Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

DEVELOPMENT OF COMBINATORIAL THINKING OF ELEMENTARY SCHOOL STUDENTS

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Abstract. In the contribution, the development of combinatorial thinking of elementary school students is discussed. A set of problems including their elaboration with different solving strategies is also presented.

1. Introduction

Combinatorics plays a very important role in the development of mathematical thinking. Its importance is primarily in the development of logical thinking and common combination skills. But it can also be considered as the basis for subsequent probabilistic problem solving.

In particular, it is a part of the high school (secondary school) curriculum, where it is restricted to classical problems of forming groups of subjects and determining the number of all groups that meet specific conditions. Usually, it is not a part of the elementary school curriculum, only in some math-focused schools their students get in touch with it.

Some elementary school students solve combinatorial problems already in the 5th grade when they take tests to get into a grammar school ("gymnázium"). Apart from the tests, they also solve them during mathematical competitions or while taking various quizzes. On the second level of elementary schools, students also solve combinatorial problems but only intuitively by using their common sense or by substituting random figures without using formulas or general combinatorial principles.

We focus on the development of combinatorial thinking on the basis of problem solving. The attention is paid to various solving strategies without the use of given combinatorial relations with progressive problems and building up the theory of combinations.

It is important to realize that combinatorics is tightly linked to other mathematical branches. We need to solve several problems dealing with combination of letters, numbers, etc. And that is why we believe that the development of combinatorial thinking is very important and should be initiated as soon as possible.

2. Basic concepts of combinatorics

In daily life we need to solve problems in which we compile certain groups of objects. We want to know the number of such groups and the order of particular elements. For example, an ice hockey coach who forms groups of players needs to form all possible groups plus to know who is going to play a certain position. Therefore the order of players in every group is very important.

When a school principal builds the schedule of classes, he does this it the same manner as the hockey coach forms groups of players. He forms groups of classes in which their order is important. Combinatorics can help us solve such problems. Combinatorics examines groups (subsets) of elements which are taken out of a certain basic group (set). Depending on whether the elements in the subsets may or may not be repeated, we differentiate several kinds of groups of elements – subsets with repetition and without repetition. We also recognize ordered and disordered subsets.

We choose k from n given elements of a finite set $N\,(k \in N, n \in N)$ of all natural numbers and form (dis)ordered k-tuples. To find all the possibilities we can apply basic combinatorial rules (the rule of sum and the rule of product), basic defined concepts (permutation, variation, and combination), or a list of all possible solutions (a table chart, a logical tree, etc.). We can also use some solving strategies based on the graph theory (graphical illustration of the problem, etc.).

Combinatorics can be also used in many other mathematical branches, particularly in algebra (the representation theory of groups), in the number theory and the game theory, in geometry, in topology, and mathematical analysis. Now we will focus on the application of basic combinatorial rules in the development of combinatorial ideas and students' solving strategies.

2.1. The rule of sum (addition principle)

The rule of sum is the first rule of combinatorics. We use it when we need to find all possible solutions of a problem or to set a rule while creating those solutions. It is good to divide the examined object (the given numbers, the number of contestants, etc.) into a few mutually disjoint groups $A_1, A_2, \dots, A_k$ and look into each group/set separately. Let $A_1, A_2, \dots, A_k$ be finite sets. If a set A_1 includes n_1 elements, a set A_2 includes n_2 elements,..., A_k includes n_k elements and if every two from these sets are disjoint ($A_i \cap A_j = \emptyset$) for $i \neq j$; $i, j = 1, 2, \dots, k$, then the number of all elements of the union of sets

$$A_1 \cup A_2 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i \tag{1}$$

is equal to

$$n_1 + n_2 + \ldots + n_k = \sum_{i=1}^{k} n_i. \tag{2}$$

At the end, we count the numbers of obtained solutions in each group.

