APPLICATION OF DARBOUX TRANSFORMATION TO MULTIDIMENSIONAL INHOMOGENEOUS PROBLEMS

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(received: 4 January 2016; revised: 15 February 2016;
accepted: 22 February 2016; published online: 28 March 2016)

Abstract: General properties of ladder operators applied to inhomogeneous problems are studied in the context of their usefulness for solving practical problems with stress put on the possibility of embedding the interwine relation onto a wider class of operators. From those general remarks an algorithm using the Darboux transform for construction of the Green function for linear partial differential equations is formed and a sample implementation thereof is shown along with some examples of solutions.

Keywords: Darboux transform, Green function, interwine relation, ladder operator, inhomogeneous partial differential equation

1. Introduction

Ladder operators in general and the Darboux transformation in particular in its numerous forms from the classic Darboux theorem [1] to abstract constructions on associative rings [2] are invaluable tools in studying both linear and nonlinear systems. The creation-annihilation operators [3] are one of the most widely used formalisms in quantum physics, whereas the Darboux transform has been one of the most important tools in constructing multisoliton solutions of numerous integrable systems [4]. One of the key setbacks for using them in a broader context is the fact that each known ladder operator can be constructed only for a narrow class of problems. An especially important step was made in [5] where the Darboux covariance property was spread to evolution $1+1$ equations.

With that in mind, the aim of this paper is to examine possible embeddings of the known entwine relations into broader systems as well as to apply ladder operators to inhomogeneous problems. The general formulae are then explicitly used to solve two-dimensional Green function problems using a basic, one-dimensional Darboux transformation. The key ideas have been laid out earlier in [2]. Ideas very close to those presented in this paper have been applied to the Green function construction for $1+1$ hyperbolic and parabolic equations with integrable potentials [6]

2. General properties

The primary aim of this section is to examine the use of ladder operators in the context of inhomogeneous problems as well as to determine a set of requirements for them to be practical in that regard. Let us start with arbitrary operators entering the intertwine relation

$$AL = L[1]A \tag{1}$$

If we take an equation

$$L\psi = r \tag{2}$$

where r and ψ are elements of the codomain and domain of operator L, respectively, with r considered as the inhomogeneity and ψ as the solution of the equation. By using the intertwine relation we readily obtain

$$\begin{aligned} AL\psi &= Ar \\ L[1]A\psi &= Ar \end{aligned} \tag{3}$$

which obviously means that $A\psi$ (from this point on denoted as $\psi[1]$) is a solution to the new equation and Ar (denoted later as $r[1]$) is the new inhomogeneity. Such a relation is of use only if we can control the right-hand side of the equation. For instance, if we want to find through an intertwine relation a solution to the equation

$$L[1]\psi[1] = r[1] \tag{4}$$

with a particular inhomogeneity, then we need to solve an additional problem for r

$$Ar = r[1] \tag{5}$$

as well. This is not strictly necessary, but it is immensely useful to be able to invert A (or more specifically just have the relation $AA^{-1} = \mathrm{I}$) and obtain r directly as

$$r = A^{-1}r[1] \tag{6}$$

At this point it is worth remarking that in case of an invertible A, we may also write

$$L[1] = ALA^{-1} \tag{7}$$

which implies that the intertwine relation can be considered as an extension of mapping of operators for non-invertible maps. It is important to stress that up to this point we did not assume linearity of the L and A operators or any other qualities except those explicitly stated. This means that inhomogeneous equations (or homogeneous as a special case of $r[1] = 0$) allow a much wider class of ladder operators than spectral problems for which they are mostly used, because

$$\begin{aligned} AL\phi &= A\lambda\phi \\ L[1]A\phi &= \lambda A\phi \end{aligned} \tag{8}$$

obviously requires A to be multiplicative (but not necessarily additive).

