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Dedicated to the Memory of Professor Zdzisław Kamont

ON THE TONELLI METHOD FOR THE DEGENERATE PARABOLIC CAUCHY PROBLEM WITH FUNCTIONAL ARGUMENT

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Abstract. The degenerate parabolic Cauchy problem is considered. A functional argument in the equation is of the Hale type. As a limit of piecewise classical solutions we obtain a viscosity solution of the main problem. Presented method is an adaptation of Tonelli's constructive method to the partial differential-functional equation. It is also shown that this approach can be improved by the vanishing viscosity method and regularization process.

Keywords: viscosity solutions, parabolic equation, differential-functional equation.

Mathematics Subject Classification: 35A01, 35K15, 35K60.

1. INTRODUCTION

In this paper we study the Cauchy problem for parabolic differential-functional equations. The model presented covers retarded and deviated arguments at the unknown function. The proof of the existence theorem is based on the following observation: by introducing an additional constant delay α under an unknown function in the equation we can reduce our problem to the problem of solving a finite number of nonfunctional equations with given initial dates. Next by letting $\alpha \to 0$ we obtain a viscosity solution of the main problem. This idea comes from L. Tonelli, who introduced it in order to solve a Volterra integral equation (see [17]). In paper [3] the reader can find a spectrum of Tonelli's method applied to partial differential equations. In particular, the Cauchy problem for first order partial differential equation is studied in [2]. The paper [5] deals with the Darboux problem. Quasilinear systems of hyperbolic type are considered in [4]. In the proof of the existence theorem we combine Tonelli's method with the vanishing viscosity method ([7, 21]).

One of the first papers with functional dependence in parabolic problems is [16]. Some special forms of delayed equations modelling real life problem were considered

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earlier (e.g. [23] where an automatically controlled furnace is investigated). For more applications of partial functional differential equations we refer the reader to [24]. Viscosity solutions for differential-functional problems were first considered independently in [12] and in [18]. According to the author's knowledge this paper is the first where Tonelli's constructive method is applied to the differential-functional PDE.

There are two main ways of dealing with differential-functional equations. Sometimes we generalize the techniques used in the nonfunctional case (see for instance in [18, 19, 21]). Sometimes we reduce the original problem to the nonfunctional case and we use a fixed point technique ([12, 20, 22]). In this paper we present the second approach, but instead of constructing special spaces and operators with the fixed point property, we apply a modification of the Arzela-Ascoli theorem. Our approach is based on a priori estimations (Section 3). We put a special stress on assumptions. They are general enough to cover the model presented in Section 5. Moreover, they cover the model used in [13] ("deviated Hale's operator").

Tonelli's method seems to be particulary interesting when we study viscosity solutions. It is due to the fact that they have good limit properties. By using this method we prove an existence result for the differential-functional problem, where the delay depends on the space variable. (In contrary to [22] where fixed point method was applied).

Put $\Theta = (0,T] \times \mathbb{R}^n$, $\Theta_0 = [-\tau, 0] \times \mathbb{R}^n$, $E = \Theta \cup \Theta_0$, $\tau \geq 0$, $T > 0$ and $\mathbb{D} = [-\tau, 0] \times B(r)$, where $B(r) = \{x \in \mathbb{R}^n : |x| \leq r\}$, $r \geq 0$.

Definition 1.1 (Hale's operator)**.** For $u : E \to \mathbb{R}$ and $(t,x) \in \bar{\Theta}$ we define $u_{(t,x)} : \mathbb{D} \to \mathbb{R}$ by $u_{(t,x)}(s,y) = u(t+s, x+y)$ (see [11] for ordinary equations).

Let $f : \bar{\Theta} \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ be continuous. We consider the initial-value problem:

$$\mathcal{P}u = f(t, x, u_{(t,x)}, Du) \qquad \text{in} \quad \Theta, \tag{1.1}$$

$$u = \psi \qquad \text{in} \quad \Theta_0. \tag{1.2}$$

Here $\mathcal{P}$ is a linear degenerate parabolic operator of constant coefficient, i.e.

$$\mathcal{P}u = D_t u - \sum_{i,j=1}^{n} a_{ij} D_{ij} u, \quad a_{ij} \in \mathbb{R}, \quad \text{where} \quad \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq 0 \quad \text{for all } \xi_i \in \mathbb{R}.$$

We write Du for $D_x u$ and $D_{ij} u$ for $D_{x_i x_j} u$. To underline the functional dependence described by the symbol $u_{(t,x)}$ we write u, Du, $\mathcal{P}u$ in place of $u(t,x)$, $Du(t,x)$, $\mathcal{P}u(t,x)$. Functional dependence in (1.1) means that the right hand side of (1.1) depends on the restriction of u to $(t,x) + \mathbb{D}$.

Notice that if $\mathcal{P}u = D_t u$ then problem (1.1), (1.2) reduces to the first order Cauchy problem.

It is important that many kinds of functional dependence can be derived from our model by specializing the function f (see Section 5).

2. VISCOSITY SOLUTIONS

The notion of viscosity solutions (with no functional dependence) was first introduced by M.G. Crandall and P.L. Lions (see [7] for first order differential equations and [1,6] for second order). For viscosity solutions of differential-functional equations we refer the reader to [12, 14, 18, 19, 21, 22].

We write $u \in C^{1,2}(\Theta)$ if $D_t u, D^2 u, Du$ exist and are continuous in Θ.

Definition 2.1. A function $u \in C(E)$ is a viscosity subsolution (resp. supersolution) of (1.1), (1.2) provided for all $\phi \in C^{1,2}(\Theta)$ if $u - \phi$ attains a local maximum (resp. minimum) at $(t,x) \in \Theta$, then

$$\begin{gathered} \mathcal{P}\phi(t,x) \le f(t,x,u_{(t,x)}, D\phi(t,x)) \\ (\text{resp. } \mathcal{P}\phi(t,x) \ge f(t,x,u_{(t,x)}, D\phi(t,x))), \end{gathered} \tag{2.1}$$

and

$$u \le \psi \quad \text{in} \quad \Theta_0 \qquad (\text{resp. } u \ge \psi \quad \text{in} \quad \Theta_0). \tag{2.2}$$

Definition 2.2. A function $u \in C(E)$ is a viscosity solution of (1.1), (1.2) if u is both a viscosity subsolution and supersolution of (1.1), (1.2).

Following the arguments given in the classical theory of viscosity solutions we may assume that the maximum (resp. minimum) in Definition 2.1 is strict. Moreover we set $\phi \in C^1(\Theta)$ ($D_t\phi, D\phi$ exist and are continuous in Θ) instead of $u \in C^{1,2}(\Theta)$ if $\mathcal{P}$ has no second order part.

