Interaction of internal and surface waves in a two-layer fluid with free surface

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The interaction of internal and surface waves in a two-layer fluid with free surface has been considered. The stability of wave packets propagation on the contact surface and free surface of hydrodynamic system „layer with rigid bottom - layer with free surface" was investigated. The amplitudes of the second harmonics of the elevations of the contact surface and the free surface are investigated.

Key words: internal waves, surface waves, fluid layer, multiple scales, stability.

Introduction

The study of wave processes in different hydrodynamic systems is one of the most actual problems of hydrodynamics. And in the recent times more and more experimental researches are attached to a large number of theoretical works. Identification of the conditions of the stability is of considerable theoretical and practical interest of researchers of gravitational waves in the World Ocean as well as capillary waves in applied investigations.

The essential contribution to the study of this problem was made by the H.Segur & D.Hammakc [1], H.Yuen & B.Lake [2], M.Ablowitz & H.Segur [3], J.Whitham [4], P.Bhatnagar [5], D.Lamb [6], I.T.Selezov & S.V.Korsunsky [7], I.T.Selezov & P. Hug [8]. Propagation of wave packets in a fluid environment was described taking into account surface tension in articles [9-15].

In the article [16] the two-layer system was considered on the basis of the Euler equations when the thickness is small, the solutions were founded as series in the small thickness parameter.

Nonlinear internal waves on the surface of contact of two semi-infinite fluids with different densities were investigated in [17]. The solution was based on the Fourier series expansion of the unknown functions. The main characteristics of wave motions in some limiting cases were investigated.

Propagation of internal waves in the two-layer fluid bounded above and below by the solid lids was investigated in the article [18] without taking into account the effect of surface tension. The solution was obtained in the form of generalized power series in the parameter which depends on the value inversed to the Froude number.

A.Nayfeh [19] used the method of multiple scales to obtain a pair of partial differential equations that describe the evolution of finite-amplitude wave-packets on the interface of two semi-infinite fluids with different densities taking into account the effect of surface tension. As a result two alternative nonlinear Schrödinger equations were obtained and stability of finite amplitude wave-packets was investigated.

H.Hasimoto & H.Ono [20] used the method of multiple scales to obtain the nonlinear Schrödinger equation, which describes the evolution of gravity wave packets of finite amplitude on the surface of fluid layer.

Accounting surface tension plays an important role at investigation of the capillary waves at two-component hydrodynamic systems in laboratory researches. Also, the works [21-26] are devoted to the problem of propagation nonlinear internal waves.

In the papers published recently various aspects of the fourth approximation of the problems of evolution of nonlinear wave-packets were considered, such as, evolution equation near the cutoff critical wave number was obtained in [27], evolution equation for wave numbers far from the critical was obtained in [28], investigating the stability of the solutions of these equations in the system „layer - a half-space„ was carried out [29]. Investigation of stability for the system of „layer-layer„ was performed in the works [30].

In this paper the problem of nonlinear stability in the system „layer with rigid bottom - layer with free surface" is considered. Also, this article is devoted to the research of the interaction of internal and surface waves in a two-layer fluid with a free surface.

Materials and methods

The mathematical statement of the problem for wave-packet propagation along the interface between the upper layer with free surface and a lower layer with free surface is presented in the form

$$\nabla^2\varphi_j = 0 \text{ in } \Omega_j,$$
$$\eta_{,t} - \varphi_{j,z} = -\alpha\varphi_{j,x}\eta_{,x} \text{ at } z = \alpha\eta(x,t),$$
$$\eta_{0,t} - \varphi_{2,z} = -\alpha\varphi_{2,x}\eta_{0,x} \text{ at } z = \alpha\eta_0(x,t),$$
$$\varphi_{1,t} - \rho\varphi_{2,t} + (1-\rho)\eta + 0.5\alpha\left[(\nabla\varphi_1)^2 - \rho(\nabla\varphi_2)^2\right] -$$
$$-T\left(1+\alpha^2\eta_{,x}^2\right)^{-3/2}\eta_{,xx} = 0 \text{ at } z = \alpha\eta(x,t), \quad (1)$$
$$\varphi_{2,t} + \eta_0 + 0.5\alpha(\nabla\varphi_2)^2 -$$
$$-T_0\left(1+\alpha^2\eta_{0,x}^2\right)^{-3/2}\eta_{0,xx} = 0 \text{ at } z = \alpha\eta_0(x,t),$$
$$\varphi_{1,z} = 0 \text{ at } z = -h_1,$$