If L is invertible (in the same sense as A earlier), we may generalise the investigated problem to that of finding the inverse of $L[1]$ instead of a particular solution of an inhomogeneous equation. If we label L^{-1} as G, we can write

$$\begin{aligned} LG &= I \\ L[1]G[1] &= I \end{aligned} \tag{9}$$

G is assumed to be known in advance and $G[1]$ is obtained as follows. We start from an obvious identity

$$A = A \tag{10}$$

and insert the above shown forms of identity operators

$$ALG = L[1]G[1]A \tag{11}$$

After using the intertwine relation

$$L[1]AG = L[1]G[1]A \tag{12}$$

we can readily write

$$AG = G[1]A \tag{13}$$

keeping in mind that the last equation is true only up to the elements of kernel of $L[1]$, which is a typical degree of freedom for inverse operators anyway. It clearly shows that the inverse of L enters the same entwine relation as the operator itself. Again, for invertible A we may write

$$G[1] = AGA^{-1} \tag{14}$$

which is the form we will later use for calculation of a dressed inverse operator.

2.1. Extension

In practice it is possible to build a ladder operator A only for very specific types of operators L. Therefore, it is important to investigate the possibility of extending a given entwine relation onto a wider set of operators. One particular case which will be considered in this paper requires that A is additive. If so, then for a given operator D which commutes with A the initial relation (1) obviously implies

$$A(D + L) = (D + L[1])A \tag{15}$$

3. Differential equations

If we want to apply ladder operators to differential equations, we need to establish, how they in general interact with initial or boundary conditions. Since those conditions only represent values of a given function in specific points, a ladder operator defined on those functions cannot be applied directly. As with the inhomogeneity, it does not prevent the use of a ladder operator - it is simply an issue of controlling the initial or border condition in the resulting equation. The most straightforward method is available, when we have a general solution to the initial problem. Then, we simply transform it and solve the new initial or border conditions with respect to the parameters of the initial solution. While it is not strictly necessary, our ability to solve the resulting set of equations depends highly on the linearity of both L and A.

4. Darboux transformation

In the light of the discussed properties of ladder operators, the Darboux transformation is particularly valuable, since it is a linear and invertible operation and it also preserves the conditions of the initial problem. Therefore, even if in the most basic case, when it is defined for a specific one-dimensional eigenvalue problem, it can be used just as well for a multidimensional inverse operator problem. For instance, let us look at a Green function problem for a simple wave equation on $\mathbb{R}^2$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)g(t,t_0,x,x_0)=\delta(t-t_0)\delta(x-x_0) \tag{16}$$

A Darboux transformation built for

$$\frac{\partial^2}{\partial x^2}\phi(x)=\lambda\phi(x) \tag{17}$$

is also applicable to (16) thanks to the linearity of the Darboux transform and the resulting property (15) (if the Darboux transform operates on only the x variable, then it will obviously commute with $\frac{\partial^2}{\partial t^2}$).

Before solving this problem let us look more closely at the expression (14) in case of the Darboux transform as the ladder operator

$$\begin{aligned} A&=\phi(x)\frac{\partial}{\partial x}\frac{1}{\phi(x)}\\ A^{-1}&=\phi(x)\int\limits_{-\infty}^{x}dx_1\frac{1}{\phi(x_1)} \end{aligned} \tag{18}$$

At this point we need to take into account that in order to obtain the Green function properly, we need to explicitly transform the identity operator kernel

$$\delta(x-x_0)\delta(t-t_0) \tag{19}$$

To show the construction, we will apply subsequent operators step by step

$$\begin{aligned} &\phi(x_2)\int\limits_{-\infty}^{x_2}dx_1\frac{1}{\phi(x_1)}\delta(x_1-x_0)\delta(t_1-t_0)\\ \int\limits_{-\infty}^{\infty}dt_1\int\limits_{-\infty}^{\infty}dx_2 g(t,t_1,x,x_2)&\phi(x_2)\int\limits_{-\infty}^{x_2}dx_1\frac{1}{\phi(x_1)}\delta(x_1-x_0)\delta(t_1-t_0)\\ \phi(x)\frac{\partial}{\partial x}\frac{1}{\phi(x)}\int\limits_{-\infty}^{\infty}dt_1\int\limits_{-\infty}^{\infty}dx_2 g(t,t_1,x,x_2)&\phi(x_2)\int\limits_{-\infty}^{x_2}dx_1\frac{1}{\phi(x_1)}\delta(x_1-x_0)\delta(t_1-t_0) \end{aligned} \tag{20}$$