Let $a = [a_{i,j}]_{i,j=1,\dots,n}$. We use the symbol $SOL(f,\psi,a)$ for the set of all viscosity solutions of (1.1), (1.2). We say that u is a classical solution if $u \in C(E)$, $u \in C^{1,2}(\Theta)$ and (1.1), (1.2) are satisfied everywhere.

Remark 2.3. If $u \in C(E) \cap C^{1,2}(\Theta)$ then u is viscosity subsolution (v. supersolution, v. solution) of (1.1), (1.2) if and only if u is a classical solution (subsolution, supersolution) of (1.1), (1.2).

Theorem 2.4. *Suppose that:*

1) $\mathbb{X}(E) \subset C(E)$, $\mathbb{X}(\mathbb{D}) \subset C(\mathbb{D})$ *such that:* $u \in \mathbb{X}(E) \Rightarrow u_{(t,x)} \in \mathbb{X}(\mathbb{D})$ *for all* $(t,x) \in \Theta$,
2) $f, f_k : \Theta \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ *are continuous,* $\psi_k \in C(\Theta_0)$, $u^k \in SOL(f_k, \psi_k, a^k) \cap \mathbb{X}(E)$, $a^k = [a^k_{i,j}]_{i,j=1,\dots,n}$ *for* $k \in \mathbb{N}$,
3) $f_k \to f$ *in* $\Theta \times \mathbb{X}(\mathbb{D}) \times \mathbb{R}^n$ *uniformly on bounded subsets,* $u^k \to u$ *almost uniformly,* $a^k_{i,j} \to a_{i,j}$, $i,j = 1,\dots,n$ *as* $k \to \infty$.

Then $u \in SOL(f,\psi,a)$, *where* $\psi = u_{|\Theta_0}$, $a = [a_{i,j}]_{i,j=1,\dots,n}$.

Proof. The proof generalizes the method used in the nonfunctional case. Let us assume that $u - \phi$ has a strict local maximum at $(t,x) \in \Theta$. Then $u^k - \phi$ has a local maximum at $(t_k, x_k) \in \Theta$ and $(t_k, x_k) \to (t,x)$. Notice that

$$\|u^k_{(t_k,x_k)} - u_{(t,x)}\|_{\mathbb{D}} \le \|u^k_{(t_k,x_k)} - u_{(t_k,x_k)}\|_{\mathbb{D}} + \|u_{(t_k,x_k)} - u_{(t,x)}\|_{\mathbb{D}}.$$

By the fact that $u_k \rightrightarrows u$ in some neighborhood of $(t,x)+\mathbb{D}$, the first part on the right hand side tends to 0. The second part tends to 0 by the continuity of u ($\mathbb{D}$ is bounded). This gives

$$\|u^k_{(t_k,x_k)} - u_{(t,x)}\|_{\mathbb{D}} \to 0. \tag{2.3}$$

Moreover, $\mathcal{P}_k\phi(t_k,x_k) \to \mathcal{P}\phi(t,x)$ and $D\phi(t_k,x_k) \to D\phi(t,x)$ by regularity of ϕ and by the assumption $a^k_{i,j} \to a_{i,j}$.

Since $u_k \in SOL(f_k,\psi_k,a^k)$, we can write

$$\mathcal{P}_k\phi(t_k,x_k) \le f_k(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k)). \tag{2.4}$$

We claim that

$$f_k(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k)) \to f(t,x,u_{(t,x)},D\phi(t,x)).$$

Indeed,

$$\begin{aligned}
&|f_k(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k)) - f(t,x,u_{(t,x)},D\phi(t,x))| \le \\
&\le |f_k(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k)) - f(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k))| + \\
&\quad + |f(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k)) - f(t,x,u_{(t,x)},D\phi(t,x))|
\end{aligned}$$

Note that the sequence $(t_k,x_k,u^k_{(t_k,x_k)},D\phi(t_k,x_k))$ is contained in some bounded subset of $\Theta\times\mathbb{X}(\mathbb{D})\times\mathbb{R}^n$. Thus the first part on the right tends to 0 by 3). The second part tends to 0 by (2.3) and by the continuity of f. Now letting $k\to\infty$ in (2.4) we get (2.1). Thus u is a viscosity subsolution of (1.1), (1.2). In a similar way we can show that u is a viscosity supersolution of (1.1), (1.2). Thus $u \in SOL(f,\psi,a)$. □

As an important example of $\mathbb{X}(E)$ we can take a set of all Lipschitz continuous functions in E with a given constant L.

If $\mathbb{D} = \{(0,0)\}$ (no functional dependence in the equation) we have $\mathbb{X}(\mathbb{D}) = \mathbb{R}$ (real functions on the one point set) and every $\mathbb{X}(E) \subset C(E)$ satisfies 1). We set $\mathbb{X}(E) = \mathbb{R}$ (constant functions) and see that 1) is superfluous.

Remark 2.5. If $a^k_{i,j} = \varepsilon^k\delta_{i,j}$, $\delta_{ij}=0$, $i\ne j$, $\delta_{ii}=1$, $\varepsilon^k>0$, $\varepsilon^k\to 0$ as $k\to\infty$ and if u^k are classical solutions then Theorem 2.4 is a functional version of the "vanishing viscosity" method (see [7,21]).

3. A PRIORI ESTIMATIONS

We define a modulus as a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0^+) = \omega(0) = 0$. We write $BUC(E)$ for the set of all bounded and uniformly continuous functions in E.

Definition 3.1. We write $\sigma \in O_M$, $M \ge 0$ if $\sigma : [0,T]\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing in both variables, and if a maximal solution of the problem

$$z'(t) = \sigma(t,z(t)), \qquad z(0) = M. \tag{3.1}$$

exists in $[0,T]$. We write $\mu_\sigma(\cdot,M)$ for this solution.

Put $\mu_\sigma(T, M) = R(\sigma, M)$.

Definition 3.2. Let $\sigma \in O_M$. We write $f \in X_{\sigma, M}$ if

(i) $f(t, x, w, 0) \operatorname{sgn} w(0,0) \leq \sigma(t, \|w\|_{\mathbb{D}})$ in $\Theta \times C(D) \times \mathbb{R}^m$.
(ii) For every $R \geq 0$ there exists a modulus ω_R such that:

$$|f(t, x, w, p) - f(t, x, w, 0)| \leq \omega_R(|p|) \quad \text{in} \quad \Theta \times K(R) \times \mathbb{R}^m.$$

In view of Theorem 2 of [19] we can write the following proposition.

Proposition 3.3. *If $f \in X_{\sigma, M}$, $\|\psi\|_{\Theta_0} \leq M$ and $u \in SOL(f, \psi, a) \cap BUC(E)$, then*

$$\|u\|_{E_t} \leq \mu_\sigma(t, M) \leq R(\sigma, M) \quad \text{for} \quad t \in [0, T]. \tag{3.2}$$

In the linear case we have the following remark.