where φ_j $(j = 1,2)$ are the velocity potentials; η and η_0 are the elevations of the interface and the free surface; $\rho = \rho_2/\rho_1$ is ratio of fluids densities; $\alpha = a/L$ is the nonlinearity coefficient; the lower fluid layer $\Omega_1 = \{(x,z) : |x| < \infty, -h_1 \le z < 0\}$ and the upper fluid layer of $\Omega_2 = \{(x,z) : |x| < \infty, 0 \le z \le h_2\}$.

Dimensionless values were introduced using the characteristic L, the maximal free surface elevation a, density of the lower fluid ρ_1, the acceleration of the gravity g. The solutions of the nonlinear problem (1) is determined using the method of multiple scale expansions up to the third-order approximation [19]

$$\eta(x,t) = \sum_{n=1}^{3}\alpha^{n-1}\eta_n(x_0,x_1,x_2,t_0,t_1,t_2) + O(\alpha^3),$$
$$\eta_0(x,t) = \sum_{n=1}^{3}\alpha^{n-1}\eta_{0n}(x_0,x_1,x_2,t_0,t_1,t_2) + O(\alpha^3), \quad (2)$$
$$\varphi_j(x,z,t) = \sum_{n=1}^{3}\alpha^{n-1}\varphi_{jn}(x_0,x_1,x_2,z,t_0,t_1,t_2) + O(\alpha^3),,$$

where α is small dimensionless parameter characterizing the steepness ratio of the wave $x_n = \alpha^n x$, $t_n = \alpha^n t$. Substituting of expansions (2) into the equation (1) leads to three linear problems relatively unknown functions η_1, η_{01}, φ_{11}, φ_{21}, η_2, η_{02}, φ_{12}, φ_{22}, η_3, η_{03}, φ_{31}, φ_{32}.

Results and discussion

Analysis of the stability of wave-packets

Solutions of the first linear problem and analysis of the interaction of internal and surface waves in the first approximation are given in [31, 32]. In [33] the second linear problem solutions were found and evolution equations for wave packets enveloping were obtained and the form of the wave packet on surface of contact and on the free surface were analyzed.

Evolution equations of envelopes on the surface of contact and on the free surface

$$A_{,t} + \omega' A_{,x} - 0.5\omega'' A_{,xx} = i\alpha^2 I A^2 \bar{A}, \quad (3)$$
$$A^0_{,t} + \omega' A^0_{,x} - 0.5\omega'' A^0_{,xx} = i\alpha^2 I_0 (A^0)^2 \bar{A}^0,$$

where $\bar{A}(x_1,x_2,t_1,t_2)$ is the complex conjugate of the complex envelope $A(x_1,x_2,t_1,t_2)$, $\theta = kx_0 - \omega t$, k is the wave number and ω is the wave frequency of the center of the wave-packets, $\omega' = d\omega/dk$, $\omega'' = d^2\omega/dk^2$. According to [14, 28], equation (3) has a solution that depends only on time

$$A = a\exp(i\alpha^2 a^2 \omega^{-1} I t), \quad (4)$$
$$A^0 = a^0 \exp(i\alpha^2 (a^0)^2 \omega^{-1} I_0 t).$$

As in previous articles [19, 30] the conditions of the instable of wave-packets on the interface and free surface are in the form

$I\omega'' > 0$, $\qquad I_0\omega'' > 0$.