This form can be simplified on assumption that differentiation and integration operations can be done in arbitrary order. Firstly, the evaluation of the integral over x_1 yields

$$\phi(x)\frac{\partial}{\partial x}\frac{1}{\phi(x)}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dx_2 g(t,t_1,x,x_2)\frac{\phi(x_2)}{\phi(x_0)}\Theta(x_2-x_0)\delta(t_1-t_0) \tag{21}$$

with Θ as the Heaviside step function, which can be simply included in the integration boundaries over x_2

$$\phi(x)\frac{\partial}{\partial x}\frac{1}{\phi(x)}\int_{-\infty}^{\infty}dt_1\int_{x_0}^{\infty}dx_2 g(t,t_1,x,x_2)\frac{\phi(x_2)}{\phi(x_0)}\delta(t_1-t_0) \tag{22}$$

We can also perform the integration over t_1

$$\phi(x)\frac{\partial}{\partial x}\frac{1}{\phi(x)}\int_{x_0}^{\infty}dx_2 g(t,t_0,x,x_2)\frac{\phi(x_2)}{\phi(x_0)} \tag{23}$$

We will also apply the Darboux transform before the integration

$$\int_{x_0}^{\infty}dx_2\left(\frac{\partial g}{\partial x}(t,t_0,x,x_2)-g(t,t_0,x,x_2)\frac{\partial(\ln\phi)}{\partial x}(x)\right)\frac{\phi(x_2)}{\phi(x_0)} \tag{24}$$

This is the form of $g[1]$ that we will use for further calculations.

4.1. N-fold transform

One of the important features of the Darboux transform is the ability to construct a chain of transformations. Given a set of eigenfunctions $\{\phi_0,\ldots,\phi_{N-1}\}$ of the initial operator L, one can define a chain of N Darboux transformations

$$\begin{aligned}
A[0] &= \phi_0(x)\frac{\partial}{\partial x}\frac{1}{\phi_0(x)}\\
A[k] &= \phi_k[k](x)\frac{\partial}{\partial x}\frac{1}{\phi_k[k](x)}\\
\phi_k[k] &= A[k-1]\ldots A[0]\phi_k
\end{aligned} \tag{25}$$

Although there is a method for calculating the N-fold Darboux transform in one step, its form does not give an easy way for inverting it. For this reason we will construct the chain of Green functions iteratively. It is most convenient to prepare the seed functions in advance. Given the set $\{\phi_0,\ldots,\phi_{N-1}\}$ we need to apply $A[0]$ to all seed functions except for the first, which gives us the function needed for $A[1]$, which needs to be applied to all seed functions except for the first two ones. We proceed in the same manner to the end of the chain and obtain a full set of functions needed for performing Darboux transforms.

In order to construct the Green function we simply apply (24) iteratively. In symbolic terms, we perform

$$
\begin{aligned}
G[1] &= A[0]GA[0]^{-1} \\
G[2] &= A[1]G[1]A[1]^{-1} \\
&\dots \\
G[N] &= A[N-1]G[N-1]A[N-1]^{-1}
\end{aligned}
\tag{26}
$$

Sample implementation in Mathematica is provided in the Appendix.

It is interesting to note that we do not explicitly need to know the form of the potential for each step of the chain. We do however perform the calculations in order to solve a Green function problem for a specific multisolitonic potential. While the potential for each step can be easily calculated in a regular way

$$u[k] = u - 2\frac{\partial^2}{\partial x^2}\ln W\left(\phi_1, \phi_2, \dots, \phi_k\right) \tag{27}$$

(with W as the Wronskian determinant and u as the initial potential), there is no algorithmic method for obtaining the set of seed functions needed for construction of a given potential.