Remark 3.4. Let $\sigma(t, z) = \gamma(t) + Cz$, $\gamma : [0, T] \to \mathbb{R}$ nondecreasing $C \geq 0$, $\|\psi\|_{\Theta_0} = M$, $f \in X_{\sigma, M}$, $u \in SOL(f, \psi, a) \cap BUC(E)$. Then for $t \in [0, T]$

$$\|u\|_{E_t} \leq e^{Ct}\Big(\|\psi\|_{\Theta_0} + \int\limits_0^t \gamma(s)ds\Big) \leq R(\sigma, M), \tag{3.3}$$

where $R(\sigma, M) = e^{CT}\big(M + \int_0^T \gamma(s)ds\big)$.

Proof. We apply Proposition 3.3 to the sequence $\{\gamma_k\}$ of continuous majorants of γ such that $\int_0^t \gamma_k(s)ds \to \int_0^t \gamma(s)ds$. Next we pass to the limit. $\square$

Let $Y \subset E$. For every $z : Y \to \mathbb{R}$, we define

$$L_x[z](t) = \sup\Big\{\frac{|z(s, x) - z(s, \bar{x})|}{|x - \bar{x}|} : x \neq \bar{x},\ s \leq t\Big\},$$
$$L_t[z](t) = \sup\Big\{\frac{|z(s, x) - z(\bar{s}, x)|}{|s - \bar{s}|} :\ s \neq \bar{s},\ s, \bar{s} \leq t\Big\}.$$

We write $L_x[z] = L_x[z](T)$, $L_t[z] = L_t[z](T)$.
Define

$$C^{L,L}(Y) = \{z \in C(Y) : L_t[z] < \infty, L_x[z] < \infty\},$$
$$C^{L,0}(Y) = \{z \in C(Y) :\ L_t[z] < \infty\}$$

and

$$C^{0,L}(Y) = \{z \in C(Y) :\ L_x[z] < \infty\}.$$

In the next theorem we will use the following proposition.

Proposition 3.5. *Let $k \in \mathbb{N}$, $P_0, P_1 \in \mathbb{R}^k_+$, $M \in \mathcal{M}_{k\times k}(\mathbb{R})$ with nonnegative coefficients. Suppose that for some integrable $l : [0,T] \to \mathbb{R}^k_+$*

$$l(t) \le P_0 + \int_0^t [P_1 + Ml(s)]ds, \qquad t \in [0,T].$$

Then

$$l(t) \le e^{tM}\Big(P_0 + \int_0^t e^{-sM} P_1 ds\Big), \qquad t \in [0,T].$$

Proof. We apply a standard method for monotone integral system of inequalities. □

For $R > 0$, we define $K(R) = \{w \in C(D) : \|w\|_D \le R\}$.

Assumption 3.6. Suppose that:

1) $f \in X_{\sigma,M}$, $\sigma \in O_M$ and $R = R(\sigma, M)$,
2) there exists $C \ge 0$, such that

$$|f(t,x,w,p) - f(t,x,\bar{w},p)| \le C\|w - \bar{w}\|_{\mathbb{D}} \quad \text{in } \Theta \times K(R) \times \mathbb{R}^m,$$

3) there exist $A_k \ge 0$, $k = 1,2,3,4$, such that

$$|f(t,x,w,p) - f(t,\bar{x},w,p)| \le (A_1 + A_2 L_x[w] + A_3 L_t[w] + A_4|p|)|x - \bar{x}|$$

in $\Theta \times K(R) \cap C^{L,L}(\mathbb{D}) \times \mathbb{R}^n$ for $i \in I$,
4) there exists $B_k \ge 0$, $k = 1,2,3,4$, such that

$$|f(t,x,w,p) - f(\bar{t},x,w,p)| \le (B_1 + B_2 L_x[w] + B_3 L_t[w] + B_4|p|)|t - \bar{t}|$$

in $\bar{\Theta} \times K(R) \cap C^{L,L}(\mathbb{D}) \times \mathbb{R}^n$ for $i \in I$,
5) for every $\tilde{L} \ge 0$ there exists a modulus $\omega_{\tilde{L}}$ such that

$$|f(t,x,w,p) - f(t,x,w,\bar{p})| \le \omega_{\tilde{L}}(|p - \bar{p}|) \quad \text{in } \Theta \times K(R) \times B(\tilde{L}).$$

Remark 3.7. It follows from Definition 3.2 and continuity of f that $|f(t,x,0,0)| \le \sigma(T,0)$ in Θ. Moreover, under Assumption 3.6 1),2) in view of Proposition 3.3 using the standard retraction argument, we may assume, without loss of generality, that $|f(t,x,w,0)| \le \sigma(T,R)$ in $\Theta \times C(D)$, where $R = R(\sigma,M)$. By the same argument in view of Proposition 3.3 we may assume, without loss of generality, that Assumption 3.6 is satisfied globally in w (i.e. we may consider $C(\mathbb{D})$ in place of $K(R)$).

We write $u \in C_b^{1,2}(\bar{\Theta})$ if $D_t u, D^2 u, Du$ exist, are continuous in $\bar{\Theta}$ and $D_t u, Du$ are bounded in $\bar{\Theta}$.

Define

$$\gamma_0 = \sup_{x\in\mathbb{R}^m} \Big\{\Big| \sum_{i,j=1}^n a_{ij} D_{ij}\psi(0,x) + f(0,x,\psi_{(0,x)}, D\psi(0,x))\Big|\Big\}. \tag{3.4}$$

Lemma 3.8. *Suppose that f satisfies Assumption 3.6, $\|\psi\|_{\Theta_0} \leq M$, $\psi \in C^{L,L}(\Theta_0)$ and $u \in C_b^{1,2}(\bar{\Theta})$ is a solution of (1.1), (1.2). Then there exist $L, \tilde{L}$ depending on $\sigma, M, \gamma_0, L_t[\psi], L_x[\psi] A_i, B_i, i = 1, \ldots, 4$, such that $L_x[u] \leq L$, $L_t[u] \leq \tilde{L}$.*

Proof. Let $u \in C_b^{1,2}(\bar{\Theta})$ be a solution of (1.1), (1.2). Define $F[u](t,x,p) = f(t,x,u_{(t,x)},p)$. Since $u \in C^{L,L}(E)$, the following estimations hold:

$$
\begin{aligned}
&|F[u](t,x,p) - F[u](t,\bar{x},p)| \leq \\
&\leq \big\{A_1 + (A_2 + C)L_x[u](t) + A_3 L_t[u](t) + A_4|p|\big\}|x - \bar{x}|
\end{aligned}
\tag{3.5}
$$

and for $\bar{t} \leq t$

$$
\begin{aligned}
&|F[u](t,x,p) - F[u](\bar{t},x,p)| = |f(t,x,u_{(t,x)},p) - f(\bar{t},x,\bar{u}_{(\bar{t},x)},p)| \leq \\
&\leq \big\{B_1 + B_2 L_x[u](t) + (C + B_3)L_t[u](t) + B_4|p|\big\}|t - \bar{t}|.
\end{aligned}
\tag{3.6}
$$