Curves defined by equations

$I\omega'' = 0$ (curves "1" and "3"),

$I\omega'' \to \infty$ (curves "2" and "4"),

are presented below. The curves were constructed in a coordinate system (ρ, k), the range of wave numbers that was considered is $0 \le k \le 2.5$ for different values of thickness of lower fluid layer $h_1 \in \{1, 1.73, 2.23, 10\}$ and fixed value of $h_2 = 1$ and in the case of absence of surface tension $T = 0$, $T_0 = 0$.

The region of linear instability is separated by p = 1, thus curve with index „3" is the same vertical line. The curve „4" is contained at region of linear instable. Thereby, only curves „1" and „2" define the borders of regions of nonlinear modulational stability (MS) and modulational instability (MI), which. are alternated.

If h_1 = 10 the four curves that separate region of nonlinear stability and unstable in 7 regions were found. There are two regions of modulational stability for gravitational and capillary waves in the case of p < 1 (Fig. 1 a).

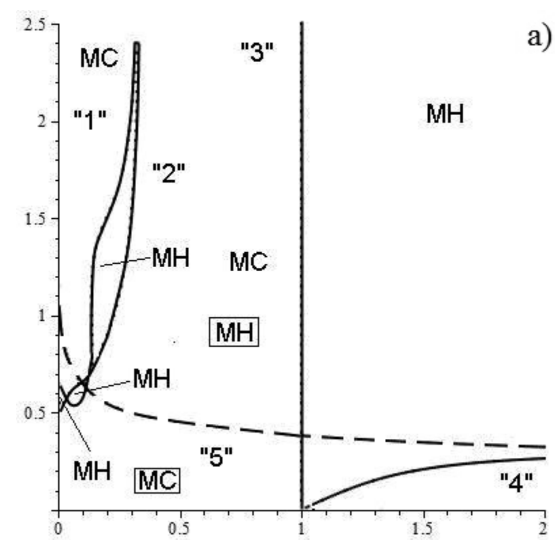

Fig. 1. Diagrams of stability: *T = 0, T_0 = 0, h_2 = 1,* a) *h_1 = 10*

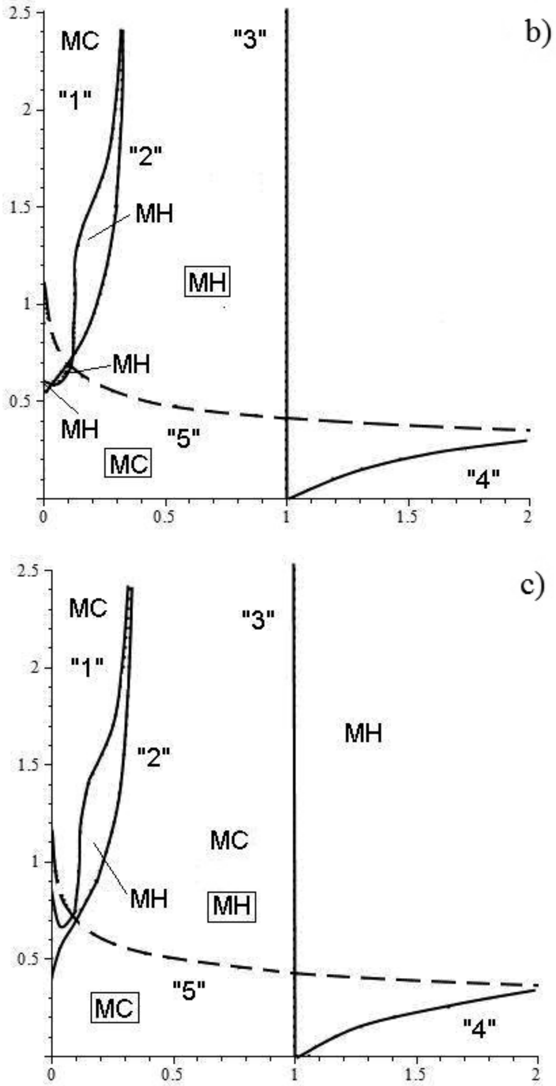

Fig. 1. Diagrams of stability: *T = 0, T₀ = 0, h₂ = 1*, b) *h₂ = 2.23*, c) *h₃ = 1.73*

Reduction of the thickness to the value $h_2 = 2.23$ leads to such result: the curve „1" was raised up, and the curve „2" was rectified near the intersection with the axis of ordinates. The curve „4" was removed up from the axis p (Fig. 1 b). Hus, the region of modulational stability has covered physically important region (gravitational and capillary waves).

