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GRAPHS WHOSE VERTEX SET CAN BE PARTITIONED INTO A TOTAL DOMINATING SET AND AN INDEPENDENT DOMINATING SET

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Abstract. A graph G whose vertex set can be partitioned into a total dominating set and an independent dominating set is called a TI-graph. We give constructions that yield infinite families of graphs that are TI-graphs, as well as constructions that yield infinite families of graphs that are not TI-graphs. We study regular graphs that are TI-graphs. Among other results, we prove that all toroidal graphs are TI-graphs.

Keywords: total domination, vertex partitions, independent domination.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

A classic 1962 result by Ore [21] shows that for any isolate-free graph G, the vertices of G can be partitioned into two dominating sets. However, this result does not necessarily extend to other types of domination. For example, although the vertices of the 4-cycle can be partitioned into two (total) dominating sets, they cannot be partitioned into an independent dominating set and a total dominating set. Further, the vertices of a 5-cycle cannot be partitioned into two total dominating sets or even into a total dominating set and a (independent) dominating set. On the other hand, Henning and Southey [16] showed that if G is a connected graph with minimum degree at least 2 and G is not the 5-cycle, then the vertex set of G can be partitioned into a total dominating set and a dominating set. Hence, a natural problem is to consider which graphs can be partitioned into two specific types of dominating sets. Such problems have been studied in [2, 4–6, 8, 12–17, 20, 23, 24] and elsewhere. In this paper we study graphs whose vertex set can be partitioned into a total dominating set and an independent dominating set.

We begin with some basic definitions. For an integer $k \geq 1$, let $[k] = \{1, 2, \ldots, k\}$. Let G be a graph with vertex set $V = V(G)$, edge set $E = E(G)$. The *open neighborhood* $N_G(v)$ of a vertex $v \in V$ is the set of vertices adjacent to v, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is $N_G(S) = \bigcup_{v \in S} N_G(v)$, while the *closed neighborhood of a set* $S \subseteq V$ is the set $N_G[S] = \bigcup_{v \in S} N_G[v]$. Two vertices are *neighbors* if they are adjacent. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* in G is a vertex of degree 0 in G. An *isolate-free graph* is a graph which contains no isolated vertex. A *trivial graph* is the graph of order 1, and a *nontrivial graph* has order at least 2. If G is clear from the context, then we will use $N(v)$, $N[v]$, $N[S]$, $N(S)$ and $\deg(v)$ in place of $N_G(v)$, $N_G[v]$, $N_G[S]$, $N_G(S)$ and $\deg_G(v)$, respectively. Let P_n denote the path on n vertices and C_n denote the cycle on n vertices.

The subgraph of G induced by a set $S \subseteq V$ is denoted by $G[S]$. A set S is a *dominating set* of a graph G if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in a graph G is the *domination number* of G and is denoted by $\gamma(G)$. A dominating set S is a *total dominating set*, abbreviated TD-set, of an isolate-free graph G if $G[S]$ has no isolated vertices, that is, $N(S) = V$. If X and Y are sets of vertices in G, where possibly $X = Y$, then the set X *totally dominates* the set Y if every vertex in Y has a neighbor in X. A dominating set S is an *independent dominating set*, abbreviated ID-set, of G if S is an independent set in G, that is, $G[S]$ consists of isolated vertices. The *independent domination number* $i(G)$ is the minimum cardinality of a ID-set of G and an ID-set of cardinality $i(G)$ is called an *i-set* of G. For other graph theory terminology not defined herein, the reader is referred to [11], and for other recent books on domination in graphs, we refer the reader to [9, 10, 19].

Here we consider graphs whose vertex sets can be partitioned into a TD-set and an ID-set, and we refer to such a partition as a *TDID-partition* of G. If G has a TDID-partition, then we say that G is a *TI-graph.* We note that since any maximal independent set is also a minimal dominating set, Ore's result [21] also implies that the vertices of any isolate-free graph G can be partitioned into an ID-set and a dominating set. However, not all graphs have a TDID-partition as can be easily seen with the cycle C_5 and the path P_5. We remark that if a graph G is a TI-graph, then every TD-set of G contains at least two vertices from every component of G and every ID-set of G contains at least one vertex from every component of G, implying that every component of G has order at least 3. In particular, if G is connected, then G has order at least 3.

We present some basic results in Section 2 followed by methods of constructing TI-graphs in Section 3. We then turn our attention to regular graphs in Section 4 and focus on two infinite families of regular graphs in the final two sections, namely toroidal graphs in Section 5 and cubic graphs in Section 6.

2. PRELIMINARY RESULTS

It remains an open problem to characterize TI-graphs. We present in this section some preliminary results. We begin with an example. The *k-corona* $H \circ P_k$ of a graph H is the graph of order $(k+1)|V(H)|$ obtained from H by attaching a path of length k to each vertex of H so that the resulting paths are vertex-disjoint. For example, the

4-corona $C_5 \circ P_4$ of a 5-cycle is illustrated in Figure 1. We note that every TD-set of the k-corona $H \circ P_k$ of a graph H contains all support vertices of $H \circ P_k$. Moreover, in order to totally dominate the support vertices, every TD-set also contains a neighbor of each support vertex. Thus, if $H \circ P_k$ has a TDID-partition $\{I, T\}$ where I is an ID-set and T is a TD-set, then every leaf is in I in order to be dominated by I and a unique partition is forced. For instance, a TDID-partition $\{I, T\}$ of the 4-corona $C_5 \circ P_4$ is shown Figure 1, where the shaded vertices are in I and the white vertices are in T.

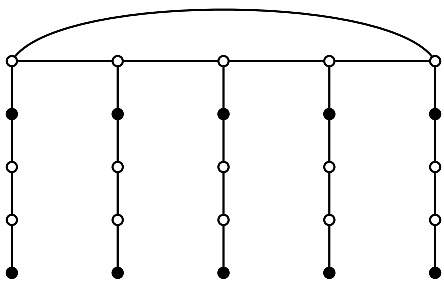


Fig. 1. A TDID-partition of the 4-corona $C_5 \circ P_4$ of a 5-cycle

This leads to the following observation.

Proposition 2.1. *If H is an isolate-free graph, then the k-corona $H \circ P_k$ is a TI-graph if and only if $k \equiv 1 \pmod 3$.*

We observe next that a graph in which every vertex belongs to a triangle is a TI-graph.

Proposition 2.2. *A graph in which every vertex belongs to a triangle is a TI-graph.*

Proof. Let G be a graph in which every vertex belongs to a triangle. Let I be an arbitrary maximal independent set in G, and let $T = V \setminus I$. Thus, I is an ID-set of G. Let v be a vertex in T. Let T_v be a triangle that contains the vertex v, and let $V(T_v) = \{v, v_1, v_2\}$. Since v_1 and v_2 are adjacent vertices, at most one of v_1 and v_2 belongs to the independent set I, implying that the vertex v has at least one neighbor in T. Thus, the subgraph of G induced by the set T is isolate-free. Moreover since every vertex belongs to a triangle, every vertex in I has at least two neighbors in T, and so T is a dominating set. Thus, T is a TD-set of G, implying that the resulting sets I and T form a TDID-partition of G. □

As a consequence of Proposition 2.2, the following families of graphs are TI-graphs.

Corollary 2.3. *The following families of graphs are TI-graphs.*

(a) *Maximal outerplanar graphs.*
(b) *Claw-free graphs with minimum degree at least* 3.