Fix $\xi \in \mathbb{R}^n$. Since $v(t,x) = u(t,x+\xi) - u(t,x)$ is a solution of

$$
\mathcal{P}v = g(t,x,Dv) \qquad \text{in} \quad \Theta, \tag{3.7}
$$

$$
v = \tilde{\psi} \qquad \text{in} \quad E_0, \tag{3.8}
$$

where

$$
g(t,x,p) = F[u](t,x+\xi,p+Dv) - F[u](t,x,Dv))
$$

and

$$
\tilde{\psi}(t,x) = \psi(t,x+\xi) - \psi(t,x).
$$

By (3.5) and Remark 3.4, we obtain

$$
|u(t,x+\xi) - u(t,x)| \leq \|\tilde{\psi}\|_0 + \int_0^t \big\{A_1 + (A_2+C)L_x[u](s) + A_3 L_t[u](s) + A_4\|Du\|_s\big\} ds |\xi|
$$

and consequently

$$
L_x[u](t) \leq L_x[\psi] + \int_0^t \big\{A_1 + (A_2 + C + A_4)L_x[u](s) + A_3 L_t[u](s)\big\} ds. \tag{3.9}
$$

In a similar way we obtain an inequality for $L_t[u](t)$. Indeed, fix $h_0 > 0$ and set $h_0 > h > 0$. Of course, $\bar{v}(t,x) = u(t+h,x) - u(t,x)$ is a solution of (3.7), (3.8) in Θ_{T-h_0} with $g(t,x,p) = F[u](t+h,x,p+Dv) - F[u](t,x,Dv))$.

By (3.6) and Remark 3.4, we obtain

$$
\begin{aligned}
&|u(t+h,x) - u(t,x)| \leq \\
&\leq |u(h,x) - u(0,x)| + h \int_0^t \big\{B_1 + B_2 L_x[u](s) + (C + B_3)L_t[u](s) + B_4\|Du\|_s\big\} dt \leq \\
&\leq \Big[\|D_t u\|_h + \int_0^t \big\{B_1 + (B_2 + B_4)L_x[u](s) + (C + B_3)L_t[u](s)\big\} dt\Big] h.
\end{aligned}
$$

After dividing both sides by h and letting $h \to 0(\|D_t u\|_h \to \|D_t u\|_0)$ we see that

$$\|D_t u\|_t = \|D_t u\|_t \leq \gamma_0 + \int_0^t \big\{B_1 + (B_2 + B_4)L_x[u](s) + (C + B_3)L_t[u](s)\big\}dt$$

for $t \in [0, T - h_0]$. This gives

$$L_t[v](t) \leq \gamma_0 + \int_0^t \big\{B_1 + B_2 L_x[u](s) + (C + B_3)L_t[u](s) + B_4\|Dv\|_s\big\}dt. \tag{3.10}$$

in $[0, T]$. By Proposition 3.5, we get

$$\begin{bmatrix} L_x[u](t) \\ L_t[u](t) \end{bmatrix} \leq e^{tM}\Big(\begin{bmatrix} L_x[\psi] \\ \gamma^0 \end{bmatrix} + \int_0^t e^{-sM}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}ds\Big), \qquad t \in [0, T],$$

where

$$M = \begin{bmatrix} A_2 + C + A_4, & A_3 \\ B_2 + B_4, & C + B_3 \end{bmatrix}.$$

The proof is completed by setting $t = T$. □

Since we use the space $C^{L,L}(\mathbb{D})$ in Assumption 3.6 3), 4), we can apply our results to equations with a retarded and deviated argument. It would be impossible if we considered the space $C(\mathbb{D})$ instead of $C^{L,L}(\mathbb{D})$ leaving out $L_x[u], L_t[u]$ in 3), 4). Of course, the assumption would be stronger in this case, general enough to cover only differential-integral equation and constant retarded and deviated argument (see Section 5).

Remark 3.9. If Assumption 3.6 3) is satisfied with $A_3 = 0$, then Assumption 3.6 1)–3), 5) implies that there exists L depending only on A_1, A_2, A_4, C, L_0, T such that $L_x[u] \leq L$. In this case we can assume that Assumption 3.6 4) holds with $L_x[w]$, $|p| \leq L$ and $B_2 = B_4 = 0$.

Proof. In case $A_3 = 0$ we can treat (3.9) and (3.10) separately. First we obtain a uniform bound on $L_x[u]$ using (3.9). □

Remark 3.10. By a similar argument, we can assume that Assumption 3.6 3) is satisfied locally in $L_t[w]$ ($A^3 = 0$) if Assumption 3.6 4) is satisfied with $B^2 = B^4 = 0$.

4. THE EXISTENCE THEOREM

Let $\tilde{\mathbb{D}} = [-\tau - 1, 0] \times B(r)$ and $0 \leq \alpha < 1$. For $w \in C(\tilde{\mathbb{D}})$ we define $w_{-\alpha} \in C(\mathbb{D})$ by the formula: $w_{-\alpha}(s, y) = w(s - \alpha, y)$ (in this notation $w_{-0} = w_{|\mathbb{D}}$).

We define $f_\alpha : \Theta \times C(\tilde{\mathbb{D}}) \times \mathbb{R}^n \to \mathbb{R}$ by $f_\alpha(t,x,w,p) = f(t,x,w_{-\alpha},p)$.

Put $\tilde{\Theta}_0 = [-\tau-1,0] \times \mathbb{R}^n$, $\tilde{E} = \tilde{\Theta}_0 \cup \Theta$. Let $\tilde{\psi} : \tilde{\Theta}_0 \to \mathbb{R}$ be equal to ψ in Θ_0 and $\tilde{\psi}(t,x) = \psi(-\tau,x)$ in $\tilde{\Theta}_0 \setminus \Theta_0$. Consider the problem:

$$\mathcal{P}u = f_\alpha(t,x,u_{(t,x)},Du) \qquad \text{in} \quad \Theta, \tag{4.1}$$

$$u = \tilde{\psi} \qquad \text{in} \quad \tilde{\Theta}_0, \tag{4.2}$$

where the Hale operator is defined for $\tilde{\mathbb{D}}$ and solutions are defined in $\tilde{E}$.

Remark 4.1. The initial-value problem (1.1), (1.2) is equivalent to (4.1), (4.2) with $\alpha = 0$ i.e. solutions of both problems are equal in E.

