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Influence of discrete model parameters on the KZK equation solution using the finite difference method

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ABSTRACT

Khokhlov - Zabolotskaya - Kuznetsov (KZK) nonlinear parabolic wave equation describes the acoustic pressure changes in the nonlinear and dissipative medium along the sound beam. This equation is solving numerically. On the basis of the finite difference method computer programs have been worked out. These programs allow to conduct different numerical investigations. The paper presents a mathematical model of finite amplitude wave propagation problem. The results of numerical investigations refer to influence of discrete model parameters on accuracy of numerical calculations are studied.

INTRODUCTION

The finite amplitude wave propagation problem is described basing on continuity, motion and state equations. The wave distortion is observed along its propagation path. It means that the harmonic wave shape changes step by step during wave propagation. It is impossible to use linear equations to describe this phenomenon. The system of above mentioned equations is converted to a nonlinear partial differential equation called the nonlinear equation of acoustics [4]. This equation has not exact analytical solution till now. Moreover it has rather complicated form. Therefore the equations which have easier form are used to solve the finite amplitude wave propagation problem in practice. Using the quasi-optical assumption the nonlinear equation of acoustic is converted to the KZK equation. It describes the changes in acoustic pressure along the sound beam. This equation allows to include nonlinearity, dissipation of medium and sound beam diffraction. Similarly as the nonlinear equation of acoustics, the KZK equation has not exact solution. There are known only asymptotic solutions of it. So there is necessary to solve this equation approximately. The analytic, half-analytic methods and numerical one are used to solve the KZK equation. The method of successive approximations [3] to find the KZK equation solution can be used when the nonlinear effects are not very big ($Re_a<1$, where Re_a - Reynold's number). The numerical methods allow to find the solution of the KZK equation in opposite situation. The finite difference method is one of numerical methods which can be used to solve this equation [1,2].

In this paper some difficulties connected with the numerical calculations using the finite difference method have been presented.

MATHEMATICAL MODEL

We assume that the circular piston with a fixed radius a is the finite amplitude wave source. Figure 1 shows this source and a coordinate system. The wave source is placed in plane yOz and the wave is propagated in the x direction.

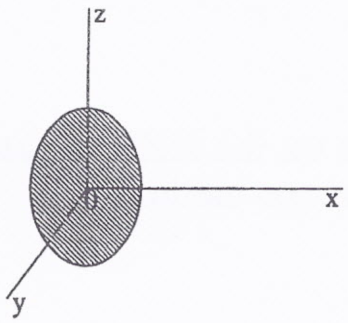

Fig.1 The wave source and a coordinate system

The mathematical model is built on basis of the KZK equation

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p'}{\partial x}-\frac{\varepsilon}{\rho_o c_o^3}p'\frac{\partial p'}{\partial\tau}-\frac{b}{2\rho_o c_o^3}\frac{\partial^2 p'}{\partial\tau^2}\right)=\frac{c_o}{2}\left(\frac{\partial^2 p'}{\partial y^2}+\frac{\partial^2 p'}{\partial z^2}\right) \qquad (1)$$

where $p'=p-p_o$ is an acoustic pressure, $\tau=t-x/c_o$ - the time in the coordinate system fixed in the zero phase of the propagating wave, y and z - axes ortogonal to an axis x, ρ_o - medium density at rest, c_o - speed of sound, b - dissipation coefficient of the medium, ϵ - nonlinerity parameter. This equation describes the changes in acoustic pressure in nonlinear and dissipative medium along a sound beam.

When the wave source is circular and amplitude of harmonic piston distribution is only function of variable $r=(y^2+z^2)^{1/2}$ it is comfortably solve this problem as an axial symmetric.