2.2. The rule of product (multiplication principle)

The second combinatorial rule, the rule of product, is a bit more complicated than the first one. When forming groups of two elements, we often know how many ways are there to select the first and the second element, while the number of options of selecting the second element does not depend on the way the first element was selected. Let the first element can be selected by m ways and the second element by n ways. Then the couple of those elements (m, n) may be selected by $m \cdot n$ ways. This characteristic may be generalized for the selection of k-tuples. If a set A_1 consists of n_1 elements, a set A_2 of n_2 elements, and a set A_k of n_k elements, then the number of all possible ordered k-tuples is equal to

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k = \prod_{i=1}^{k} n_i \tag{3}$$

2.3. The concept of order and duplication of elements

Understanding the concept of a dis-/ordered k-tuple consisting of certain n elements with or without the possibility of repetition of the elements is very important. The correct comprehension may be further developed on the basis of problem solving without the use of combinatorial rules and principles.

For elementary school students, working with numbers is the most natural activity – formation of numbers according to given rules. While doing that, they practice terminology (a number, a digit, a single/double digit number, etc.). We can use tasks focusing on formation of numbers due to their divisibility, parity, the order of digits, etc.

We must not forget the students motivation – the way we set problems, their relevance and application in daily life, and the whole elaboration of them. There is a set of problems below in which there are some examples of problems. To solve them, basic combinatorial rules may be used.

3. Set of problems

U1 Form all possible double digit numbers by using the digits 1, 2, 3, 4. The digits must not be repeated.

U2 How many even natural numbers can be formed by using the digits given below? Each digit can be used only once.

a) 1, 2, 3, 4, 5

b) 0, 1, 2, 3, 5

U3 How many six-digit natural numbers can we get by using the digits 1 and 2 if no twos can be next to each other?

U4 There are five tickets available. There is the digit "1" written on three of them, the digit "2" on one of them, and the digit "3" on another one. How many five-digit numbers can we get using the tickets?

U5 Determine how many three-digit natural numbers are there:

a) if each digit in their decimal record occurs only once

b) if some digit in their decimal record occurs at least twice

U6 How many natural numbers bigger than 15 can be formed using the digits 1, 2, 3, 4, 5? Each digit can be used only once (must not be repeated).

U7 Vašek forgot his schoolmate's phone number. All he remembers is that it is a nine-digit number and starts with 23. No digits can be repeated and it is divisible by 25. Determine how many possible phone numbers are there?

When solving given problems, we focus on illustration. Pupils are given e.g. playing cards or cards with numbers and are led to systematically record all possibilities.

4. Examples of problem solving

U1

Students are given multi-colored cards with digits 1, 2, 3, and 4 on them. While they are forming pairs, they need to realize that the digits must not be repeated (i.e. no two pairs can have the same color, Figure 1 (right)).

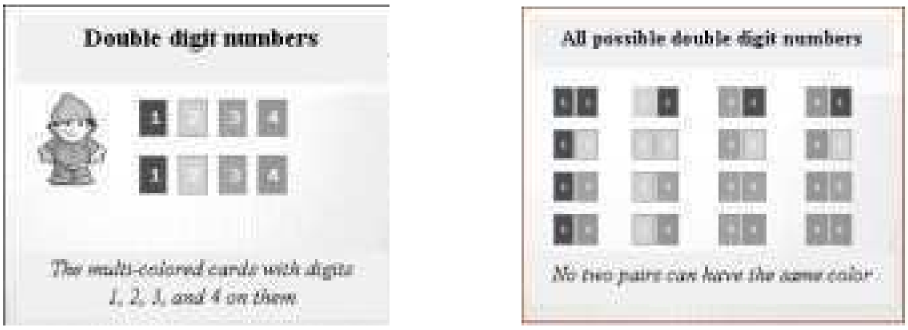

Figure 1: Double digit numbers – U1

Another possibility is to use a table diagram. This method is very well-arranged. In an arranged diagram a student crosses out numbers in which the same digit is used twice.

U5

In the following example, the emphasis is put on the clarification of the basic concepts which are used by teachers and students. At first, we read the task and then we take a closer look at it to prevent potential misunderstanding of the task. Everything is done illustratively to encourage the kids to ask questions. They then use logic to solve the task.

A teacher asks questions such as:

- How many digits does a three-digit number have?
- What does "the decimal notation" mean?
- What is in the first, second, and third place of a three-digit number?
- Read the given number and tell me how many units, tens, and hundreds it has.
- What is the smallest/largest three-digit number?