Proposition 4.2. *If $f(t,x,w,p)$ is Lipschitz in $w \in C(\mathbb{D})$ with a constant $C \geq 0$ and $\alpha, \beta \geq 0$. Then for $w \in C^{L,0}(\tilde{\mathbb{D}})$*

$$|f_\alpha(t,x,w,p) - f_\beta(t,x,w,p)| \leq CL_t[w]|\alpha - \beta|.$$

Proof.

$$\begin{aligned}|f_\alpha(t,x,w,p) - f_\beta(t,x,w,p)| &= |f(t,x,w_{-\alpha},p) - f(t,x,w_{-\beta},p)| \leq \\ &\leq C\|w_{-\alpha} - w_{-\beta}\|_{\mathbb{D}} = \\ &= C \sup_{(s,y)\in\mathbb{D}} |w(s-\alpha,y) - w(s-\beta,y)| \leq \\ &\leq CL_t[w]|\alpha-\beta|.\end{aligned}$$

□

Let $C^{L,0}(\tilde{\mathbb{D}}, L_1) = \{w \in C^{L,0}(\tilde{\mathbb{D}}) : L_t[w] \leq L_1\}$.

Corollary 4.3. *By setting $\beta = 0$ we obtain $f_\alpha \rightrightarrows f_0$ in $\Theta \times C^{L,0}(\tilde{\mathbb{D}}, L_1) \times \mathbb{R}^n$ as $\alpha \to 0$, where $L_1 \geq 0$.*

Remark 4.4. If f satisfies Assumption 3.6, then f_α satisfies Assumption 3.6 with the same constants (with $\tilde{\mathbb{D}}$ in place of $\mathbb{D}$). A global estimation on the solution and its Lipschitz constant is valid for the problem (4.1), (4.2).

We say that $\mathcal{P}$ defined in Section 1 is a strictly parabolic operator if there exists $\kappa > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \kappa|\xi|^2, \qquad \xi = (\xi_1, \ldots \xi_n).$$

We denote by $C^{1+\beta/2,2+\beta}(\bar{\Theta})$, $\beta \in (0,1)$, the space of all functions $u \in C^{1,2}(\bar{\Theta})$ such that Du, D^2u, D_tu exist and are continuous in $\bar{\Theta}$, D^2u satisfies a Hölder condition in x with an exponent β and D_tu satisfies a Hölder condition in t with an exponent $\beta/2$. It is well known that $C^{1+\beta/2,2+\beta}(\bar{\Theta})$ is a Banach space with some norm $\|\cdot\|_{2+\beta}$ (see [15]). It is important here that $C^{1+\beta/2,2+\beta}(\bar{\Theta}) \subset C_b^{1,2}(\bar{\Theta})$. We write $z \in C^{2+\beta}(\mathbb{R}^m)$ if $\tilde{z}$ defined by $\tilde{z}(t,x) = z(x)$ belongs to $C^{1+\beta/2,2+\beta}(\bar{\Theta})$.

Theorem 4.5. *Suppose that P is strictly parabolic, $\psi \in C^{L,L}(\Theta_0)$, $\psi(0,\cdot) \in C^{2+\beta}(\mathbb{R}^m)$ for some $\beta \in (0,1)$. Let $M = \|\psi\|_\Theta$. If Assumption 3.6 with $w_{\tilde{L}}(r) = C_{\tilde{L}}r$, $C_{\tilde{L}} \geq 0$ in 5) is satisfied, then for every $0 < \alpha < 1$ the problem (4.1), (4.2) has exactly one solution $u^\alpha \in C^{1+\beta/2,2+\beta}(\bar{\Theta}) \cap C^{L,L}(\tilde{E})$.*

Proof. Fix $0 < \alpha < 1$. We apply a step by step method to problem (4.1), (4.2). Set $u^0 = \tilde{\psi}$ in $\tilde{\Theta}_0$ and $u^0(t,x) = u^0(0,x)$ in $\Theta = \tilde{E} \setminus \tilde{\Theta}_0$. $\square$

Let $N \in \mathbb{N}$ such that $(N-1)\alpha < T \leq N\alpha$. Define $t_i = i\alpha$ for $i = 0, 1, \ldots, N-1$ and $t_N = T$. Put $\Theta_i = (t_{i-1}, t_i] \times \mathbb{R}^n$, $\tilde{\Theta}_i^0 = [t_{i-1} - \tau, t_{i-1}] \times \mathbb{R}^n$ for $i = 1, 2 \ldots, N$. Consider the problem

$$\mathcal{P}u = f_\alpha(t, x, u^{i-1}_{(t,x)}, Du) \qquad \text{in} \quad \Theta_i, \tag{4.3}$$

$$u = u^{i-1} \qquad \text{in} \quad \tilde{\Theta}_i^0. \tag{4.4}$$

for $i = 1, \ldots, N$. In view of the classical theory of nonfunctional equations (see [15]) problem (4.3), (4.4) has a solution u^i in $C^{1+\beta/2, 2+\beta}(\bar{\Theta}_i)$ for each $i = 1, 2 \ldots, N$. Set $u^i(t,x) = u^i(t_i, x)$ in $(t_i, T] \times \mathbb{R}^n$ and $u^i(t,x) = u^{i-1}(t,x)$ for $t \leq t_{i-1}$. Define $u^\alpha = u^N$. It is immediate that $u^\alpha \in C^{1+\beta/2, 2+\beta}(\bar{\Theta})$ and u^α is a classical solution of (4.1), (4.2).

The uniqueness follows from Proposition 3.3. Indeed, if $u, v \in C^{1+\beta/2, 2+\beta}(\bar{\Theta}) \cap C^{L,L}(\tilde{E})$ are classical solutions of (4.1), (4.2), then $u - v$ is a classical solution of the problem

$$\mathcal{P}z = g(t, x, z_{(t,x)}, Dz) \qquad \text{in} \quad \Theta, \tag{4.5}$$

$$z = 0 \qquad \text{in} \quad \tilde{\Theta}_0, \tag{4.6}$$

where

$$g(t,x,w,p) = f_\alpha(t, x, w + v_{(t,x)}, p + Dv) - f_\alpha(t, x, v_{(t,x)}, Dv).$$

It is not difficult to verify that the hypothesis of Proposition 3.3 is satisfied for g. (see Remark 4.4)

The idea of passing to the limit $\alpha \to 0$ in (4.1), (4.2) is an adaptation of the Tonelli method. In the following we will combine this method with a regularization process where the "vanishing viscosity" method plays an important role.

Theorem 4.6. *Suppose that $\mathcal{P}$ is a degenerate parabolic operator. Let $\psi \in BUC(\Theta_0)$, $M = \|\psi\|_{\Theta_0}$ and there exists a sequence of f_k such that $f_k \rightrightarrows f$ and f_k satisfies Assumption 3.6 with σ and C independent of k. Then there exists a viscosity solution of* (1.1), (1.2).

Proof. Let $\psi_k \in C^{L,L}(\Theta_0)$ such that $\psi_k \rightrightarrows \psi$ and $\|\psi_k\|_{\Theta_0} \leq M$.