An *efficient dominating set* $S \subseteq V$ in a graph $G = (V, E)$ is a dominating set with the additional property that the closed neighborhood $N[v]$ of every vertex $v \in V$ contains exactly one vertex in S.

Proposition 2.4. *If G is a graph with $\delta(G) \geq 2$ and G has an efficient dominating set, then G is a TI-graph.*

Proof. Let G be a graph with $\delta(G) \geq 2$ and an efficient dominating set I. We note that I is an ID-set of G such that every pair of vertices in I are distance at least 3 apart. Let $T = V \setminus I$. Since every vertex in I has at least two neighbors in T, the set T is a dominating set of G. Further, since $\delta(G) \geq 2$ and no vertex of T has two neighbors in I, it follows that the induced subgraph $G[T]$ is isolate-free, that is, T is a TD-set of G. Hence, G is a TI-graph. □

A graph G is *idomatic* if V has a partition $\pi = \{V_1, \dots, V_k\}$ in which every subset V_i is an ID-set for all $i \in [k]$. Such a partition π is called an *independent domatic partition* of G. Thus, an independent domatic partition of G is a collection of ID-sets and is also a proper coloring of the vertices of G. The *idomatic number*, denoted $\mathrm{idom}(G)$, equals the maximum order of an independent domatic partition of G. If a graph G does not have an independent domatic partition, then we define $\mathrm{idom}(G) = 0$.

Proposition 2.5. *Every graph G with $\mathrm{idom}(G) \geq 3$ is a TI-graph.*

Proof. Let G be a graph with $\mathrm{idom}(G) \geq 3$, and let $\pi = \{V_1, V_2, \dots, V_{\mathrm{idom}(G)}\}$ be a partition of V into $\mathrm{idom}(G)$ ID-sets of G. The requirement that $\mathrm{idom}(G) \geq 3$ guarantees that the set $V \setminus V_i$ is a TD-set of G for all $i \in [\mathrm{idom}(G)]$. Hence, the sets V_i and $V \setminus V_i$ form a TDID-partition of G for all $i \in [\mathrm{idom}(G)]$. □

As a consequence of Proposition 2.5, every complete k-partite graph where $k \geq 3$ is a TI-graph.

Corollary 2.6. *Every complete k-partite graph $K_{n_1,n_2,\dots,n_k}$ where $k \geq 3$ is a TI-graph.*

3. CONSTRUCTING TI-GRAPHS

In this section, we present methods to construct a TI-graph from two smaller TI-graphs. We also observe that the *union* $G \cup H$ of two graphs G and H is a TI-graph if and only if G and H are TI-graphs. The *join* of two graphs G and H, denoted $G \oplus H$, is constructed from their disjoint union by adding edges making every vertex in G adjacent to every vertex in H. If at least one of G and H is isolate-free, then the join $G \oplus H$ is a TI-graph. To see this, suppose that G is an isolate-free graph and consider the join $G \oplus H$ for any graph H. Any ID-set of G is also an ID-set of $G \oplus H$. Further, since $V(G) \setminus I \neq \emptyset$, every vertex in $G \oplus H$ has neighbor in $V(G \oplus H) \setminus I$, that is, $V(G \oplus H) \setminus I$ is a TD-set of $G \oplus H$. We state this formally as follows.

Proposition 3.1. *If G and H are two graphs, then the following properties hold.*

(a) *The union $G \cup H$ is a TI-graph if and only if G and H are TI-graphs.*
(b) *If at least one of G and H is isolate-free, then the join $G \oplus H$ is a TI-graph.*

Let G_i be a graph having a TDID-partition $\{I_i, T_i\}$ where I_i is an ID-set of G_i and T_i is a TD-set of G_i for $i \in [2]$. We build a larger TI-graph G from $G_1 \cup G_2$ by applying one of the three operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ as shown in Figures 2, 3, and 4, respectively. In these figures, the vertices of T_i are white and the vertices of I_i are shaded. Beginning with $G_1 \cup G_2$, the operations build a graph G having a TDID-partition $\{I, T\}$, where I is an ID-set and T is a TD-set and where (as shown in Figures 2–4) the shaded vertices belong to the set I and the white vertices to the set T.

– **Operation $\mathcal{O}_1$.** Add edge uv where vertex $u \in T_1$ and vertex $v \in T_2$. Let $I = I_1 \cup I_2$ and $T = T_1 \cup T_2$.

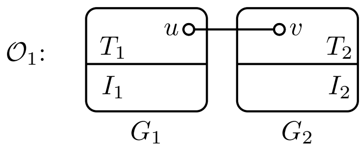


Fig. 2. The operation $\mathcal{O}_1$

– **Operation $\mathcal{O}_2$.** Add a path xy and the edges xu and yv where vertex $u \in I_1$ and vertex $v \in I_2$. Let $I = I_1 \cup I_2$ and $T = (T_1 \cup T_2) \cup \{x, y\}$.

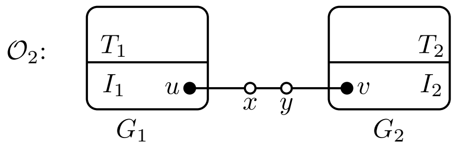


Fig. 3. The operation $\mathcal{O}_2$

– **Operation $\mathcal{O}_3$.** Add a path xyz and the edges ux and vz where $u \in I_1$ and $v \in T_2$. Let $I = (I_1 \cup I_2) \cup \{z\}$ and $T = (T_1 \cup T_2) \cup \{x, y\}$.

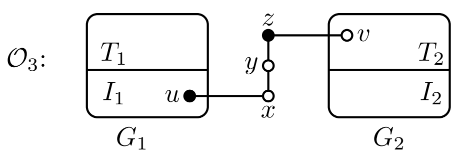


Fig. 4. The operation $\mathcal{O}_3$

We note that $\{I, T\}$ is a TDID-partition for the graphs G constructed by operations $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_3$. Hence, in each case G is a TI-graph.

4. REGULAR GRAPHS

As observed earlier, no 1-regular graph is a TI-graph. The connected 2-regular TI-graphs were determined in [4].

Proposition 4.1 ([4]). *A cycle C_n is a TI-graph if and only if $n \equiv 0 \pmod 3$.*

In this section, we consider r-regular graphs where $r \geq 3$. Since the only ID-set of a complete bipartite graph $K_{p,q}$, for $1 \leq p \leq q$, is one of its partite sets and the remaining partite set is not a TD-set, the graph $K_{p,q}$ is not a TI-graph. In particular, the graph $K_{r,r}$ is not a TI-graph.

We begin with a simple operation $\mathcal{R}$ on a TI-graph G' to build another TI-graph G. Although the operation works in general graphs, we note that in particular if G' is an r-regular TI-graph, then the regularity of G' can be preserved in G using operation $\mathcal{R}$ by adding the complete graph K_{r+1} minus an edge. Let $K_k - e$ denote a complete graph on k vertices minus an edge e. Let $\{I', T'\}$ be an TDID-partition of the vertices of G' where I' is an ID-set of G' and T' is a TD-set of G'.

Operation $\mathcal{R}$. Let $uv \in E(G')$. Replace $u'v'$ with a complete graph $K_k - u'v'$ where $u'v'$ is an edge in K_k and $k \geq 3$. Add edges uu' and vv' to form graph G. See Figure 5.