4.2. Examples

Let us consider the wave operator Green function (16) on a $\mathbb{R}^2$ plane. The initial Green function will take the form

$$g(t,t_0,x,x_0) = \frac{1}{4}\Theta(t-t_0+x-x_0)\Theta(t-t_0-x+x_0) \tag{28}$$

with Θ as the Heaviside step function. If we take a seed function

$$\phi_k = \cosh(x) \tag{29}$$

we obtain a well known soliton potential

$$u[1](x) = -2\,\mathrm{sech}^2(x) \tag{30}$$

and the new Green function

$$
\begin{aligned}
g[1](t,t_0,x,x_0) = \frac{e^{t+t_0+x+x_0}}{2}&\mathrm{sech}(x)\,\mathrm{sech}(x_0)\left[\cosh(t-t_0)+\sinh(x)\sinh(x_0)\right] \\
&\times\Theta(t-t_0+x-x_0)\Theta(t-t_0-x+x_0)
\end{aligned}
\tag{31}
$$

This method can also be used for Green functions described through a formal sum. Let us consider a heat conduction equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)g(t,x,x_0) = \delta(t)\delta(x-x_0) \tag{32}$$

with the periodic boundary conditions

$$
\begin{aligned}
g(t,x+2,x_0) &= g(t,x,x_0) \\
g(t,x,x_0+2) &= g(t,x,x_0)
\end{aligned}
\tag{33}
$$

The Green function can be expressed as

$$g(t,x,x_0) = \sum_{k=-\infty}^{\infty} e^{-\pi^2 k^2 t + i\pi k(x-x_0)}\Theta(t) \tag{34}$$

with Θ as the Heaviside step function. Let us take a pair of arbitrary seed functions

$$\begin{aligned} \phi_1(x) &= 3e^{i\pi x} + e^{-i\pi x} \\ \phi_2(x) &= e^{2i\pi x} - e^{-2i\pi x} \end{aligned} \tag{35}$$

which result in new potentials

$$\begin{aligned} u[1](x) &= \frac{6\pi^2}{(2\cos(\pi x) + i\sin(\pi x))^2}, \\ u[2](x) &= \frac{3\pi^2(9+20\cos(2\pi x)+3\cos(4\pi x)+16i\sin(2\pi x))}{3\cos(\pi x)+\cos(3\pi x)-2i\sin(\pi x)^3}^2 \end{aligned} \tag{36}$$

The Green function after a single transformation will take the form

$$\begin{aligned} g[1](x,x_0,t) = -\Theta(t) \sum_{k=\mathbb{Z}\{-1,1\}} & \frac{1+3e^{2i\pi x}(-1+k)+k}{(1+3e^{2i\pi x})(1+3e^{2i\pi x_0})(-1+k^2)} \\ & \left\{e^{-\pi[k^2\pi t - ik(-1+x) - i(-1+x_0)]}(2+4k)\right. \\ & \left. + e^{-k\pi[k\pi t + i(-x+x_0)]}[1-k-3e^{2i\pi x_0}(1+k)]\right\} \end{aligned} \tag{37}$$

which is clearly a convergent series. The second transformation gives us

$$\begin{aligned} g[2](x,x_0,t) = -\Theta(t) \sum_{k=\mathbb{Z}\{-2,-1,1,2\}} & -\left\{1+3e^{2i\pi x_0}\right\}\left\{3e^{6i\pi x}[k-2][k-1]3e^{4i\pi x}[k-2][1+k]\right. \\ & \left. + [1+k][2+k] + 9e^{2i\pi x}[k+k^2-2]\right\}\left\{4e^{-k\pi[k\pi t - i(x-1)] + 2i\pi x_0}[k^2-1]\right. \\ & - \frac{e^{k\pi[-k\pi t + i(x-x_0-1)]}}{1+3e^{2i\pi x_0}}[e^{ik\pi}(k-2)(k-1) + 9e^{i\pi(k+2x_0)}(k-2)(1+k) \\ & + 3e^{i\pi(k+6x_0)}(1+k)(2+k) + 2e^{i(1+k)\pi x_0}(k-2)(2+k)(1+2k) \\ & \left. - 2e^{i(5+k)\pi x_0}(k-2)(2+k)(1+2k) + 3e^{i\pi(k+4x_0)}(k+k^2-2)]\right\} \\ & \left\{1+3e^{2i\pi x}[3+e^{2i\pi x}+e^{4i\pi x}]\right\}^{-1}\left\{1+3e^{2i\pi x_0}[3+e^{2i\pi x_0}+e^{4i\pi x_0}]\right\}^{-1} \\ & \left\{k^2-4\right\}^{-1}\left\{k^2-1\right\}^{-1} \end{aligned} \tag{38}$$

As with the wave equation example, further steps down the Darboux chain are possible, yet of limited practicality, since the complexity of solutions will grow in a largely uncontrolled way.