Fix $\delta > 0$. Define $\rho_\delta \in C_0^\infty(\mathbb{R}^n)$ such that $\int\limits_{\mathbb{R}^n} \rho_\delta = 1$, $\operatorname{supp} \rho_\delta \subset B(\delta)$, $B(\delta)$ ball of radius δ in $\mathbb{R}^n$. For $L > 0$, define

$$f_\delta^{Lk}(t,x,w,p) = \omega_\delta^p * f_k^L(t,x,w,\cdot)(p),$$

where $f_k^L(t,x,w,p) = f_k(t,x,w,I_L(p))$, $I_L(p) = p$ if $|p| \leq L$ and $I_L(p) = \frac{L}{|p|}p$ if $|p| \geq L$.

Set $f_{\delta,\alpha}^{Lk} = (f_\delta^{L,k})_\alpha$. We can verify that for each k $f_{\delta,\alpha}^{L,k} \to f_k$ almost uniformly as $\delta, \alpha \to 0$, $L \to \infty$. Moreover, $f_{\delta,\alpha}^{L,k} \rightrightarrows f_{\delta,\alpha}^L$ as $k \to \infty$ uniformly in δ, α, L.

For $\varepsilon > 0$ consider problem

$$\mathcal{P}_\varepsilon u = f^{L,k}_{\delta,\alpha}(t,x,u_{(t,x)},Du) \qquad \text{in} \quad \Theta, \tag{4.7}$$

$$u = \tilde{\psi}_k \qquad \text{in} \quad \tilde{\Theta}_0, \tag{4.8}$$

where $\mathcal{P}_\varepsilon$ is a strictly parabolic operator defined by

$$\mathcal{P}_\varepsilon u = D_t u - \sum_{i,j=1}^{n} (\varepsilon\delta_{ij} + a_{ij}) D_{ij} u, \qquad \delta_{ij} = 0,\ i \neq j,\ \delta_{ii} = 1.$$

We can verify that for each k $f^{L,k}_{\delta,\alpha}$ satisfy Assumption 3.6 with constants in $2), 3), 4)$ and comparison function σ independent of δ, α, L. Moreover, for each k the hypothesis of Theorem 4.5 is satisfied with $\mathcal{P}$ replaced by $\mathcal{P}_\varepsilon$, ψ replaced by ψ_k and f replaced by $f^{L,k}_{\delta,\alpha}$. Then there exists a unique classical solution of (4.7), (4.8). Denote this solution by $u^{L,\varepsilon,k}_{\alpha,\delta}$. By Proposition 3.3 a family of functions $\{u^{L,\varepsilon,k}_{\alpha,\delta}\}$ is uniformly bounded. By Lemma 3.8 it is also equicontinuous for fixed k.

Note that $u^{L,\varepsilon,k}_{\alpha,\delta} - u^{L,\varepsilon,\bar{k}}_{\alpha,\delta} \in SOL(g, \psi_k - \psi_{\bar{k}}, a^\varepsilon)$ (since it is in fact a classical solution), where $a^\varepsilon_{ij} = \varepsilon\delta_{ij} + a_{ij}$ and

$$g(t,x,w,p) = f^{L,k}_{\delta,\alpha}(t,x,w+(u^{L,\varepsilon,\bar{k}}_{\alpha,\delta})_{(t,x)}, p+Du^{L,\varepsilon,\bar{k}}_{\alpha,\delta}) - f^{L,\bar{k}}_{\delta,\alpha}(t,x,(u^{L,\varepsilon,\bar{k}}_{\alpha,\delta})_{(t,x)}, Du^{L,\varepsilon,\bar{k}}_{\alpha,\delta}).$$

Moreover, for every $\rho > 0$ there exists δ_1 such that for $0 < k, \bar{k} < \delta_1$

$$|g(t,x,w,0)| \le \rho + C\|w\|_{\mathbb{D}}, \quad \|\psi_k - \psi_{\bar{k}}\|_{\Theta_0} < \rho, \quad \|f^{L,k}_{\alpha,\delta} - f^{L,\bar{k}}_{\alpha,\delta}\|_{\bar{\Theta}\times C(D)\times\mathbb{R}^n} < \rho.$$

In view of Proposition 3.3 we obtain

$$\|u^{L,\varepsilon,k}_{\alpha,\delta} - u^{L,\varepsilon,\bar{k}}_{\alpha,\delta}\|_E \le e^{CT}(\rho + T\rho). \tag{4.9}$$

Define $u^m_k = u^{m,\frac{1}{m},k}_{\frac{1}{m},\frac{1}{m}}$, $f^m_k = f^{m,k}_{\frac{1}{m},\frac{1}{m}}$, where $m, k \in \mathbb{N}$. Since u^m_1 is equicontinuous and uniformly bounded then by the Arzela-Ascoli theorem in unbounded domains there exists a subsequence $\sigma_m(1)$ of $\frac{1}{m}$ such that $u_1^{\sigma_m(1)} \to u_1$ almost uniformly. Since $u_2^{\sigma_m(1)}$ is equicontinuous and uniformly bounded then by a similar argument there exists a subsequence $\sigma_m(2)$ of $\sigma_m(1)$ such that $u_2^{\sigma_m(2)} \to u_2$ almost uniformly. In this way we define $\sigma_m(k)$ such that $\sigma_m(k) \to 0$ and $u_k^{\sigma_m(k)} \to u_k$ almost uniformly. It is not difficult to verify that for each k $f_k^{\sigma_m(k)} \to f_k$ as $m \to \infty$ in a bounded subset of $\Theta \times \mathbb{X}_k(\mathbb{D}) \times \mathbb{R}^n$ as $m \to \infty$, where $\mathbb{X}_k(\mathbb{D}) = C^{L,0}(\tilde{\mathbb{D}}, L_k)$ and L_k is a constant $\tilde{L}$ given in Lemma 3.8 applied to f_k (see Corollary 4.3). By Theorem 2.4, we have $u_k \in SOL(f_k, \psi_k, a)$.

Setting $\alpha = \varepsilon = \delta = \sigma_m(k)$ $k > \bar{k}$, in (4.9) and letting $m \to \infty$ we conclude that for every $\rho > 0$ $\|u_k - u_{\bar{k}}\|_E \le e^{CT}(\rho + T\rho)$ if $k, \bar{k}$ are large. ($\sigma_m(k)$ is a subsequence of $\sigma_m(\bar{k})$ hence $u_{\bar{k}}^{\sigma_m(k)} \to u_{\bar{k}}$). This gives $u_k \rightrightarrows u$ and by Theorem 2.4 $u \in SOL(f, \psi, \Theta)$. This completes the proof. $\square$

Remark 4.7. If $\mathcal{P}$ is a strictly parabolic operator we set $\varepsilon = 0$. If $\psi \in C^{L,L}(\Theta_0)$ than the solutions are in $C^{L,L}(\Theta_0)$ (we set $\psi_k = \psi$). If f satisfies Assumption 3.6, we set $f_k = f$.