(a) If $u \in T'$ and $v \in T'$, then let $I = I' \cup \{x\}$ where x is any vertex of the added $K_k - u'v'$ except u' and v', and let $T = V(G) \setminus I$.
(b) If $u \in T'$ and $v \in I'$, then let $I = I' \cup \{u'\}$ and $T = V(G) \setminus I$.

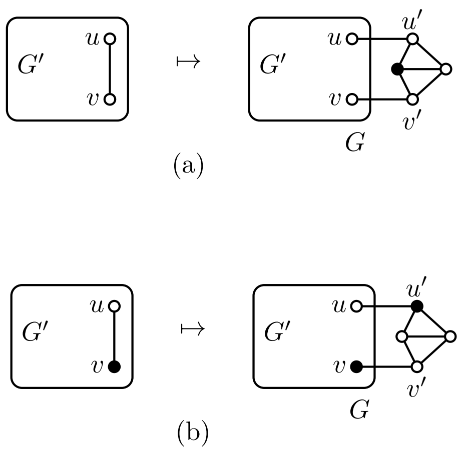


Fig. 5. Illustration of operation $\mathcal{R}$ when $k = 4$

Note that $\{I, T\}$ is a TDID-partition for the graphs G constructed by operation $\mathcal{R}$, and so G is a TI-graph.

Next we construct an infinite family of r-regular TI-graphs. For $r \geq 3$ and $k \geq 1$, let $\mathcal{N}_{\text{regular}}$ be the family of r-regular graphs $N_{r,k}$ constructed as follows. Let $H_1, \dots, H_k$ be k vertex disjoint copies of $K_{r,r} - e$ where H_i has partite sets $X_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,r}\}$ and $Y_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,r}\}$ and where the missing edge e in H_i is the edge $x_{i,1}y_{i,1}$ for $i \in [k]$. Let $N_{r,k}$ be the obtained from the disjoint union of the graphs $H_1, \dots, H_k$ by adding the edges $x_{i,1}y_{i+1,1}$ where addition is taken modulo k. We note that $N_{2,1} = K_{2,2} = C_4$ and $N_{r,1} = K_{r,r}$. When $r = 4$ and $k = 3$ the graph $N_{4,3}$, for example, in the family $\mathcal{N}_{\text{regular}}$ is illustrated in Figure 6, where the shaded vertices belong to the set I and the white vertices to the set T. We remark that repeating this pattern for each three consecutive copies H_j, H_{j+1}, and H_{j+2}, results in sets I and T such that $\{I, T\}$ is TDID-partition of G. We state this formally as follows.

Proposition 4.2. *For $r \geq 3$ and $k \geq 3$ and $k \equiv 0 \pmod 3$, the graph $N_{r,k} \in \mathcal{N}_{\text{regular}}$ is an r-regular TI-graph.*

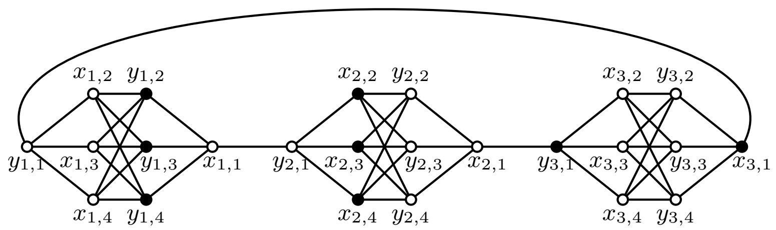


Fig. 6. A TDID-partition of the 4-regular graph $N_{4,3}$

We show next that for $r \geq 3$, if $k \geq 1$ and $k \not\equiv 0 \pmod 3$, then the graph $N_{r,k} \in \mathcal{N}_{\text{regular}}$ is not a TI-graph.

Proposition 4.3. *For $r \geq 2$ and $k \bmod 3 \in \{1, 2\}$, the graph $N_{r,k} \in \mathcal{N}_{\text{regular}}$ is not a TI-graph.*

Proof. Let $N_{r,k}$ be a graph in the family $\mathcal{N}_{\text{regular}}$ for some $k \geq 1$ where $k \bmod 3 \in \{1, 2\}$. If $k = 1$, then the graph $N_{r,1}$ is the graph $K_{r,r}$, which as observed earlier is not a TI-graph. Hence, we may assume that $k \geq 2$ (and $k \bmod 3 \in \{1, 2\}$). Recall that H_i has partite sets $X_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,r}\}$ and $Y_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,r}\}$ and where the missing edge e in H_i is the edge $x_{i,1}y_{i,1}$ for $i \in [k]$. Let $X'_i = X_i \setminus \{x_{i,1}\}$ and let $Y'_i = Y_i \setminus \{y_{i,1}\}$. We note that if $x \in I$ for some $x \in X'_i$, then since all neighbors of x belong to the set T and since every two vertices in X'_i have the same neighborhood, we infer that $X'_i \subseteq I$ in order for the set I to dominate the vertices in X'_i. Analogously, if $y \in I$ for some $y \in Y'_i$, then $Y'_i \subseteq T$. It follows that if $x \in T$ for some $x \in X'_i$, then $X'_i \subseteq T$ and that if $y \in T$ for some $y \in Y'_i$, then $Y'_i \subseteq T$. Throughout the proof we take addition modulo k. Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G.

We show firstly that at most one of $x_{i,1}$ and $y_{i+1,1}$ belongs to T for all $i \in [k]$. Suppose, to the contrary, that $x_{i,1} \in T$ and $y_{i+1,1} \in T$ for some $i \in [k]$. For notational convenience, we may assume that $x_{1,1} \in T$ and $y_{2,1} \in T$. Renaming vertices if necessary, we may assume that $y_{1,2} \in I$ in order to dominate the vertex $x_{1,1}$, and $x_{2,2} \in I$ in order to dominate the vertex $y_{2,1}$. Thus, by our earlier observations, $Y'_1 \subseteq I$ and $X'_2 \subseteq I$, implying that $X'_1 \subseteq T$ and $Y'_2 \subseteq T$. This in turn implies that $y_{1,1} \in T$ in order for the set T to totally dominate the vertices in X'_1, and $x_{2,1} \in T$ in order for the set T to totally dominate the vertices in Y'_2. Thus, $y_{3,1} \in I$ in order for I to dominate the vertex $x_{2,1}$. This in turn implies that $X'_3 \cup Y'_3 \subset T$ and $x_{3,1} \in I$. In the special case when $r = 3$, we illustrate these sets in the graph shown in Figure 7 which is a subgraph of $G = N_{3,k}$, where the shaded vertices belong to the set I and the white vertices to the set T. The sets I and T are now determined, noting that this pattern repeats itself. From this we infer that necessarily $k \equiv 0 \pmod 3$, contradicting our supposition that $k \bmod 3 \in \{1, 2\}$.