The solution of Eq. (1) is looked for inside a cylinder with radius R (Fig. 2a). The pressure p' is a function of three coordinates (x, r, τ) so the solution of the KZK equation is looked for in a domain D (Fig. 2b)

(a)

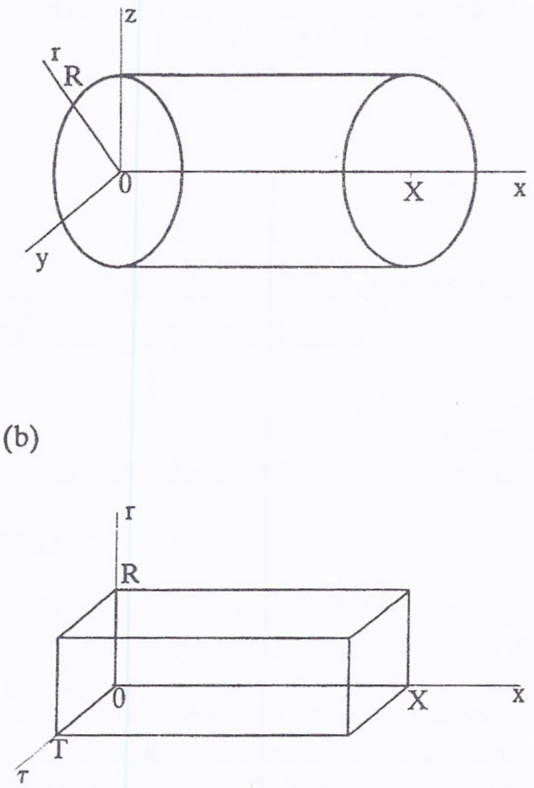

Fig.2 (a) Three - dimensional space and (b) domain D

$$D=\{(x,r,\tau)\in\mathbb{R}^3:\ x\in[0,X],\ r\in[0,R],\ \tau\in[0,T]\} \qquad (2)$$

To complete the problem the boundary conditions are added

$$\frac{\partial p'}{\partial r}\Big|_{r=0}=0 \qquad (3)$$

$$\begin{cases} p'(0,r,\tau)=-P(r)\sin(\omega\tau), & r\le a \\ p'(0,r,\tau)=0, & r>a \end{cases} \qquad (4)$$

where $\omega=2\pi f$, f - fundamental wave frequency. First of this conditions is connected with axialsymmetry and the second one describes the fundamental wave distribution on the source. Additionally it is assumed that

1. $p'(x,r,\tau)=0$ for $r>R$
2. p' is a periodic function of the coordinate τ.

NUMERICAL SOLUTION

The finite amplitude wave propagation problem is solved in fixed region and fixed time interval. The pressure changes along the sound beam are obtained after computer calculations.

To solve Eq. (1) numerically function $p'(x,r,\tau)$ is discretized in both space and time. The rectangular net is constructed in the domain D. Now let n designate the nth step in the x direction, k the kth step in the r direction and m the mth step time. So the net is defined in following form

$$x_n=n\Delta x,\quad r_k=k\Delta r,\quad \tau_m=m\Delta\tau \\ \Delta x=\frac{X}{N_x},\quad \Delta r=\frac{R}{N_r},\quad \Delta\tau=\frac{T}{N_\tau} \qquad (5)$$

where n=0, 1,..., N_x - 1, k=0, 1,..., N_r - 1, m=1, 2,..., N_τ. For fixed values of physical parameters (static pressure, density, speed of sound, nonlinear parameter, dissipation coefficient) the calculation's accuracy depends on step sizes. This accuracy depends on the value of cylinder radius R, too. It should be so large that the space can be considered as a half-infinite one. Moreover similar results are obtained for different types of pressure distribution on the source and different values of physical parameters.

The waveform change is equivalent with spectrum change. The harmonic analysis is very often used to investigate wave distortion. To find spectrum of the time waveform obtained during numerical calculations the fast Fourier transform (FFT) is used.