Proposition 4.8. *Suppose that Assumption 3.6 1), 2), 5) holds and there exists modulus ω such that*

$$|f(t,x,w,p) - f(t,\bar{x},w,p)| \leq \omega((1+|p|)(|t-\bar{t}| + |x-\bar{x}|))$$

in $\bar{\Theta} \times K(R) \cup C(D) \times \mathbb{R}^n$. Then there exists a sequence f_k such that $f_k \rightrightarrows f$ and f_k satisfies Assumption 3.6 with σ and C independent of k.

Proof. Since we can assume that $\omega(z)$ is nondecreasing and subadditive, we have the following

$$\omega(z) \leq \frac{\omega(d)}{d} z + \omega(d), \quad z \geq 0,\ d > 0. \tag{4.10}$$

We can also assume that for every k there exists $d_k > 0$ such that

$$m_k = \frac{\omega(d_k)}{d_k} \to \infty, \quad d_k \to 0^+.$$

Indeed, if this not true, (4.10) implies that $\omega(z) \leq Az$ for some $A \geq 0$ and Assumption 3.6 3),4) is satisfied. In this case we set $f_k = f$. Define

$$\begin{aligned} g_k(s,y,t,x,w,p) &= f(s,y,w,p) + m_k(1+|p|)(|t-s| + |x-y|). \\ f_k(t,x,w,p) &= \inf_{(s,y)\in\Theta_t} g_k(s,y,t,x,w,p). \end{aligned}$$

We can verify that for every k f_k satisfies Assumption 3.6. First we demonstrate 1)

$$g_k(s,y,t,x,w,0) = f(s,y,w,0) + m_k(|t-s| + |x-y|).$$

For $w(0,0) > 0$, we have

$$g_k(s,y,t,x,w,0) \leq \sigma(s, \|w\|_{\mathbb{D}}) + m_k(|t-s| + |x-y|).$$

Taking infimum in $(s,y) \in \Theta_t$ we get $f_k(t,x,w,0) \leq \sigma(t, \|w\|_{\mathbb{D}})$.

For $w(0,0) < 0$, we have

$$g_k(s,y,t,x,w,0) \geq -\sigma(s, \|w\|_{\mathbb{D}}) + m_k(|t-s| + |x-y|) \geq -\sigma(t, \|w\|_{\mathbb{D}})$$

and $f_k(t,x,w,0) \geq -\sigma(t, \|w\|_{\mathbb{D}})$.

Point (ii) of Definition 3.2 follows from the fact that for $w \in K(R)$,

$$-\omega_R(|p|) + m_k|p| \leq g_k(s,y,t,x,w,p) - g_k(s,y,t,x,w,0) \leq \omega_R(|p|) + m_k|p|.$$

Consider now Assumption 3.6 2). Since

$$f_k(t,x,w,p) - g_k(s,y,t,x,\bar{w},p) \leq g_k(s,y,t,x,w,p) - g_k(s,y,t,x,\bar{w},p) \leq C\|w - \bar{w}\|_{\mathbb{D}},$$

we obtain by taking supremum in $(s,y)\in\Theta_t$

$$f_k(t,x,w,p)-f_k(t,x,\bar{w},p)\le C\|w-\bar{w}\|_{\mathbb{D}}.$$

By replacing w and $\bar{w}$ we obtain Assumption 3.6 2) with C independent of k.

By a similar argument Assumption 3.6 3), 4) can be derived from

$$g_k(s,y,t,x,w,p)-g_k(s,y,\bar{t},\bar{x},w,p)\le m_k(1+|p|)(|t-\bar{t}|+|x-\bar{x}|)$$

and Assumption 3.6 5) follows from

$$g_k(s,y,t,x,w,p)-g_k(s,y,t,x,w,\bar{p})\le \omega_{\tilde{L}}(|p-\bar{p}|)+m_k|p-\bar{p}|$$

for $p,\bar{p}\in B(\tilde{L})$. Now we will show that $f_k\rightrightarrows f$ uniformly in $\Theta\times C(D)\times\mathbb{R}^n$.

Indeed, we see that $f_k(t,x,w,p)\le f(t,x,w,p)$ and

$$\begin{aligned}
&f_k(t,x,w,p)-f(t,x,w,p)=\\
&=\inf_{(s,y)\in\Theta_t}\{f(s,y,w,p)-f(t,x,w,p)+m_k(1+|p|)(|t-s|+|x-y|)\}\ge\\
&\ge\inf_{(s,y)\in\Theta}\{-\omega(\ (1+|p|)(|s-t|+|y-x|))+m_k(1+|p|)(|t-s|+|x-y|)\}\ge\\
&\ge\inf_{(s,y)\in\Theta}\Big\{-\frac{\omega(d_k)}{d_k}(1+|p|)(|t-s|+|x-y|)-\omega(d_k)+\\
&\qquad\qquad+m_k(1+|p|)(|t-s|+|x-y|)\Big\}=\\
&=-\omega(d_k).
\end{aligned}$$

□

Of course the method presented in this paper does not guarantee that the viscosity solutions of (1.1), (1.2) are unique. For the uniqueness results we refer to [18, 22].

Remark 4.9. It is possible to obtain result similar to this in Theorem 4.6 by adopting the method used in [8] to the functional equations. The proof is however much more complicated (only the first order equation is considered).

5. INTEGRO-DIFFERENTIAL EQUATION WITH A RETARDED ARGUMENT

Let $\tilde{D}=[-\tau_1,0]\times B(r_1)$, $\tau_1,r_1\ge 0$. Given are: $K:\Theta\times\tilde{D}\times\mathbb{R}\to\mathbb{R}$ of variable (t,x,s,y,u), $F:\Theta\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ of variable (t,x,q,p) and $\mu:\Theta\to\mathbb{R}$, $\nu:\Theta\to\mathbb{R}^n$ such that $t-\tau_2\le\mu(t,x)\le\ t$, $|\nu(t,x)-x|\le\ r_2$, where $\tau_2,r_2\ge 0$. Put $\tau=\tau_1+\tau_2$ and define Θ_0,Θ,E. We consider the equation:

$$\mathcal{P}u=F\Big(t,x,u,\int_{\tilde{D}}K(t,x,s,y,u(\mu(t,x)+s,\nu(t,x)+y))dsdy,\ Du\Big)\quad\text{in }\Theta.\tag{5.1}$$

Set $\mathbb{D}=[-\tau,0]\times B(r)$, where $r=r_1+r_2$. We reduce the initial-value problem for (5.1) to (1.1), (1.2) by setting $f:\Theta\times C(\mathbb{D})\times\mathbb{R}^n\to R$:

$$f(t,x,w,p) = \\ = F\Big(t,x,w(0,0), \int\limits_{\tilde{D}} K(t,x,s,y,w(\mu(t,x)-t+s,\nu(t,x)-x+y))dsdy, p\Big). \tag{5.2}$$

It is easily seen that putting $\mu(t,x)=t$, $\nu(t,x)=x$ we obtain an integro-differential equation. In the similar way we can treat equations with a retarded argument (with no integrals).

