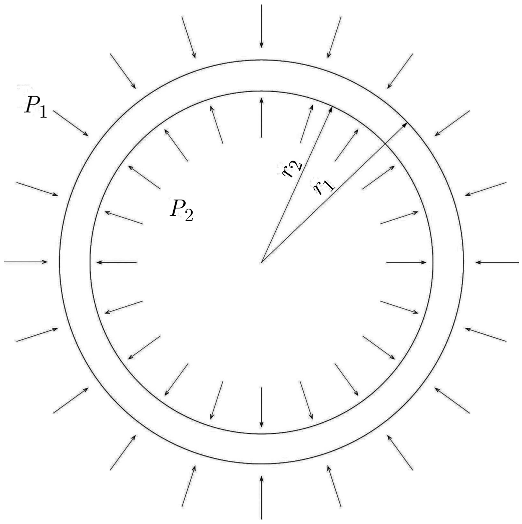

Fig. 1.

4.2. A shell bounded by concentric spherical surfaces

We consider a body bounded by concentric spherical surfaces under the action of internal and external pressure. We denote by r_1 and r_2 the radius of the external and internal boundaries, respectively. Let p_1 be the pressure on the external spherical surface and p_2 the pressure on the internal spherical surface (Fig. 1).

Boundary conditions (3.6) become

$$\begin{aligned} &\sigma_{rr} = -p_1 \qquad &&\sigma_{\varphi r} = \sigma_{\vartheta r} = 0 \qquad &&h_r = 0 \qquad &&\text{on} \quad r = r_1 \\ &\sigma_{rr} = -p_2 \qquad &&\sigma_{\varphi r} = \sigma_{\vartheta r} = 0 \qquad &&h_r = 0 \qquad &&\text{on} \quad r = r_2 \end{aligned} \tag{4.15}$$

From (3.4) and (3.9), it follows that

$$h_r = \alpha\xi[C_1 i_1(\xi r) - C_2 k_1(\xi r)] \tag{4.16}$$

For $r \neq 0$, the function i_1 and k_1 assume finite and non-zero values, so that boundary conditions (4.15) imply

$$C_1 = C_2 = 0 \tag{4.17}$$

Taking into account (3.9), (3.12), (3.4) and (4.6), the stress component σ_{rr} has the form

$$\sigma_{rr} = cB_1 + 4\mu\frac{1}{r^3}B_2 \tag{4.18}$$

With the use of (4.15), we find

$$B_1 = \frac{1}{c}\frac{p_2 r_2^3 - p_1 r_1^3}{r_1^3 - r_2^3} \qquad\qquad B_2 = \frac{p_1 - p_2}{4\mu}\frac{r_1^3 r_2^3}{r_1^3 - r_2^3} \tag{4.19}$$

From (3.9), (3.12), (3.4) and (4.19), we obtain the solution in an explicit form

$$\begin{aligned} u &= \frac{1}{r_1^3 - r_2^3}\Big[\frac{1}{3K}(p_1 r_1^3 - p_2 r_2^3)r - \frac{1}{4\mu}(p_1 - p_2)\frac{r_1^3 r_2^3}{r^2}\Big] \\ \psi &= \frac{1}{K^*}\frac{p_1 r_1^3 - p_2 r_2^3}{r_1^3 - r_2^3} \end{aligned} \tag{4.20}$$

Further

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r_1^3 - r_2^3}\Big[(p_1 - p_2)\frac{r_1^3 r_2^3}{r^3} - p_1 r_1^3 + p_2 r_2^3\Big] \\
\sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} &= -\frac{1}{r_1^3 - r_2^3}\Big[\frac{1}{2}(p_1 - p_2)\frac{r_1^3 r_2^3}{r^3} + p_1 r_1^3 - p_2 r_2^3\Big] \\
h_r &= 0 \qquad\qquad p = 0
\end{aligned} \tag{4.21}$$

Relations (4.20) and (4.21) generalize the solution of the analogous problem in the classical elasticity (Love, 1926).