5. Conclusions

Ladder operators in general and the Darboux transform in particular can be used to solve a significantly richer class of problems than those for which they were originally constructed. Not only can the class of operators for which the ladder operators work be embedded in a larger one, but there is a direct method for solving inhomogeneous problems in case of invertible ladder operators. The provided examples show that while general solutions to inhomogeneous problems can be obtained, they are not necessarily simple or usable. The complexity of these solutions is not a strict constraint, but for the applied sciences, practicality is an important factor [7].

Acknowledgements

This work would not have been possible without Sergey Leble, who provided ideas and with whom the subject of this paper was discussed at length.

Appendix: Implementation

The shown implementation of the method described in this paper allows us to calculate the N-fold Darboux transform of a given two-dimensional Green function although the algorithm is not sensitive to the number of variables, hence, one can use a more complex Green function as the base solution as well. At the start, we need to provide the base solution, a set of seed functions for the Darboux transform (in this implementation, they all are assumed to work on the same variable) and the initial potential, which is not strictly necessary, but the program provides a chain of dressed potentials along with the chain of dressed Green functions for the sake of completeness.

In the presented form the program is assuming the domain in the variable, on which the Darboux transform is dependent, to be infinite, but it is possible to use the same code for problems on an interval (as has been shown in one of the examples). This does require the user to manually change the integration limits in all relevant places.

Main setbacks for using the program come from the complexity of the emerging results and the related limits of the Mathematica environment, which cannot handle too large symbolic expressions properly.

```
Clear[in, int, dar, rad, u, f, f0, phi, temp, i, L, test];
(*Definition of the seed function set*)
in={E^(#1)+E^(-#1)&,E^(2 #1)-E^(-2 #1)&};
(*Definition of the Darboux transform and its inverse in the next line*)
dar[f1_, f2_, x_, t_] := f2[x, t] D[f1[x, t]/f2[x, t], x];
rad[f1_, f2_, x_, t_] :=
  f2[x, t] Integrate[f1[s, t]/f2[s, t], {s, -Infinity, x},
  Assumptions ->
    x \[Element] Reals && x0 \[Element] Reals && x1 \[Element] Reals];
int = in;
(*Iterative transformation of the seed functions for all steps
  of the dressing chain after the first*)
Do[Do[temp[x_, t_] = FullSimplify[dar[int[[j]], int[[i]], x, t]];
  int[[j]] = Evaluate[temp[#1, #2]] &, {j, i + 1, Length[int]}], {i,
    1, Length[int] - 1}]
(*Initial potential*)
u = {0 #1 #2 &};
(*Construction of potentials for all steps of the dressing chain*)
Do[AppendTo[u,
  Evaluate[
    FullSimplify[
      u[[i]][#1, #2] - 2 D[Log[int[[i]][#1, #2]], #1, #1]]] &], {i, 1,
  Length[int]}];
```

```
(*Sought inhomogeneity. Time-dependent part omitted along with relevant
  integration, since it does not influence the form of Green function*)
f0 = DiracDelta[#1 - x1] &;
(*Initial Green function*)
g = {(1/4)*HeavisideTheta[#2-t0+#1 - x0]HeavisideTheta[#2-t0-#1 + x0] &};
(*Iterative construction of Green functions*)
Do[f = Evaluate[rad[f0, int[[i]], #1, #2]]
  &; Print["r", i];
  temp = Evaluate[
    Integrate[f[x0] g[[i]][#1, #2], {x0, -Infinity, Infinity},
      Assumptions -> (#1 > 0 && #1 < Pi && x1 \[Element] Reals &&
        t0 \[Element] Reals && #2 \[Element]
        Reals)] /. {HeavisideTheta[0] -> 1/2, x1 -> x0}] &;
  Print["m", i]; AppendTo[g, Evaluate[dar[temp, int[[i]], #1, #2]] &];
  Print["l", i], {i, 1, Length[int]}];
```

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