If the wave distortion is not very large the first and second harmonic amplitude changes can be observed using the method of successive approximations. Assuming that function P in formula (4) is a polynomial defined by

$$P(r)=-p_o\left(1-\frac{r^2}{a^2}\right)^2 \qquad (6)$$

the first harmonic amplitude evaluation as a function of range is defined in following form

$$|p_1(z)|=p_o\{[1+32z^2(\cos(\frac{1}{4z})-1)]^2+ \\ +[8z-32z^2\sin\frac{1}{4z}]^2\}^{1/2} \qquad (7)$$

where $z=x/2ka^2$, k - wave number [1].

Similar analytical formulas are known for another kinds of primary wave distributions on the source, especially when the first of conditions in the formula (4) is defined in following form [1,3]

$$p'(0,r,\tau)=-p_o\sin(\omega\tau) \qquad (8)$$

NUMERICAL INVESTIGATIONS

The numerical investigations were referred to influence of the discrete model parameters on calculation accuracy. This calculations were carried out using owns computer programs.

Figure 3 shows the first harmonic amplitude evaluation as a function of range on beam axis for R equal 2a and 4a respectively. During calculations it was assumed that the fundamental wave frequency was equal f=1 MHz and its distribution was defined by formula (6) with pressure p_o=10 kPa. The dissipation coefficient of the medium b=0, nonlinearity parameter ϵ=3.5. The dashed line was theoretically calculated using formula (7) and the solid one presents numerically calculated first harmonic amplitude changes. The source radius in this situation is equal a=2.5 cm. This figure shows calculating errors for distance larger than x=0.7 m when radius R=2a and too small changes of the first harmonic amplitude for distances near source. The differences observed for large distances are connected with the source beam diffraction and the second kind of numerical errors is connected with choose of time and space step sizes.

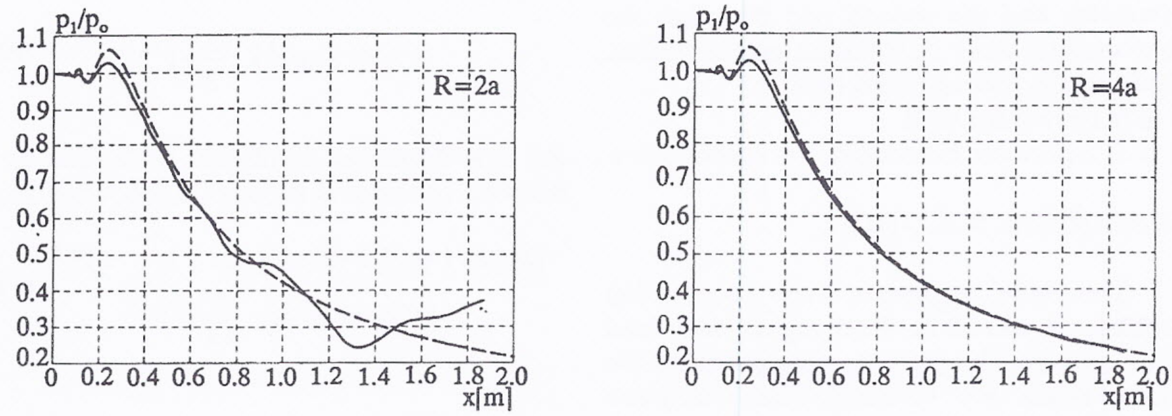

Fig.3 The first harmonic amplitude evaluation as a function of range on the beam axis (p_o=10 kPa) calculated numerically (solid line) and theoretically (dashed line)

Figures 4 and 5 shows the influence of value of radius R on the waveform. First of this figures demonstrates the wave shape as a function of time in fixed distances of source calculated using the same values of all physical parameters as previously and the second one presents similar results when pressure p_o=150 kPa.