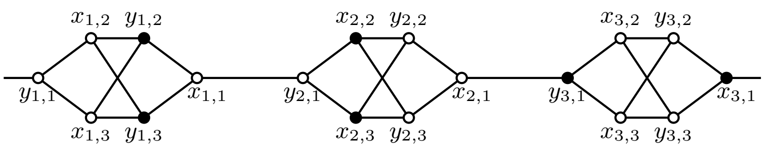


Fig. 7. The subgraph of $N_{3,k}$ in the proof of Proposition 4.3

Hence, at most one of $x_{i,1}$ and $y_{i+1,1}$ belongs to T for all $i \in [k]$. Since the ID-set I contains at most one of $x_{i,1}$ and $y_{i+1,1}$ for all $i \in [k]$, we observe that exactly one of $x_{i,1}$ and $y_{i+1,1}$ belongs to set I and the other to the set T. For notional convenience and by symmetry, we may assume that $x_{1,1} \in I$. Thus, $N(x_{1,1}) = Y'_1 \cup \{y_{2,1}\} \subseteq T$. In order to totally dominate the neighbors of $x_{1,1}$, we may assume renaming vertices if necessary, that $\{x_{1,2}, x_{2,2}\} \subset T$. From our previous comments, $X'_1 \cup X'_2 \subseteq T$. Hence, $y_{1,1} \in I$ in order to dominate the vertex $x_{1,2}$, and the set I contains at least one vertex in Y'_2 in order to dominate the vertex $x_{2,2}$, implying by our earlier observations that $Y'_2 \subset I$. This in turn implies that $x_{2,1} \in T$ and so $X_2 \subset T$. In order to totally dominate the vertex $x_{2,1}$, we have $y_{3,1} \in T$. We now infer that $X'_3 \subset I$ and $Y'_3 \subset T$. This in turn yields $x_{3,1} \in T$. In the special case when $r = 3$, we illustrate these sets in the graph shown in Figure 8 which is a subgraph of $G = N_{3,k}$, where the shaded vertices belong to the set I and the white vertices to the set T.

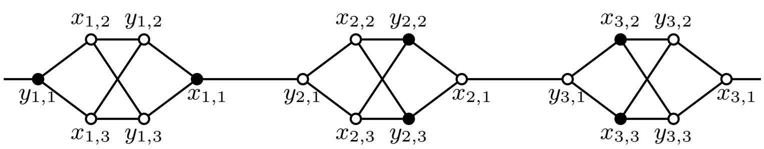


Fig. 8. The subgraph of $N_{3,k}$ in the proof of Proposition 4.3

The sets I and T are now determined, noting that this pattern repeats itself. Therefore, $k \equiv 0 \pmod 3$, contradicting our supposition that $k \bmod 3 \in \{1, 2\}$. □

As a consequence of Propositions 4.2 and 4.3, we have the following result.

Proposition 4.4. *For $r \geq 2$ and $k \geq 1$, the graph $N_{r,k} \in \mathcal{N}_{\text{regular}}$ is a TI-graph if and only if $k \geq 3$ and $k \equiv 0 \pmod 3$.*

Next we give a sufficient condition for a connected r-regular graph to be a TI-graph. A graph is C_4-*free* if it contains no induced 4-cycle.

Theorem 4.5. *For $r \geq 3$, if G is connected, r-regular graph that is C_4-free and satisfies $i(G) \leq r$, then G is a TI-graph.*

Proof. For $r \geq 3$, let G be a connected r-regular graph that is C_4-free and satisfies $i(G) \leq r$. Let I be an i-set of G. By the regularity of G, every vertex in I has r neighbors in $V \setminus I$, and so $V \setminus I$ is a dominating set of G. If $V \setminus I$ is a TD-set of G, then $\{I, V \setminus I\}$ is a TDID-partition of G. Hence, we may assume that $V \setminus I$ is not a TD-set of G, for otherwise the desired result follows. Thus, there exists a vertex $x \in V \setminus I$ such that $N(x) \subseteq I$. This implies that $r \geq i(G) = |I| \geq |N(x)| = r$. Consequently, $I = N(x)$ and $|I| = r$.

Let $N(x) = \{x_1, x_2, \ldots, x_r\}$. Now each vertex x_i has exactly r neighbors in $V \setminus I$ for all $i \in [r]$. For $i \in [r]$, let $X_i = N(x_i) \setminus \{x\}$ and let $X = \cup_{i=1}^{r} X_i$. Since I is an ID-set of G and $I = N(x)$, every vertex in X belongs to one of the sets X_i for some $i \in [r]$. Since G contains no induced 4-cycle, $X_i \cap X_j = \emptyset$ for all $i, j \in [r]$ and $i \neq j$. Thus, the sets $\{X_1, X_2, \ldots, X_r\}$ partition the set X and the sets $\{I, X, \{x\}\}$ partition the set V. In particular, $V = I \cup X \cup \{x\}$. Further, $|X_i| = r - 1$ and for all $i \in [r]$. Since every vertex in X belongs to exactly one of the sets X_i for $i \in [r]$ and is therefore adjacent to exactly one vertex that belongs to the set I, we note that the subgraph $G_X = G[X]$ induced by the set X is an $(r-1)$-regular graph.

Let I_X be an i-set of G_X. Suppose that $|I_X \cap X_i| \leq r - 2$ for all $i \in [r]$. In this case, we consider the set $I^* = I_X \cup \{x\}$. Every vertex in I has at least one neighbor that belongs to $X \setminus I_X$, and therefore has at least one neighbor in $V \setminus I^*$. As observed earlier, every vertex in X has a neighbor in I. In particular, every vertex in $X \setminus I^*$ has a neighbor in I, and therefore has at least one neighbor in $V \setminus I^*$. Hence, $G[V \setminus I^*]$ is an isolate-free graph. Moreover, $V \setminus I^*$ dominates the graph G, implying that $V \setminus I^*$ is a TD-set of G. Thus, $\{I^*, V \setminus I^*\}$ is a TDID-partition of G.

Hence, we may assume that $|I_X \cap X_i| \geq r - 1$ for some $i \in [r]$. Since $|X_j| = r - 1$ for all $j \in [r]$, we infer that $X_i \subseteq I_X$. Renaming sets if necessary, we may assume that $i = 1$, that is, $X_1 \subseteq I_X$. Hence, X_1 is an independent set. Therefore, each vertex in X_1 has $r - 1$ neighbors in $X \setminus X_1$. Thus, there are exactly $(r-1)|X_1| = (r-1)(r-1)$ edges between the vertices of X_1 and the vertices of $X \setminus X_1$. Since G is C_4-free and X_1 is an independent set, every vertex in X_i is adjacent to at most one vertex in X_1 for all $i \in [r] \setminus \{1\}$. Thus, there are at most $(r-1)(r-1)$ edges between the vertices of X_1 and the vertices of $X \setminus X_1$, implying that each vertex in X_i is adjacent to exactly one vertex in X_1 for all $i \in [r] \setminus \{1\}$. As observed earlier, G_X is an $(r-1)$-regular graph. Hence, every vertex in X_1 is adjacent to exactly one vertex in X_i for all $i \in [r] \setminus \{1\}$. From these observations, we infer that the edges between the sets X_1 and X_i induce a perfect matching for all $i \in [r] \setminus \{1\}$. This implies that X_1 is an ID-set of G_X and so $X_1 = I_X$. We now consider the set

$$I^* = I_X \cup (I \setminus \{x_1\}).$$

By our earlier observations, the set I^* is an ID-set of G. Moreover, since $r \geq 3$, every vertex in $X \setminus I_X$ has $r - 2 \geq 1$ neighbors in $X \setminus I_X$. We also note that x and x_1 are adjacent vertices that belong to the set $V \setminus I^*$. Therefore, the set $V \setminus I^*$ is isolate-free. By our earlier observations, the set $V \setminus I^*$ is a dominating set of G, implying that $V \setminus I^*$ is a TD-set of G. Thus, $\{I^*, V \setminus I^*\}$ is a TDID-partition of G. $\square$

As an illustration of our proof in Theorem 4.5, consider the Petersen graph $G = P(5, 2)$ with the vertices named as in Figure 9. Adopting the notation in Theorem 4.5, the set $I = \{x_1, x_2, x_3\}$ is an i-set of G such that $I = N(x)$. As illustrated in Figure 9, $X_i = \{a_i, b_i\}$ for $i \in [3]$. The set $I_X = X_1$ and the set $I^* = I_X \cup (I \setminus \{x_1\}) = \{a_1, b_1, x_2, x_3\}$ given by the shaded vertices form an ID-set in the graph and the set $V \setminus I^*$ given by the white vertices form a TD-set in G. Thus, $\{I^*, V \setminus I^*\}$ is a TDID-partition of the Petersen graph G.