4.3. A sphere with a rigid core

Now, we consider a spherical porous body B of radius r_1 in which a concentric rigid sphere of radius r_2 has been inserted. The body is in equilibrium under the action of uniform pressure $-t$ ($t > 0$), Fig. 2.

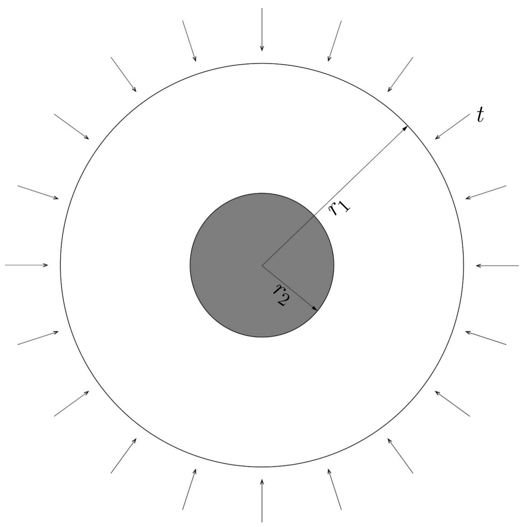

Fig. 2.

The boundary conditions are

$$\begin{aligned}
\sigma_{rr} &= -t \qquad & h_r &= 0 \qquad & \text{for} \quad r &= r_1 \\
u &= 0 \qquad & \psi &= 0 \qquad & \text{for} \quad r &= r_2
\end{aligned} \tag{4.22}$$

We introduce the following functions

$$\begin{aligned}
Q_1(\xi r) &= \frac{i_0(\xi r)k_1(\xi r_1) + k_0(\xi r)i_1(\xi r_1)}{i_1(\xi r_1)} \\
Q_2(\xi r) &= \frac{i_1(\xi r)k_1(\xi r_1) - i_1(\xi r_1)k_1(\xi r)}{i_1(\xi r_1)}
\end{aligned} \tag{4.23}$$

It follows from (3.4), (3.9) and (3.12) that boundary conditions (4.22) are satisfied if and only if

$$\begin{aligned}
C_1 &= \frac{k_1(\xi r_1)}{i_1(\xi r_1)}C_2 \qquad & C_2 &= -\frac{1}{D}t \\
B_1 &= \frac{\alpha\xi^2}{\beta}Q_1(\xi r_2)C_2 \qquad & B_2 &= \Big[\frac{\zeta}{3\beta}Q_1(\xi r_2)r_2^3 - \nu Q_2(\xi r_2)r_2^2\Big]C_2
\end{aligned} \tag{4.24}$$

where

$$D = Q_1(\xi r_2)\Big(K^* + \frac{4\mu\zeta}{3\beta}\frac{r_2^3}{r_1^3}\Big) - 4\mu\nu Q_2(\xi r_2)\frac{r_2^3}{r_1^3} \tag{4.25}$$

Replacing constants (4.24) in relations (3.9), (3.12), (3.3) and (3.4), the solution can be rewritten in an explicit form. The kinematic variables are

$$\begin{aligned} u &= -\frac{t}{D}\Big\{\frac{\zeta}{3\beta}Q_1(\xi r_2)\Big(r - \frac{r_2^3}{r^2}\Big) + \nu\Big[Q_2(\xi r_2)\frac{r_2^2}{r^2} - Q_2(\xi r)\Big]\Big\} \\ \psi &= -\frac{t}{D}[Q_1(\xi r) - Q_1(\xi r_2)] \end{aligned} \tag{4.26}$$

The components of the stress are expressed by

$$\begin{aligned} \sigma_{rr} &= -\frac{t}{D}\Big\{Q_1(\xi r_2)\Big(K^* + \frac{4\mu\zeta}{3\beta}\frac{r_2^3}{r^3}\Big) + 4\mu\nu\Big[Q_2(\xi r)\frac{1}{r} - Q_2(\xi r_2)\frac{r_2^2}{r^3}\Big]\Big\} \\ \sigma_{\vartheta\vartheta} &= \sigma_{\varphi\varphi} = -\frac{t}{D}\Big\{c\frac{k_1(\xi r_1)}{i_1(\xi r_1)} - 2\mu\Big[\frac{\zeta}{3\beta}Q_1(\xi r_2)\frac{r_2^3}{r^3} - \nu Q_1(\xi r)\Big] \\ &\quad + 2\mu\nu\Big[Q_2(\xi r_2)\frac{r_2^2}{r^3} - Q_2(\xi r)\frac{1}{r}\Big]\Big\} \\ h_r &= -\frac{t}{D}\alpha\xi Q_2(r) \qquad\qquad p = -\frac{t}{D}\alpha\xi^2 Q_1(r) \end{aligned} \tag{4.27}$$