If $h_3 = 1.73$ the curve „2" became even smoother. In this case three regions of modulational instability have merged (Fig. 1 c). Also, curve „4" is further away from the axis p.

Analysis of the second approximation problem solutions

Here are some of the results obtained in previous works. Solution of the first linear approximation problem and the dispersion equation are as follows

$$\eta_1 = A\,\mathrm{e}^{i\theta} + \bar{A}\,\mathrm{e}^{-i\theta}, \qquad (5)$$

$$\eta_{01} = \frac{\omega^2 (A\,\mathrm{e}^{i\theta} + \bar{A}\,\mathrm{e}^{-i\theta})}{\omega^2\,\mathrm{ch}(kh_2) - (k + T_0 k^3)\,\mathrm{sh}(kh_2)},$$

$$\omega^2\,\mathrm{cth}(kh_1) + \rho\omega^2 \left(\frac{\omega^2 - (k + T_0 k^3)\,\mathrm{cth}(kh_2)}{\omega^2\,\mathrm{cth}(kh_2) - (k + T_0 k^3)} \right) =$$

$$= (1-\rho)k + Tk^3.$$

The solutions to the problem of the second linear approximation

$$\eta_2 = \frac{0.5\omega^2}{1-\rho}\Bigg(1 - \rho - \mathrm{cth}^2(kh_1) +$$

$$+\rho\left[\frac{(1-\rho)k + Tk^3 - \omega^2\,\mathrm{cth}(kh_1)}{\rho\omega^2}\right]^2\Bigg) A\bar{A} +$$

$$+\Lambda\,\mathrm{e}^{2i\theta} A^2 + cc, \qquad (6)$$

$$\eta_{02} = \frac{0.5\omega^2 \left(\omega^4 - (k + T_0 k^3)^2\right)}{\omega^2\,\mathrm{ch}(kh_2) - (k + T_0 k^3)\,\mathrm{sh}(kh_2)} +$$

$$+\Lambda_0 e^{2i\theta} A^2 + cc.$$

where cc is the complex conjugate value of the preceding expression

In the first approximation the interaction of internal and surface waves was investigated based on expression [31]

$$\eta_1 = A_1 \cos(kx - \omega_1 t) + A_2^0 a_2 \cos(kx - \omega_2 t),$$

$$\eta_{01} = A_1 a_1 \cos(kx - \omega_1 t) + A_2^0 \cos(kx - \omega_2 t), \qquad (7)$$

where

$$a_1 = \frac{\omega_1^{\,2}}{\omega_1^{\,2}\,\mathrm{ch}(kh_2) - (k + T_0 k^3)\,\mathrm{sh}(kh_2)},$$

$$a_2 = \left(\frac{\omega_2^{\,2}}{\omega_2^{\,2}\,\mathrm{ch}(kh_2) - (k + T_0 k^3)\,\mathrm{sh}(kh_2)}\right)^{-1}.$$

Let us analyze the amplitudes of the second approximation of elevation of the free surface $\eta_0(x,t)$ and of the second approximation of elevation of the contact surface $\eta(x,t)$ corresponding to the pairs of frequencies $\pm 2\omega_1$ and $\pm 2\omega_2$. We denote these ratios c_1 and c_2 respectively

$$c_1 = \frac{\Lambda_0(2\omega_1)}{\Lambda(2\omega_1)}, \qquad c_2 = \left(\frac{\Lambda_0(2\omega_2)}{\Lambda(2\omega_2)}\right)^{-1}. \qquad (8)$$

The value c_1 characterizes the contribution of waves with a frequency $2\omega_1$ into surface wave movement and value c_2 characterizes the contribution of waves with a frequency $2\omega_2$ into wave movement at the interface between two fluid layers.