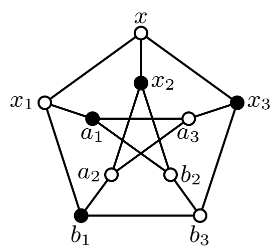


Fig. 9. A TDID-partition in the Petersen graph $P(5, 2)$

5. TOROIDAL GRAPHS

In this section, we consider a special class of 4-regular graphs, called toroidal graphs. A *toroidal graph*, or simply a *torus*, is a Cartesian product of two cycles. Thus, a torus is a Cartesian product of the form $C_m \,\square\, C_n$ where $m, n \geq 3$. We denote such a torus by $G_{m,n}$, and we define its vertex set by $V(G_{m,n}) = \{(i, j) : i \in [m], j \in [n]\}$, where (i, j) is adjacent to (k, ℓ) if $i = k$ and $|j - \ell| \in \{1, n - 1\}$ or $j = \ell$ and $|i - k| \in \{1, m - 1\}$. For a fixed value of i, the set of vertices of the form (i, j) where $j \in [n]$, is called the i^{th} *row* of $G_{m,n}$, and for a fixed value of j, the set of vertices of the form (i, j) where $i \in [m]$, is called the j-th *column* of $G_{m,n}$. Thus, the vertex (i, j) is placed in the ith row and jth column of the grid. In this section, we show that every torus is a TI-graph.

Theorem 5.1. *The torus $C_m \,\square\, C_n$ is a TI-graph for all $m, n \geq 3$.*

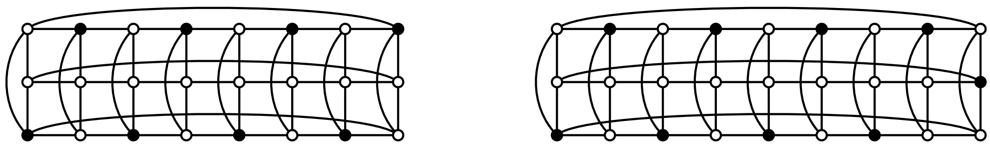
Proof. For $m, n \geq 3$, let $G_{m,n}$ be the torus $C_m \,\square\, C_n$ and let $V = V(G_{m,n})$. Assume $n \geq m = 3$. For n even, let

$$\begin{aligned} I_{\text{even}} = &\{(1,j) \colon j \text{ where } j \in [n-1] \text{ is odd}\} \\ &\cup \{(3,j) \colon j \text{ where } j \in [n] \text{ is even}\}, \end{aligned}$$

and for n odd, let

$$\begin{aligned} I_{\text{odd}} = &\{(1,j) \colon j \text{ where } j \in [n-2] \text{ is odd}\} \\ &\cup \{(3,j) \colon j \text{ where } j \in [n-1] \text{ is even}\} \\ &\cup \{(2,n)\}. \end{aligned}$$

As an example, the shaded vertices in Figure 10(a) form the set I_{even} in the torus $G_{3,8}$, and the shaded vertices in Figure 10(b) form the set I_{odd} in the torus $G_{3,9}$. For $n \geq 4$ even, the set I_{even} is an ID-set in $G_{3,n}$ whose complement $V \setminus I_{\text{even}}$ is a TD-set in $G_{3,n}$, and for $n \geq 3$ odd, the set I_{odd} is an ID-set in $G_{3,n}$ whose complement $V \setminus I_{\text{odd}}$ is a TD-set in $G_{3,n}$. The torus $G_{3,n}$ is therefore a TI-graph for all $n \geq 3$.



(a) $G_{3,8}$ (b) $G_{3,9}$

Fig. 10. TDID-partitions in the torus $G_{3,8}$ and $G_{3,9}$

Hence, we may assume in the remainder of the proof that $m \geq 4$ and $n \geq 4$. Further, by symmetry we may assume $n \geq m$. We consider cases based on the values of m and n modulo 4. We first define a set $I_{k,\ell}$ of vertices in a torus $G_{m,n}$ where $k \leq m$ and $\ell \leq n$. For $i \in [k]$ and $j \in [\ell]$, let

$$\begin{aligned} I_{k,\ell} = &\{(i,j) \,:\, i \equiv 0 \pmod 4 \text{ and } j \equiv 2 \pmod 4\} \\ &\cup \{(i,j) \,:\, i \equiv 1 \pmod 4 \text{ and } j \equiv 3 \pmod 4\} \\ &\cup \{(i,j) \,:\, i \equiv 2 \pmod 4 \text{ and } j \equiv 1 \pmod 4\} \\ &\cup \{(i,j) \,:\, i \equiv 3 \pmod 4 \text{ and } j \equiv 0 \pmod 4\}. \end{aligned}$$

For example, the shaded vertices in Figure 11 form the set $I_{4,4}$ in the torus $G_{4,4}$. We note that the set $I_{4,4}$ is an ID-set of $G_{4,4}$ and $\{I_{4,4}, V \setminus I_{4,4}\}$ is a TDID-partition of the vertices of $G_{4,4}$.

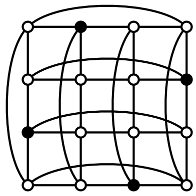


Fig. 11. A TDID-partition in the torus $G_{4,4}$

We now consider cases based on the values of m and n modulo 4. In each case, we give a TDID-partition of $G_{m,n}$.

Case 1. $m \equiv 0 \pmod 4$ *and* $n \equiv 0 \pmod 4$. In this case when m and n are equivalent to 0 modulo 4, the set $I_{m,n}$ is an ID-set of $G_{m,n}$ and $\{I_{m,n}, V \setminus I_{m,n}\}$ is a TDID-partition of the vertices of $G_{m,n}$. For example, the shaded vertices in Figure 12 form the ID-set $I_{8,12}$ in the torus $G_{8,12}$ and the white vertices (in $V \setminus I_{8,12}$) form a TD-set in $G_{8,12}$.

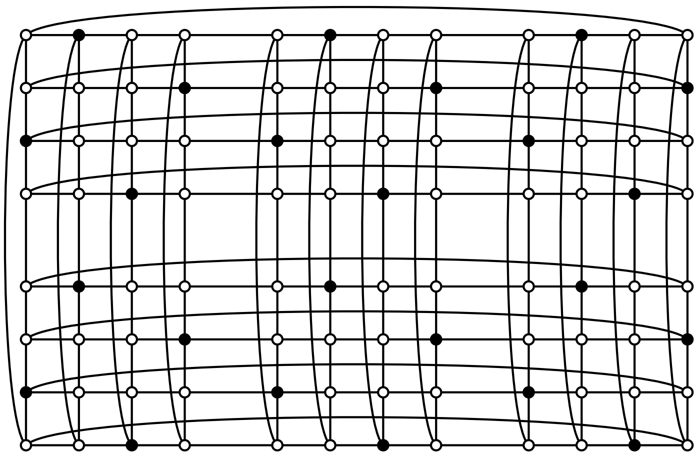


Fig. 12. A TDID-partition in the torus $G_{8,12}$

In the remaining cases, we define a set I in the torus $G_{m,n}$ satisfying the property that the partition $\{I, V \setminus I\}$ is a TDID-partition of $G_{m,n}$ where I is an ID-set and $V \setminus I$ is a TD-set of $G_{m,n}$.