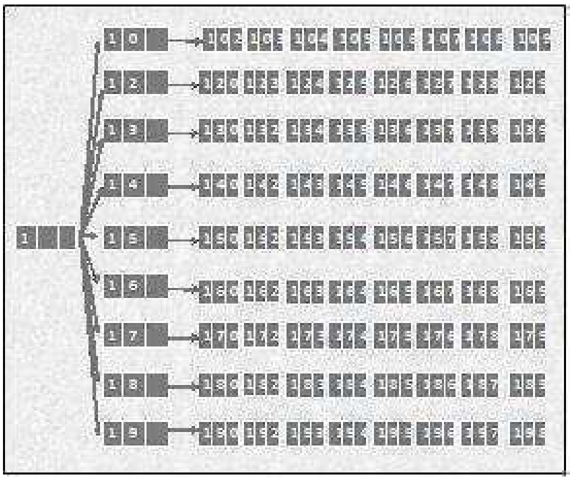

Figure 2: Logical tree – U5

Note: *We explain to the kids that between these double digit numbers there are all three-digit numbers and that we will take only these ones that meet the conditions of the task.*

The assignment: **Determine the number of all three-digit natural numbers such that in their decimal notation each digit appears only once.**

Procedure of problem solving:

- We determine how many different digits can be in the place of hundreds, tens, and units.
- We tell the kids that in the first place there must not be the digit 0 because if it was there, then it would not be a three-digit but a double digit number.
- In the second place there may be all the digits 0-9 except for the digit which is already in the first place. That way we meet the assignment that says that each digit can appear only once.

Another procedure:
We make a logic tree which will help the kids to find all possible solutions (Figure 2) and to understand the principle of the given task. We ask them questions, so they use logic to find solutions. We ask them questions such as:

- What digit can be in the first place?

- If the digit 1 is in the first place, what digits can be in the second place?
- Can the digit 1 be both in the first and in the second place? Why/why not?
- Can the digit 0 be in the first place? Why/why not?
- What digits can be in the third place if there is the digit 1 in the first place and the digit 2 in the second place?
- How many three-digit numbers which meet the conditions are there?

The kids must keep in mind that each digit can appear only once, i.e. it must not be repeated. In the picture (Figure 2), possible steps how to record the creation of numbers are shown.

The kids will notice that there are 9 possibilities of the record of the first place of a three-digit number (digits 1–9). There are also 9 possibilities in the second place (digits 0–9 except for the digit which is already in the first place, i.e. $10 - 1 = 9$). And in the third place, there are 8 possibilities of the record (digits 0–9 except for the digits in the first and in the second place, i.e. $10 - 2 = 8$). In every chain oriented by the first digit, there is $9 \cdot 8 = 72$ possible solutions. In the end they will count the total of all solutions which is $9 \cdot 72 = 648$ solutions.

U7

In this last example of problem solving, we will pay attention to students' motivation – there is a real situation which they are familiar with. The use of combinatorial rules and principles is fairly obvious. As another element of motivation, a colored picture is used. Different possibilities are in different colors (Figure 3).

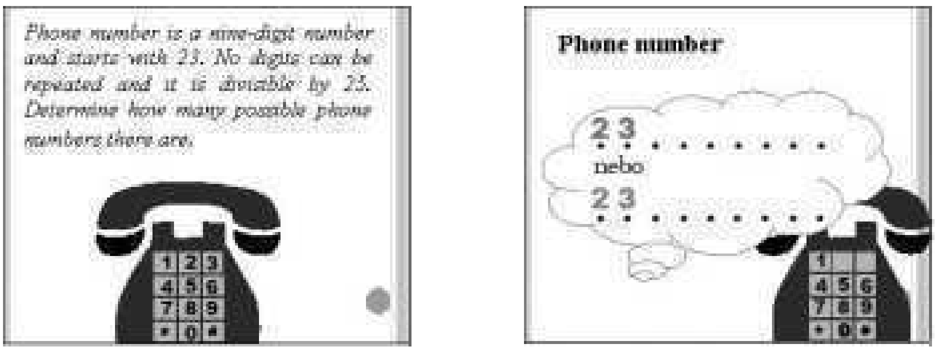

Figure 3: Phone number – U7

5. Conclusion

In the contribution, a possible procedure how to develop combinatorial thinking of primary school students and to build their knowledge structure without the understanding of basic combinatorics is outlined. The emphasis is put on the development of solving strategies, creative approach of a teacher and a student, motivation, and practical application of gained knowledge.

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