In Fig. 2 the dependence of variable c_1 on the thickness of lower layer h_1 by changing the ratio of densities ρ is presented for the following parameters of the system $h_2 = 1$, $\rho \in \{0.6, 0.7, 0.8, 0.9\}$, $T = T_0 = 0$ and wave num-

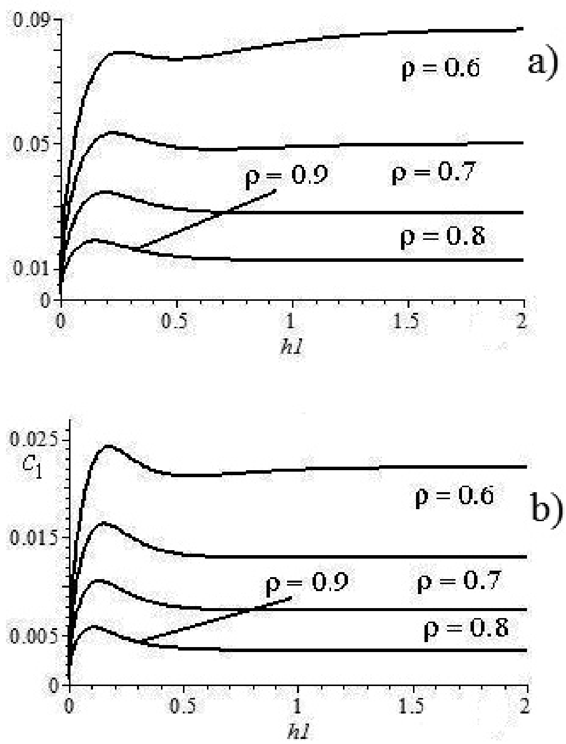

Fig. 2. The dependence of c_1 on the thickness of lower layer: a) k = 1.5; b) k = 2

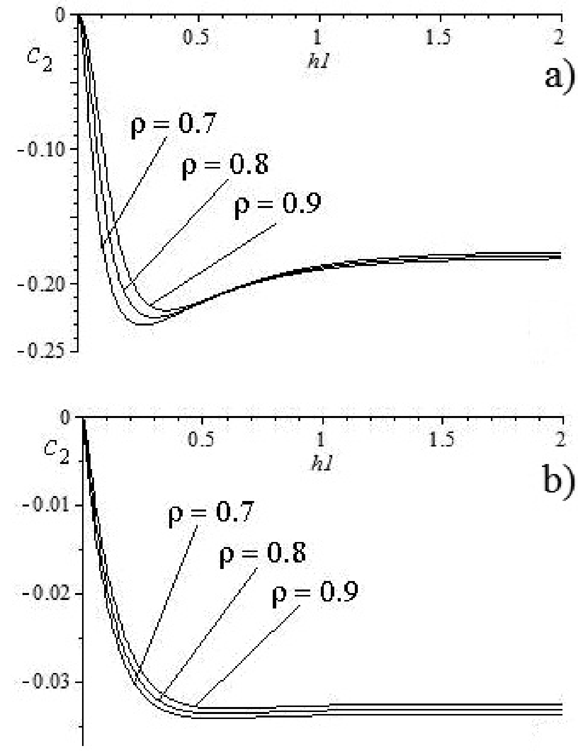

Fig. 3. The dependence of c_2 on the thickness of lower layer: a) k = 1.5; b) k = 2

bers $k = 1.5$ (Fig. 2a) and $k = 2$ (Fig. 2b). Changing the ratio of densities ρ affects the value of c_1. Moreover, with increasing of the wave number k value c_1 becomes smaller, although still greater than zero. In each case, there is a certain boundary value to which c_1 tends with increasing of thickness of the lower layer.