Case 2. $m \equiv 0 \pmod 4$ *and* $n \equiv 1 \pmod 4$. Let

$$I = I_{m,n-1} \cup \{(i,n) : i \equiv 0 \pmod 4 \text{ and } i \in [m]\}.$$

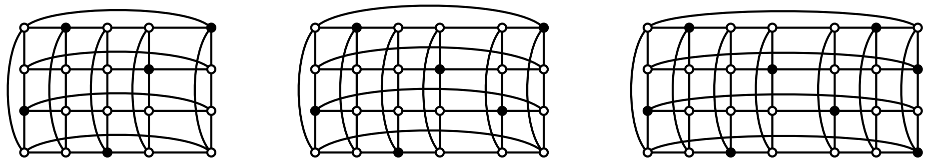
For example, the shaded vertices in Figure 13(a) form the set I in the torus $G_{4,5}$.

Case 3. $m \equiv 0 \pmod 4$ *and* $n \equiv 2 \pmod 4$. Let $I = I_{m,n}$. For example, the shaded vertices in Figure 13(b) form the set I in the torus $G_{4,6}$.

Case 4. $m \equiv 0 \pmod 4$ *and* $n \equiv 3 \pmod 4$. Let

$$I = I_{m,n} \cup \{(i,n) : i \equiv 3 \pmod 4 \text{ and } i \in [m])\}.$$

For example, the shaded vertices in Figure 13(c) form the set I in the torus $G_{4,7}$.



(a) $G_{4,5}$ (b) $G_{4,6}$ (c) $G_{4,7}$

Fig. 13. TDID-partitions in the torus $G_{4,5}$, $G_{4,6}$ and $G_{4,7}$

Case 5. $m \equiv 1 \pmod 4$ *and* $n \equiv 1 \pmod 4$. Let

$$\begin{aligned} I = I_{m-1,n-1} &\cup \{(m,n)\} \\ &\cup \{(m,j) : j \equiv 0 \pmod 4 \text{ and } j \in [n-2]\} \\ &\cup \{(i,n) : i \equiv 0 \pmod 4 \text{ and } i \in [m-2]\}. \end{aligned}$$

For example, the shaded vertices in Figure 14(a) form the set I in the torus $G_{5,5}$.

Case 6. $m \equiv 1 \pmod 4$ *and* $n \equiv 2 \pmod 4$. Let

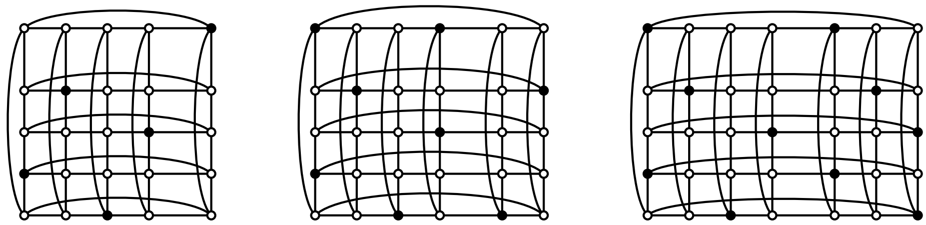
$$\begin{aligned} I = I_{m-1,n-2} & \\ &\cup \{(m,j) : j \equiv 1 \pmod 4 \text{ and } j \in [n-2]\} \\ &\cup \{(i,n-1) : i \equiv 1 \pmod 4 \text{ and } i \in [m-1]\} \\ &\cup \{(i,n) : i \equiv 0 \pmod 4 \text{ and } i \in [m-1]\} \\ &\cup \{(m,n-2)\}. \end{aligned}$$

For example, the shaded vertices in Figure 14(b) form the set I in the torus $G_{5,6}$.

Case 7. $m \equiv 1 \pmod 4$ *and* $n \equiv 3 \pmod 4$. Let

$$\begin{aligned} I = I_{m-1,n} & \\ &\cup \{(m,j) : j \equiv 1 \pmod 4 \text{ and } j \in [n]\} \\ &\cup \{(i,n) : i \equiv 3 \pmod 4 \text{ and } i \in [m-1]\}. \end{aligned}$$

For example, the shaded vertices in Figure 14(c) form the set I in the torus $G_{5,7}$.



(a) $G_{5,5}$ (b) $G_{5,6}$ (c) $G_{5,7}$

Fig. 14. TDID-partitions in the torus $G_{5,5}$, $G_{5,6}$ and $G_{5,7}$

Case 8. $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$. Let

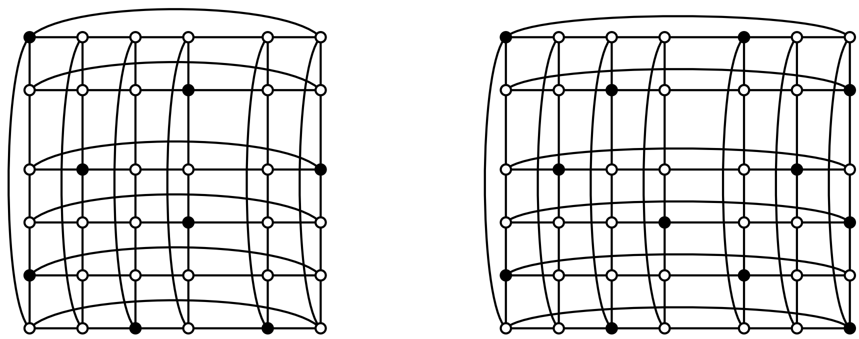
$$\begin{aligned} I = I_{m-2,n-2} \\ &\cup \{(m-1,j) : j \equiv 0 \pmod 4 \text{ and } j \in [n-2]\} \\ &\cup \{(m,j) : j \equiv 1 \pmod 4 \text{ and } j \in [n-2]\} \\ &\cup \{(i,n-1) : i \equiv 1 \pmod 4 \text{ and } i \in [n-2]\} \\ &\cup \{(i,n) : i \equiv 0 \pmod 4 \text{ and } i \in [m-2]\}. \end{aligned}$$

For example, the shaded vertices in Figure 15(a) form the set I in the torus $G_{6,6}$.

Case 9. $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$. Let

$$I = I_{m,n} \cup \{(i,n) : i \equiv 3 \pmod 4 \text{ and } i \in [m-1]\}.$$

For example, the shaded vertices in Figure 15(b) form the set I in the torus $G_{6,7}$.



(a) $G_{6,6}$ (b) $G_{6,7}$

Fig. 15. TDID-partitions in the torus $G_{6,6}$ and $G_{6,7}$

Case 10. $m \equiv 3 \pmod 4$ *and* $n \equiv 3 \pmod 4$. Let

$$\begin{aligned} I = I_{m,n} \\ &\cup \{(m,j) : j \equiv 2 \pmod 4 \text{ and } j \in [n-1]\} \\ &\cup \{(i,n) : i \equiv 3 \pmod 4 \text{ for } i \in [m-3]\}. \end{aligned}$$

For example, the shaded vertices in Figure 15(b) form the set I in the torus $G_{7,7}$.