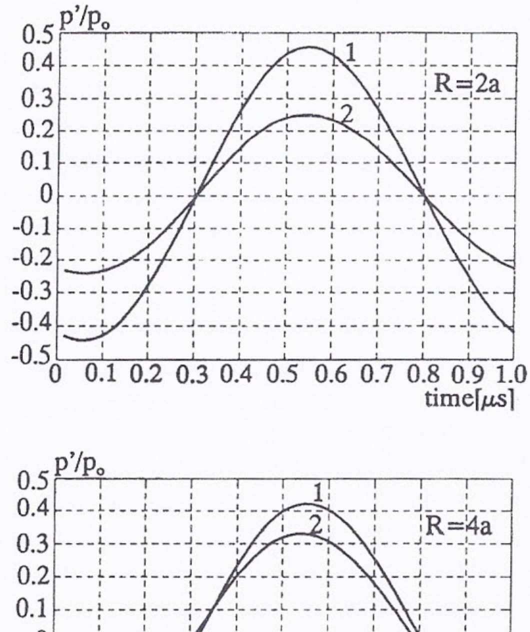

Fig.4 The waveform as a function of time on the beam axis for fixed distances of source (p_o=10 kPa): 1 - x=1 m, 2 - x=1.3 m

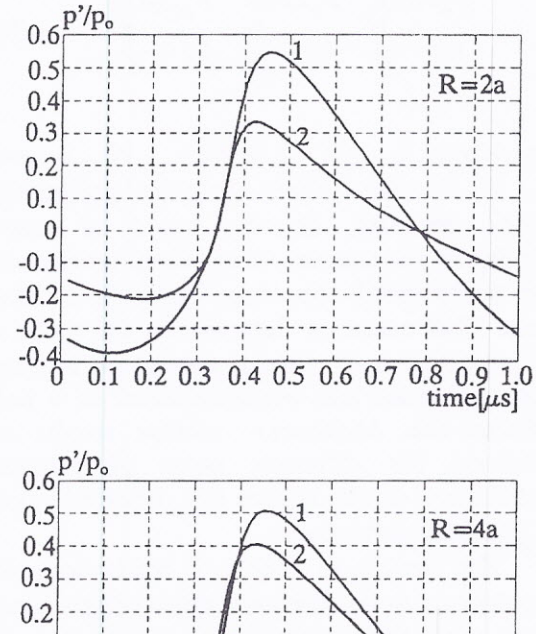

Fig.5 The waveform as a function of time on the beam axis for fixed distances of source (p_o=150 kPa): 1 - x=1 m, 2 - x=1.3 m

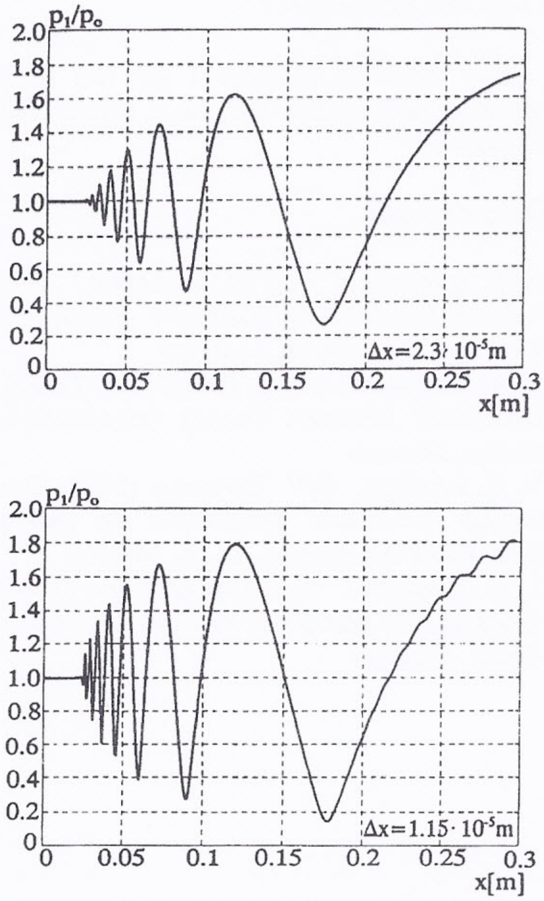

Fig. 6 The first harmonic amplitude evaluation as a function of range on the beam axis for different step sizes Δx