With increasing h_1 from 0 to 0.5 it is noticeable that for each of the wave number k and densities ratio ρ the value c_1 reaches a local maximum on this interval. This means that contribution of waves with frequency $2\omega_1$ in surface movement is the highest possible one for the gravitational waves in such system parameters. In Fig. 3 the dependence of c_2 on the thickness of lower layer h_1 by changing of densities ratio ρ is represented for the following parameters of the system $h_2 = 1$, $T = T_0 = 0$, $\rho \in \{0.7, 0.8, 0.9\}$ and wave numbers $k = 1.5$ (Fig. 3a) and $k = 2$ (Fig. 3b).

Changing ratio of densities slightly affects on the change in value c_2, whereby it remains negative at different wave numbers. Should be noted that analyzing the amplitude ratio in the first approximation, the opposite phenomenon is observed (value a_1 corresponding to wave with frequency ω_1 was always negative and the value a_2 corresponding to wave with frequency ω_2 was positive, and also it almost does not depend on changes in ratio of densities).

With the growing h_1 from 0 to 0.5 the local minimum at different of densities ratios ρ takes place that was observed for value c_1. That is, such system parameters contribution of waves with frequency $2\omega_2$ of wave motion on the surface of contact between two layers is maximized for capillary waves. Elevation of the contact surface and elevation of the free surface in the second approximation are in the form

$$\eta_2 = 2A^2\left(B(2\omega_1) + \Lambda(2\omega_1)\cos(2kx - 2\omega t)\right) +$$
$$+2\left(A_2^0\right)^2\left(B(2\omega_2) + + c_2\Lambda(2\omega_2)\cos(2kx - 2\omega t)\right), \quad (9)$$
$$\eta_{02} = 2A_1^2\left(C(2\omega_1) + c_1\Lambda_0(2\omega_1)\cos(2kx - 2\omega t)\right) +$$
$$+2\left(A_2^0\right)^2\left(C(2\omega_2) + + \Lambda_0(2\omega_2)\cos(2kx - 2\omega t)\right),$$

where c_1 and c_2 are given by (8) and

$$B(\omega_{1,2}) = \frac{0.5\omega_{1,2}^2}{1-\rho}\left(1 - \rho - \mathrm{cth}^2(kh_1) + \right.$$
$$\left. + \rho\left[\frac{(1-\rho)k + Tk^3 - \omega_{1,2}^2\,\mathrm{cth}(kh_1)}{\rho\omega_{1,2}^2}\right]^2\right),$$
$$C(\omega_{1,2}) = \frac{0.5\omega_{1,2}^2\left(\omega_{1,2}^4 - (k + T_0k^3)^2\right)}{\omega_{1,2}^2\,\mathrm{ch}(kh_2) - (k + T_0k^3)\,\mathrm{sh}(kh_2)}.$$