We deduce from the above ten cases that the torus $G_{m,n}$ is a TI-graph. This completes the proof of Theorem 5.1. □

6. CUBIC GRAPHS

In this section, we present examples and constructions of cubic graphs that are TI-graphs, as well as examples and constructions of cubic graphs that are not TI-graphs.

6.1. CUBIC TI-GRAPHS

In this section, we present examples of (connected) cubic graphs of small order that are TI-graphs. We also present infinite families of connected cubic graphs where every graph in the family is a TI-graph. As shown earlier, the Petersen graph $P(5,2)$ illustrated in Figure 9 is a cubic TI-graph. The nonplanar cubic graph $G_{8.1}$ of order 8 illustrated in Figure 16 is another example of a cubic TI-graph of small order, where the shaded vertices form an ID-set and the white vertices form a TD-set in the graph. Moreover, these two sets partition the vertex set of $G_{8.1}$, yielding a TI-graph.

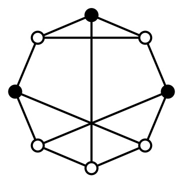


Fig. 16. The nonplanar cubic graph $G_{8.1}$ of order 8

For $\ell \geq 1$, let $\mathcal{G}^1_{\text{cubic}}$ be the family of cubic graphs constructed in [7] by taking a copy of a cycle $C_{3\ell}$ with vertex sequence $a_1b_1c_1 \dots a_\ell b_\ell c_\ell$, and for each $i \in [\ell]$, adding the vertices $\{w_i, x_i, y_i, z_i^1, z_i^2\}$, and joining a_i to w_i, b_i to x_i, and c_i to y_i, and further for each $j \in [2]$, joining z_i^j to each of the vertices w_i, x_i, and y_i. A graph in the family $\mathcal{G}^1_{\text{cubic}}$ is illustrated in Figure 17, where the shaded vertices form an independent set I and the white vertices form a TD-set T, yielding the TDID-partition $\{I, T\}$ of G. We state this formally as follows.

Proposition 6.1. *Every graph in the family* $\mathcal{G}^1_{\text{cubic}}$ *is a TI-graph.*

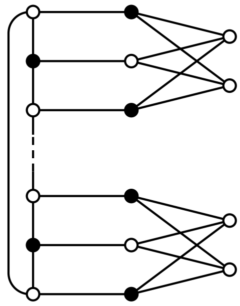


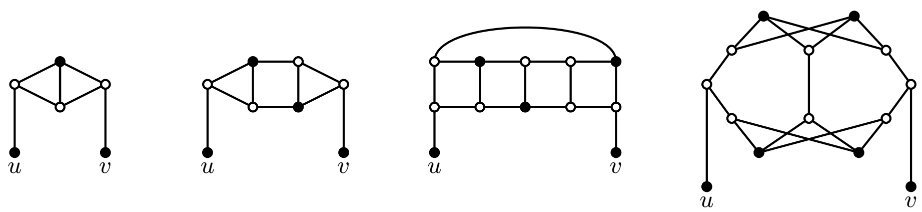
Fig. 17. A graph in the family $\mathcal{G}^1_{\rm cubic}$

We construct next other infinite families of cubic graphs that are TI-graphs. For this purpose, associated with an arbitrary edge $e = uv$ of a cubic graph G, let $G_{e,1}$, $G_{e,2}$, $G_{e,3}$, and $G_{e,4}$ be the four graphs given in Figures 18(a), 18(b), 18(c), and 18(d), respectively. We call these four graphs *gadgets* associated with the edge e in G and note that they are samples of many graphs that could be used as gadgets. If G is an arbitrary cubic graph, then let G^* be the graph obtained from G by replacing every edge e in G with one of the gadgets $G_{e,1}$, $G_{e,2}$, $G_{e,3}$, and $G_{e,4}$. Let $\{I_e, T_e\}$ be the TDID-partition of the gadget $G_{e,i}$ where $i \in [4]$ as given in Figure 18, where the shaded vertices form the ID-set I_e and the white vertices the TD-set T_e of the gadget $G_{e,i}$. Let

$$I^* = \bigcup_{e\in E(G)} I_e \quad \text{and} \quad T^* = \bigcup_{e\in E(G)} T_e.$$

We note that $V(G) \subset I^*$. The set I^* is an ID-set of G^* and the set T^* is a TD-set of G^*, implying that the partition $\{I^*, T^*\}$ of the newly constructed cubic graph G^* built from the cubic graph G is a TDID-partition of G^*. We state this formally as follows.

Proposition 6.2. *If G is an arbitrary cubic graph, then the graph obtained from G by replacing every edge e of G with any one of the gadgets $G_{e,1}$, $G_{e,2}$, $G_{e,3}$ and $G_{e,4}$ in Figure 18 is a cubic TI-graph.*



(a) $G_{e,1}$ (b) $G_{e,2}$ (c) $G_{e,3}$ (c) $G_{e,4}$

Fig. 18. Gadgets associated with an edge $e = uv$ in a cubic graph

6.2. CUBIC GRAPHS THAT ARE NOT TI-GRAPHS

In this section, we present examples of (connected) cubic graphs that are not TI-graphs. We begin with examples of cubic graphs of small order that are not TI-graphs. As observed earlier, the graph $K_{3,3}$ shown in Figure 19(a) is not a TI-graph.

Proposition 6.3. *The graph $G_{8.2}$ shown in Figure 19(b) is not a TI-graph.*

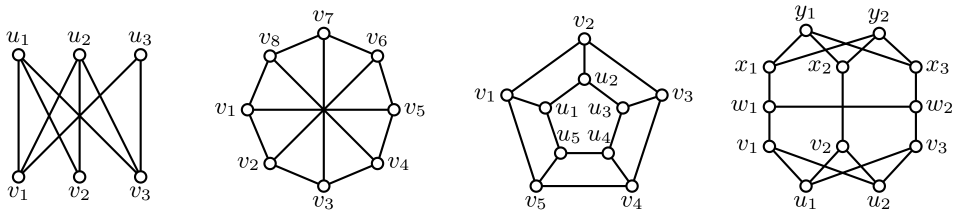
Proof. Let G be the nonplanar cubic graph $G_{8.2}$ shown in Figure 19(a). Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G. We note that G is vertex-transitive. Renaming the vertices if necessary, we may assume that $v_1 \in I$, implying that $N(v_1) = \{v_2, v_5, v_8\} \subseteq T$. Suppose that v_3 or v_7 belongs to the set I. By symmetry, we may assume that $v_3 \in I$, implying that $\{v_4, v_7\} \subset T$. In this case, $N(v_6) = \{v_2, v_5, v_7\} \subset T$, implying that $v_6 \in I$ in order for the set I to dominate the vertex v_6. But then $N(v_2) \subseteq I$, and so the vertex v_2 is not totally dominated by the set T, a contradiction. Hence $\{v_3, v_7\} \subset T$. This implies that $\{v_4, v_6\} \subset I$ in order for the set I to dominate the vertices v_3 and v_7. However, then, $N(v_5) \subseteq I$, and so the vertex v_5 is not totally dominated by the set T, a contradiction. □

The 5-prism $C_5 \,\square\, K_2$ shown in Figure 19(c) is another example of a cubic graph of small order that is not a TI-graph. A proof of this property of the 5-prism is along similar lines to that of Proposition 6.3, and hence we omit a proof.