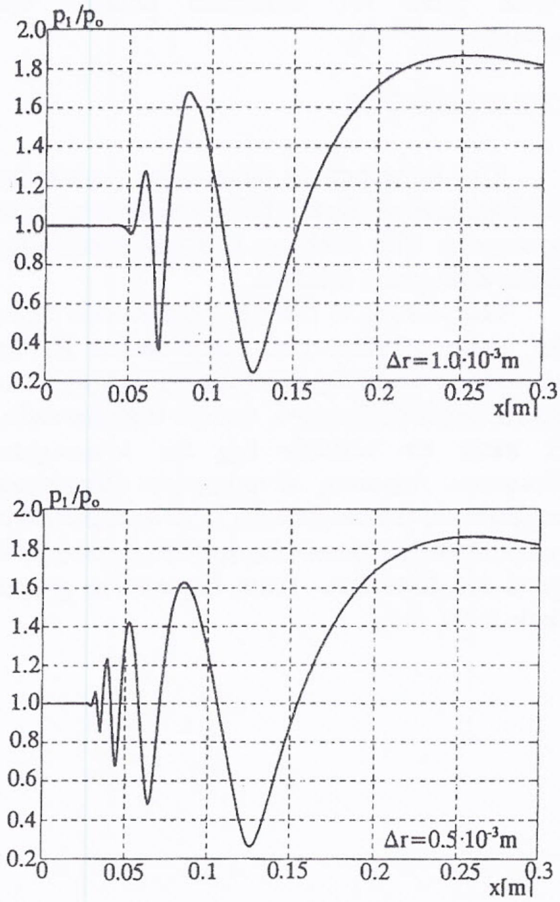

Fig. 7 The first harmonic amplitude evaluation as a function of range on the beam axis for different step sizes Δr

The results of numerical investigations of the space step sizes Δx and Δr influence on calculation accuracy are presented in Figs. 6 and 7 respectively.

Figure 6 shows the first harmonic amplitude evaluation as a function of range on the beam axis for different step sizes Δx. In this situation it was assumed that the fundamental wave distribution on the source was defined by formula (8) with frequency f=1 MHz and amplitude p_o=150 kPa.

The first harmonic amplitude evaluation as a function of range on the beam axis is presented in Fig.7. Calculations were made for different step sizes Δr. During these calculations was assumed that the distribution on the source was defined by (8) with fundamental wave frequency f=600 kHz. The step size $\Delta x=3.9\cdot 10^{-5}$ m. The values of the other parameters and conditions are similar as earlier, except source radius a. Now it is equal 2.5 cm (Fig. 6 presents results for a=2.3 cm).

The acoustic pressure changes near source are very fast. Both space steps Δx and Δr have influence on accuracy of calculations. The step size Δx has influence on numerically calculated value of harmonic amplitude maxima and step size Δr has influence where these maxima are situated. Lowering of step sizes make that the accuracy of numerical calculations increases. It means that the reality is better modeled then.

Another one effect of numerical calculations can be observed in presented figures. There are too small changes of first harmonic amplitude in distance near source in comparison with reality. It is consequence of quasi-optical assumption which is used to obtain the KZK equation. The quasi-optical approximation holds good for distances greater than $x_0=0.5a(ka)^{1/3}$ [3].

CONCLUSIONS

The influence of numerical parameters solving numerically the KZK equation has been presented. The problem was calculated using finite difference method.

The solution of the KZK equation is looked for inside cylinder with fixed radius R. The calculation accuracy depends on its value. The sound beam diffraction causes that the radius R must be suitably big for investigated distances. Accuracy of numerical calculations depends on the step sizes (5), too. It should be noted that the parabolic approximation holds good for distances from the source greater than fixed one.

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