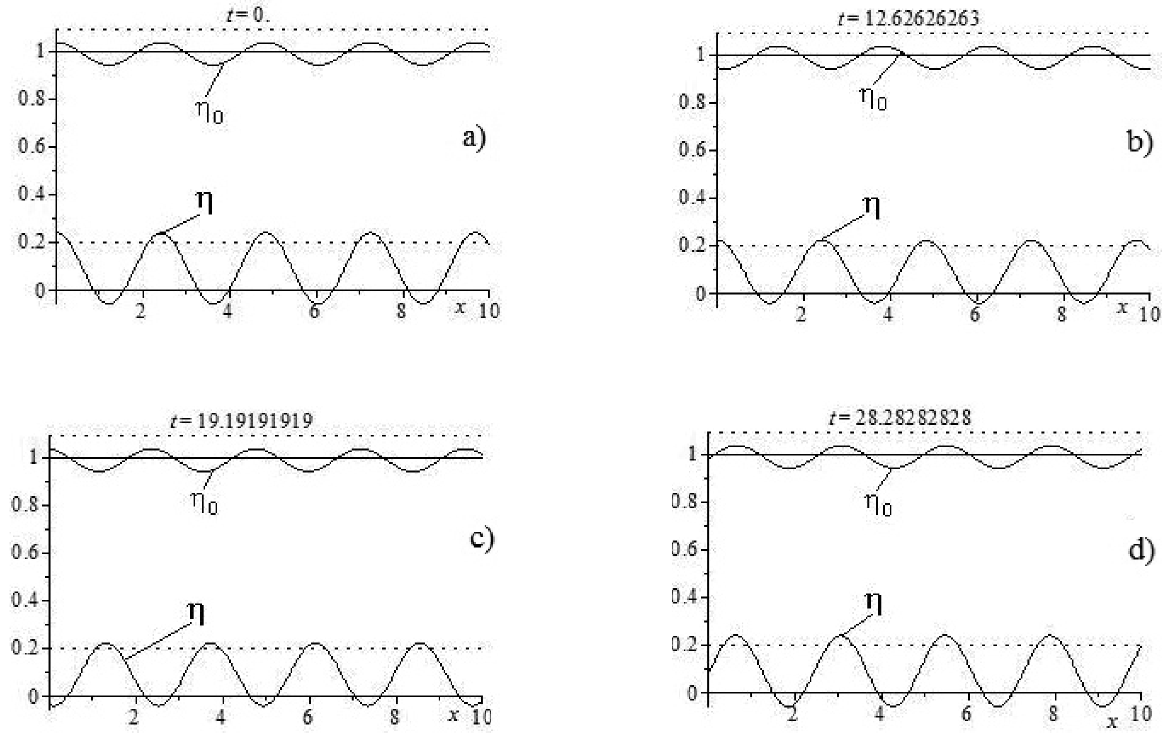

Fig. 4. Second approximation of the contact surface elevation η_2 and free surface elevation η_{02} at different times

In Fig. 4 graphics of η_2 and η_{02} are presented at different times for the following system parameters $h_2 = 1$, $h_1 = 3$, $T = T_0 = 0$, $k = 1.3$, $\rho = 0.9$, $A_1 = 0.2$, $A_2^0 = 0.09$. Levels $A_1 = 0.2$ and $A_2^0 + h_2 = 1.09$ are marked by dotted line.

The elevation η_{02} changes over time is faster than elevation η_2 changes over time, so, rotation of minima and maxima η_{02} and η_2 takes place (Fig. 4c and 4d). Also, increase or decrease the amplitude of the wave on the contact surface takes place with the change of time (Fig. 4a and 4b). For free surface the similar picture is observed to a lesser degree.

Interaction of internal and surface waves

Substituting (7) and (9) the solutions of evolution equations (4) and given expressions for the elevation of the contact surface and the free surface (2) and performing the necessary transformations we obtain

$$\eta = a\cos(kx - \tilde{\omega}_1 t) + a^0 a_2 \cos(kx - \tilde{\omega}_2 t) +$$
$$2\alpha a^2 \left(B(2\omega_1) + \Lambda(2\omega_1)\cos(2kx - 2\tilde{\omega}_1 t)\right) +$$
$$+2\alpha \left(a^0\right)^2 \left(B(2\omega_2) + c_2\,\Lambda(2\omega_2)\cos(2kx - 2\tilde{\omega}_2 t)\right),$$
$$\eta_0 = aa_1 \cos(kx - \tilde{\omega}_1 t) + a^0 \cos(kx - \tilde{\omega}_2 t) +$$
$$+2\alpha a^2 \left(B(2\omega_1) + +c_1\Lambda(2\omega_1)\cos(2kx - 2\tilde{\omega}_1 t)\right) +$$
$$+2\alpha \left(a^0\right)^2 \left(B(2\omega_2) + \Lambda(2\omega_2)\cos(2kx - 2\tilde{\omega}_2 t)\right),$$

where

$$\tilde{\omega}_1 = \omega_1 - \alpha^2 a^2 \omega_1^{-1} I,\ \tilde{\omega}_2 = \omega_2 - \alpha^2 (a^0)^2 \omega_2^{-1} I_0 .$$

In Fig. 5 and Fig. 6 graphs η and η_0 are presented at different times for the following system parameters $h_2 = 1$, $h_1 = 3$, $T = T_0 = 0$, $k = 1.3$, $\rho = 0.9$, $a = 0.2$, $a^0 = 0.2$, $\alpha = 0.1$. Levels $a = 0.2$ and $a^0 + h_2 = 1.2$ are marked by dotted line.