The maximum value of σ_{rr} occurs at $r = r_2$

$$\sigma_{rr}^{max} = -\frac{t}{D}\frac{\zeta(2\mu+\lambda) - \beta^2}{\beta}Q_1(\xi r_2) \tag{4.28}$$

If we put $\beta = 0$ into equations (4.26) and (4.27), we obtain the solution of the problem in classical elastostatics

$$u^0 = -\frac{t}{D^*}\Big(r - \frac{r_2^3}{r^2}\Big) \qquad\qquad \psi^0 = 0 \tag{4.29}$$

and

$$\begin{aligned} \sigma_{rr}^0 &= -\frac{t}{D^*}\Big(2\mu + 3\lambda + 4\mu\frac{r_2^3}{r^3}\Big) \qquad\qquad \sigma_{\varphi\varphi}^0 = \sigma_{\vartheta\vartheta}^0 = \frac{t}{D^*}2\mu\frac{r_2^3}{r^3} \\ h_r &= 0 \qquad\qquad p = 0 \end{aligned} \tag{4.30}$$

where we have introduced the notation

$$D^* = 2\mu + 3\lambda + 4\mu\frac{r_2^3}{r_1^3} \tag{4.31}$$

The maximum stress occurs at $r = r_2$

$$\sigma_{rr}^{0\,max} = -\frac{t}{D^*}(6\mu + 3\lambda) \tag{4.32}$$

If we put into (4.32) $r_2 = sr_1$, $0 < s \leqslant 1$, the relation can be rewritten in the form

$$\sigma_{rr}^{0\,max} = -3t\frac{1-\nu^*}{1+\nu^* + 2(1-2\nu)s^3} \tag{4.33}$$

where ν^* is the Poisson ratio.

The ratio $-\sigma_{rr}^{max}/t$ defines the stress concentration factor. For the problem under consideration, we have

$$\kappa = \frac{1}{D} Q_1(\xi r_2)\Big(K^* + \frac{4\mu\zeta}{3\beta}\Big) \tag{4.34}$$

Expression (4.34) generalizes the stress concentration factor for the analogous problem in classical elastostatics

$$\kappa^0 = \frac{3(2\mu + \lambda)}{D^*} \tag{4.35}$$

5. Concluding remarks

- In the theory presented here, the volume fraction as an independent kinematical variable serves to distinguish the mechanical behaviour of materials with voids from ordinary elastic materials. The theory is closely related to microstructural theories and can be considered a special case of the microstretch continuum theory formulated by Eringen (1999).
- We solve the problem of spherical bodies under normal pressure. The solution is obtained in a closed form, and explicit formulas for the displacement, volume fraction function and stress distribution are given.
- The solution is applied to study three special cases. The results are compared with those predicted by the classical elasticity for the same problems. It is interesting to note that in the first and second case, the stresses are not affected by the voids, whereas the radial displacement field is modified from the value predicted by the classical elasticity. Contrary to the previous cases, the third application exhibits both radial displacement and stress field influenced by the voids.
- The solution of the porous spherical shell under external and internal normal pressure coincides with that established by Cowin and Nunziato (1983) with a different approach.
- We derive the bulk modulus of an elastic material with voids and show that it is smaller than the bulk modulus of the elastic material of the skeleton. In the case of the porous sphere with a rigid nucleus, the maximum value of tensile stress and stress concentration factor are calculated.