In the following we will assume that $\psi \in BUC(\Theta_0)$, $M = \|\psi\|_{\Theta_0}$, and f defined by (5.2) belongs to $X_{\sigma,M}$. This gives the existence of the uniform bound R for the solutions of (5.1), (1.2) and makes possible that all the assumptions on F will be restricted to the set $\bar{\Theta} \times [-R,R] \times [-R,R] \times \mathbb{R}^n$ and all the assumptions on K to the set $\bar{\Theta} \times [-R,R] \times [-R,R]$.

Theorem 5.1. *Suppose that:*

1) *There exists modulus ω such that*

$$|F(t,x,u,v,p) - F(\bar{t},\bar{x},u,v,p)| \le \omega((1+|p|)(|t-\bar{t}|+|x-\bar{x}|)).$$

$F(t,x,\cdot,\cdot,p)$ is Lipschitz continuous in with a constant C independent of (t,x,p). $F(t,x,u,v,\cdot)$ is locally uniformly continuous with a modulus independent of (t,x,u,v).

2) *$K(\cdot,\cdot,s,y,u)$ is uniformly continuous with moduli independent of (s,y,u). For each (t,x,u) $K(t,x,\cdot,\cdot,u)$ is integrable. $K(t,x,s,y,\cdot)$ is Lipschitz continuous with a constant C_K independent of (t,x,s,y).*

3) *$\mu(\cdot,\cdot)$, $\nu(\cdot,\cdot)$ are Lipschitz continuous.*

Then there exists a viscosity solution of (5.1), (1.2).

Proof. It is not difficult to verify that if $\mu(t,x)-t$, $\nu(\cdot,\cdot)-x$ are constant then we can apply Proposition 4.8 and then Theorem 4.6. In a general case we precede as follows.

Let $F_k \rightrightarrows F$, $K_k \rightrightarrows K$ (existence of such F_k, K_k can be derived from Proposition 4.8). We define f_k by formula (5.2) with F, K replaced by F_k, K_k. It is easy to show that $f_k \rightrightarrows f$. Since other points of Assumption 3.6 are rather easy we will show only that f_k satisfies Assumption 3.6 3) 4)

$$|f_k(t,x,w,p) - f_k(\bar{t},\bar{x},w,p)| \le \\ \le \Big| F_k\Big(t,x,w(0,0), \int\limits_{\tilde{D}} K_k(t,x,s,y,w(\mu(t,x)-t+s,\nu(t,x)-x+y))dsdy, p\Big) - \\ - F_k\Big(\bar{t},\bar{x},w(0,0), \int\limits_{\tilde{D}} K_k(\bar{t},\bar{x},s,y,w(\mu(\bar{t},\bar{x})-\bar{t}+s,\nu(\bar{t},\bar{x})-\bar{x}+y))dsdy, p\Big)\Big| \le$$

$$
\begin{aligned}
&\le \Big|F_k\Big(t,x,w(0,0),\int\limits_{\tilde D} K_k(t,x,s,y,w(\mu(t,x)-t+s,\nu(t,x)-x+y))dsdy,p\Big)-\\
&\quad - F_k\Big(\bar t,\bar x,w(0,0),\int\limits_{\tilde D} K_k(t,x,s,y,w(\mu(t,x)-t+s,\nu(t,x)-x+y))dsdy,p\Big)\Big|+\\
&\quad + \Big|F_k\Big(\bar t,\bar x,w(0,0),\int\limits_{\tilde D} K_k(t,x,s,y,w(\mu(t,x)-t+s,\nu(t,x)-x+y))dsdy,\ p\Big)-\\
&\quad - F_k\Big(\bar t,\bar x,w(0,0),\int\limits_{\tilde D} K_k(\bar t,\bar x,s,y,w(\mu(\bar t,\bar x)-\bar t+s,\nu(\bar t,\bar x)-\bar x+y))dsdy,\ p\Big)\Big| \le\\
&\le L_k(1+|p|)(|t-\bar t|+|x-\bar x|))+\\
&\quad + CC_K|\tilde D||w(\mu(t,x)-t+s,\nu(t,x)-x+y))-w(\mu(\bar t,\bar x)-\bar t+s,\nu(\bar t,\bar x)-\bar x+y)| \le\\
&\le L_k((1+|p|)(|t-\bar t|+|x-\bar x|))+\\
&\quad + CC_K|\tilde D|(L_t(w)|\mu(t,x)-t-\mu(\bar t,\bar x)+\bar t|+L_x(w)|\nu(t,x)-x-\nu(\bar t,\bar x)+\bar x|) \le\\
&\le L_k((1+|p|)(|t-\bar t|+|x-\bar x|))+\\
&\quad + CC_K|\tilde D|(L_t[w](L_t[\mu-id_t]|t-\bar t|+L_x[\mu]|x-\bar x|)+L_x[w])\times\\
&\qquad \times (L_t[\nu]|t-\bar t|+L_x[\nu-id_x]|x-\bar x|),
\end{aligned}
$$

where $id_t(t,x)=t,\ id_x(t,x)=x.$ $\square$

Now we give a simple example, where we can apply Theorem 5.1.

Example 5.2. Let $\mu(t,x)=t-t^2\sin|x_1x_2|$, $\nu(t,x)=(x_1+t\cos x_2, x_2+t\sin x_1)$, $x=(x_1,x_2)$.

$$
\begin{aligned}
D_tu-D^2_{x_1}u &= \sin\sqrt{D_{x_2}u}\int\limits_{-1}^{1}\int\limits_{-1}^{1}\int\limits_{-1}^{0}u^2(\mu(t,x)+s,\nu(t,x)+y)dsdy_1y_2 \quad &&\text{in } (0,1]\times\mathbb{R}^2,\\
u &= \psi &&\text{in } [-2,0]\times\mathbb{R}^2,
\end{aligned}
$$

$y=(y_1,y_2)$ (we consider the maximum norm in $\mathbb{R}^2$). It is easily seen that a priori bound for the solutions is $R=\|\psi\|_\Theta<\infty$.

Remark 5.3. Considering Assumption 3.6 in the case of the model presented in this section we see that $A_2=0$ means that $\nu(t,x)$ must have the form $x+\tilde\nu(t)$, $A_3=0$ means that $\mu(t,x)=\mu(t)$, $B_2=0$ means that $\nu(t,x)=\nu(x)$ and $B_3=0$ means that $\mu(t,x)=t+\tilde\mu(x)$.

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