Proposition 6.4. *The graph G_{12} shown in Figure 19(d) is not a TI-graph.*

Proof. Let G be the cubic graph G_{12} shown in Figure 19(d). Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G. We show firstly that $I \cap \{u_1, u_2, y_1, y_2\} = \emptyset$. Suppose that one of u_1, u_2, y_1, and y_2 belongs to the set I. By symmetry, we may assume that $y_1 \in I$. Thus, $N(y_1) = \{x_1, x_2, x_3\} \subseteq T$, implying that $y_2 \in I$ in order for the set I to dominate the vertex y_2. Moreover, $\{v_2, w_1, w_2\} \subseteq T$ in order for the set T to totally dominate the vertices x_1, x_2, and x_3. This implies that $\{v_1, v_3\} \subset I$ in order for the set I to dominate the vertices w_1 and w_2. Since all neighbors of vertices in I belong to the set T, we infer that $\{u_1, u_2\} \subset T$. But then v_2 and all its neighbors belong to the set T, and so v_2 is not dominated by the set I, a contradiction. Hence, $I \cap \{u_1, u_2, y_1, y_2\} = \emptyset$. Thus, $\{u_1, u_2, y_1, y_2\} \subseteq T$.

In order to dominate the vertices y_1 and y_2, we have $I \cap \{x_1, x_2, x_3\} \neq \emptyset$ and in order to totally dominate the vertices y_1 and y_2, we have $T \cap \{x_1, x_2, x_3\} \neq \emptyset$. If $\{x_1, x_3\} \subseteq T$, then $\{w_1, w_2\} \subseteq I$ in order for the set I to dominate the vertices x_1 and x_3. However, w_1 and w_2 are adjacent vertices, contradicting the independence of the set I. Hence, at least one of x_1 and x_3 belongs to the set I. By symmetry, we may assume that $x_1 \in I$, and so $w_1 \in T$. Thus, by our earlier assumptions, $N(v_1) = \{u_1, u_2, w_1\} \subseteq T$, implying that $v_1 \in I$ in order for the set I to dominate the vertex v_1. This in turn implies that $w_2 \in T$ in order for the set T to totally dominate the vertex w_1. Thus, $N(v_3) = \{u_1, u_2, w_2\} \subseteq T$, implying that $v_3 \in I$ in order for the set I to dominate the vertex v_3. Hence, $v_2 \in T$ in order for the set T to totally dominate the vertices u_1 and u_2. This in turn implies that $x_2 \in I$ in order for the set I to dominate the vertex v_2. As observed earlier, $T \cap \{x_1, x_2, x_3\} \neq \emptyset$, implying that $x_3 \in T$. But then x_3 and all its neighbors belong to the set T, and so x_3 is not dominated by the set I, a contradiction. □



(a) $K_{3,3}$ (b) $G_{8.2}$ (c) $C_5 \,\square\, K_2$ (d) G_{12}

Fig. 19. Examples of cubic graphs of small orders that are not TI-graphs

For $k \geq 1$, let $\mathcal{G}^2_{\text{cubic}}$ be the family of cubic graphs G_k constructed in [7] by taking two copies of the cycle C_{4k} with respective vertex sequences $u_1u_2, \ldots, u_{4k}$ and $v_1, v_2, \ldots, v_{4k}$ and adding edges as follows. Add the edges u_iv_i for $i \in [4k-2]$ and $i \equiv 1, 2 \pmod 4$ and add the edges u_iv_{i+1} and v_iu_{i+1} for $i \in [4k-1]$ and $i \equiv 3 \pmod 4$, where addition is modulo $4k$. The graph G_3 in the family $\mathcal{G}^2_{\text{cubic}}$ is illustrated in Figure 20.

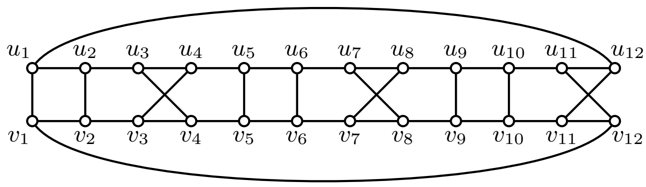


Fig. 20. The graph G_3 in the family $\mathcal{G}^2_{\text{cubic}}$

Proposition 6.5. *No graph in the family $\mathcal{G}^2_{\text{cubic}}$ is a TI-graph.*

Proof. Let G_k be a graph in the family $\mathcal{G}^2_{\text{cubic}}$ for some $k \geq 1$, and so G_k has order $8k$. If $k = 1$, then the graph G_1 is the graph $G_{8.2}$ shown in Figure 19(b). As shown in Proposition 6.3, the graph G_1 is not a TI-graph. Hence, we may assume that $G = G_k$ for some $k \geq 2$. Throughout the proof we take the indices modulo $4k$. Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G. If I contains no vertex from the set $\{u_7, v_7, u_8, v_8\}$, then u_6 and v_6 are in I to dominate the vertices u_7 and v_7. However, then the set I is not an independent set, a contradiction. Hence, I contains at least one vertex from the set $\{u_7, v_7, u_8, v_8\}$.

By symmetry, we may assume that $u_7 \in I$, and so $N(u_7) = \{u_6, u_8, v_8\} \subseteq T$. If $v_7 \in I$, then $\{u_9 v_9\} \subset T$ in order for the set T to totally dominate the vertices u_8 and v_8. However, this would imply that $\{u_{10}, v_{10}\} \subset I$ in order for the set I to dominate the vertices u_9 and v_9, and so I would contain two adjacent vertices, namely u_{10} and v_{10}, a contradiction. Hence, $v_7 \in T$. In order to dominate the vertex v_7, the set I therefore contains the vertex v_6, and so the neighbor v_5 of v_6 belongs to the set T. In order to totally dominate the vertex u_6, the set T contains the vertex u_5. This in turn implies that the set I contains the vertex u_4 in order to dominate the vertex u_5, and so $\{u_3, v_3\} \subset N(u_4) \subset T$. Thus, I contains the vertex v_4 in order to dominate the vertex v_4. Hence, $\{u_2, v_2\} \subset T$ in order for the set T to totally dominate the vertices u_3 and v_3. However, this would imply that $\{u_1, v_1\} \subset I$ in order for the set I to dominate the vertices u_2 and v_2, and so again I would contain two adjacent vertices, namely u_1 and v_1, a contradiction. □

6.3. CONCLUDING REMARKS

Using a link between hypergraphs and regular graphs established by Thomassen [25], Henning and Yeo [18] proved that the vertex set of every r-regular graph, for $r \geq 4$, can be partitioned into two disjoint TD-sets. However, this is not true for cubic graphs as observed by Seymour [22] and Alon and Bregman [1] who showed that the hypergraph equivalent of this result is not true for 3-uniform hypergraphs (as may be seen by considering, for example, the Fano plane). Indeed, there are infinitely many (connected) cubic graphs whose vertex sets cannot be partitioned into two TD-sets.

Theorem 6.6 ([1, 18, 22, 25])**.** *For every r-regular graph G, for $r \geq 4$, $V(G)$ can be partitioned into two TD-sets. However, there are infinitely many (connected) cubic graphs whose vertex set cannot be partitioned into two TD-sets.*

In contrast, not all r-regular graphs for any $r \geq 4$ are TI-graphs. In this paper, we have presented infinite families of r-regular graphs for any fixed $r \geq 3$ where every graph in the family is a TI-graph and also infinite families of r-regular graphs for any $r \geq 3$ where every graph in the family is not a TI-graph. We close with the following problem.

Problem 6.7. *Characterize the r-regular graphs, for $r \geq 3$, that are TI-graphs.*

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