A. Appendix

A.1. The volume fraction concept

In many branches of engineering, for example, mechanics and biomechanics, we address solids which contain pores, such as rocks, and ceramics as well as bones. The pores can be empty or filled with fluids. The exact location of the pores is impossible to describe, so that we suppose they are statistically distributed in order to create a homogenized continuum. In the N-C theory, the pores are empty, and the material is composed by a skeleton matrix and voids. The theory of materials with voids is based on the volume fraction concept.

We consider an element of volume dV_0 in a point X_0 in the reference configuration. Let dy_0 be the volume of the skeleton matrix in P_0, then we define the volume fraction field by the ratio

$$\sigma_0 = \frac{dv_0}{dV_0}$$

If we denote by σ the volume fraction field in a generic deformed configuration, the difference

$$\psi = \sigma - \sigma_0$$

is a scalar function measuring a change in the volume fraction.

In the N-C theory, the kinematic variables are four: three components of the displacement u_i ($i = 1, 2, 3$) and a change in the volume fraction ψ. When $\psi = 0$, the theory reduces to the classical theory of elasticity.

A.2. Balance equations

The generalized theories of elasticity are characterized by a number of kinematic variables greater than three. To formulate theories that are determinate, the number of equilibrium equations must be equal to that of kinematic variables.

In the N-C theory, the additional equilibrium equation is

$$h_{i,i} - p + q = 0 \tag{A.1}$$

The equations of motion can be easily derived from the energy conservation law (Nunziato and Cowin, 1979)

$$\frac{d}{dt}\int_r \rho_0\Big(e + \frac{1}{2\dot{u}_j\dot{u}_j} + \frac{1}{2}\kappa\dot{\psi}^2\Big)\,dV_0 = \int_\Omega (\mathbf{f}_i\dot{\mathbf{u}}_i + q\dot{\psi})\,dV_0 + \int_{\partial\Omega} (\mathbf{t}_k\dot{\mathbf{u}}_k + h\dot{\psi})\,dA_0 \tag{A.2}$$

where e is the internal energy per unit mass and k is the equilibrated inertia.

The previous equation is also true when $\dot{\mathbf{u}}$ is replaced by $\dot{\mathbf{u}} + \mathbf{a}$, where $\mathbf{a}$ is an arbitrary constant vector with all the other terms being unaltered. By subtraction, we get

$$\Big(\int_\Omega f_i\,dV_0 + \int_{\partial\Omega} t_i\,dA_0 - \int_\Omega \rho_0\ddot{\mathbf{u}}_i\,dV_0\Big) - \mathbf{a}_i = \mathbf{0} \tag{A.3}$$

for all arbitrary constant vectors $\mathbf{a}$. The quantities in the square brackets are independent of $\mathbf{a}$, then it follows that

$$\int_\Omega \rho_0\ddot{\mathbf{u}}_i\,dV_0 = \int_{\partial\Omega} t_i\,dA_0 + \int_\Omega f_i\,dV_0 \tag{A.4}$$

From the usual methods, we obtain

$$t_{ji,j} + f_i = \rho_0\ddot{\mathbf{u}}_i \tag{A.5}$$

In view of (A.4), relation (A.1) reduces to

$$\int_\Omega \rho_0(\dot{e} + \kappa\dot{\psi}\ddot{\psi}\,dV_0 = \int_\Omega (t_{ij}\dot{e}_{ij} + q\dot{\psi})\,dV_0 + \int_{\partial\Omega} h\dot{\psi}\,dA_0 \tag{A.6}$$

If the region Ω is a tetrahedron bounded by the coordinate planes through the point X and by a plane whose unit normal is $\mathbf{n}$, we obtain

$$(h - h_i n_i)\dot{\psi} = 0 \tag{A.7}$$

Using (A.5) and (A.6) and applying the resulting equation to an arbitrary region, we obtain the load form of the conservation of energy

$$\rho_0\dot{e} = T_{ij}\dot{e}_{ij} + h_i\dot{\psi}_{,i} + p\dot{\psi} \tag{A.8}$$

where p satisfies the equation

$$h_{i,i} - p + q = \rho_0\kappa\ddot{\psi} \tag{A.9}$$

The last equation is called the balance of equilibrated force and describes dynamical changes in the void volume.

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