As significantly from the figure the amplitude of internal and surface waves is increased then decreased as compared to the unperturbed state $a = a_0 = 0.2$.

Taking into account both pairs of frequencies in the propagation of waves, we notice that the internal waves leads to periodic sharpening and smoothing the wave crest. That is, the wave crest moving faster than the base at first collapses (Fig. 6a) and then under the action of dispersion it smoothed (Fig. 6a, ab). This is due to the balance between nonlinearity and dispersion.

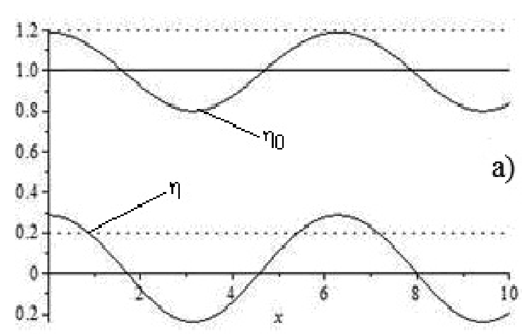

Fig. 5. Internal and surface waves at different times: a) $t = 0$

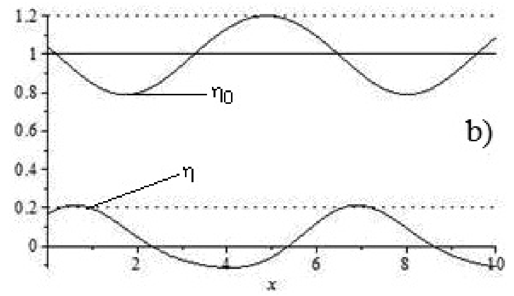

Fig. 5. Internal and surface waves at different times: a) *t* = 5

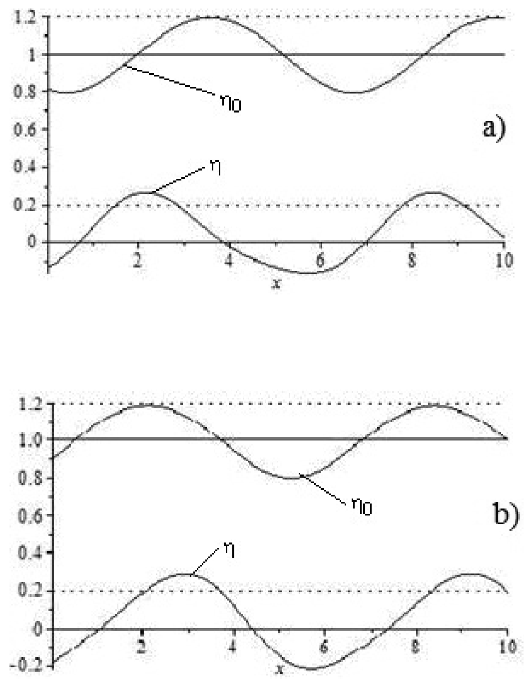

Fig. 5. Internal and surface waves at different times: a) *t* = 10, b) *t* = 15

Conclusions

The stability of wave packets propagation on the contact surface and free surface of hydrodynamic system „layer with rigid bottom – layer with free surface" was investigated. The diagrams for nonlinear modulational stability for different thicknesses of lower layer were constructed. The presence of large regions of nonlinear modulational stability for capillary and gravitational waves for different ratios of density and different thicknesses of the two fluid layers was obtained. It was noted that the region of modulational nonlinear instability of the wave packet increased with decreasing of thickness of the lower layer.

The interaction of internal and surface waves in a two-layer fluid with free surface has been considered. The amplitudes of the second harmonics of the elevations of the contact surface and the free surface for both pairs of frequencies of the center of the wave packet are investigated. It is specified on the influence of nonlinearity and dispersion in the propagation of internal and surface waves.

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