Opuscula Math. 42, no. 3 (2022), 393–413
https://doi.org/10.7494/OpMath.2022.42.3.393 Opuscula Mathematica

NEW ASPECTS FOR THE OSCILLATION OF FIRST-ORDER DIFFERENCE EQUATIONS WITH DEVIATING ARGUMENTS

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Communicated by Josef Diblík

Abstract. We study the oscillation of first-order linear difference equations with non-monotone deviating arguments. Iterative oscillation criteria are obtained which essentially improve, extend, and simplify some known conditions. These results will be applied to some numerical examples.

Keywords: difference equations, oscillation, non-monotone advanced arguments.

Mathematics Subject Classification: 39A10, 39A21.

1. INTRODUCTION

In this work, we focus our attention on the oscillatory character of the first-order linear difference equation with variable advanced argument

$$\nabla u(r) - a(r)u(\xi(r)) = 0, \quad r \in \mathbb{N}, \tag{1.1}$$

its dual retarded difference equation

$$\Delta u(r) + \overline{a}(r)u(\overline{\xi}(r)) = 0, \quad r \in \mathbb{N}_0, \tag{1.2}$$

where $\mathbb{N}$ and $\mathbb{N}_0$ are respectively the set of all positive and nonnegative integers, $(a(r))_{r\geq 1}$ and $(\overline{a}(r))_{r\geq 0}$ are sequences of nonnegative real numbers and $(\xi(r))_{r\geq 1}$ and $(\overline{\xi}(r))_{r\geq 0}$ are sequences of integers such that

$$\xi(r) \geq r+1, \quad r \in \mathbb{N}, \qquad \overline{\xi}(r) \leq r-1, \quad r \in \mathbb{N}_0,$$

also, $\nabla u(r) = u(r) - u(r-1)$ and $\Delta u(r) = u(r+1) - u(r)$. Throughout this paper, we assume that there exist nondecreasing sequences of integers $(\zeta(r))_{r\geq 1}$ and $(\overline{\zeta}(r))_{r\geq 0}$ such that

$$\xi(r) \geq \zeta(r) \geq r+1, \quad r \in \mathbb{N} \quad \text{and} \quad \overline{\xi}(r) \leq \overline{\zeta}(r) \leq r-1, \quad r \in \mathbb{N}_0.$$

Furthermore, we will use the following notation:

$$\sum_{j=s}^{s-1} D(j) = 0, \qquad \prod_{j=s}^{s-1} D(j) = 1,$$

$$\rho(r) = \min_{j\geq r} \xi(j), \qquad \overline{\rho}(r) = \max_{0\leq j\leq r} \overline{\xi}(j), \tag{1.3}$$

$$\beta = \liminf_{r\to\infty} \sum_{j=r+1}^{\xi(r)} a(j), \quad \gamma = \liminf_{r\to\infty} \sum_{j=r+1}^{\zeta(r)} a(j),$$

$$\overline{\beta} = \liminf_{r\to\infty} \sum_{j=\overline{\xi}(r)}^{r-1} \overline{a}(j), \quad \overline{\gamma} = \liminf_{r\to\infty} \sum_{j=\overline{\zeta}(r)}^{r-1} \overline{a}(j)$$

and

$$A(\upsilon) = \begin{cases} 0 & \text{if} \quad \upsilon > \frac{1}{\mathrm{e}}, \\ \dfrac{1-\upsilon-\sqrt{1-2\upsilon-\upsilon^2}}{2} & \text{if} \quad \upsilon \in \left[0, \frac{1}{\mathrm{e}}\right]. \end{cases}$$

Also, we assume that $\lambda(u)$ is the smaller root of $\lambda = e^{u\lambda}$.

The qualitative properties of delay differential equations and their discrete analogues (i.e., difference equations with retarded arguments) have attracted the attention of many mathematicians. In fact, these properties give more insight into the understanding of the dynamics of these equations. The oscillation property can be considered as one of the important features that appear in many applications. As a result, this topic has received much attention from researchers, see for example [1, 2, 5–25] and the references cited therein.

The incomplete theoretical understanding of the oscillation theory of Eq. (1.1) and its dual, Eq. (1.2), has encouraged many researchers to investigate this property. Very recently, there have been great efforts to establish new sufficient oscillation conditions. This motivates us to develop and extend some techniques introduced by Attia [2] to study the oscillation of equations (1.1) and (1.2). Several oscillation criteria for the latter equations are obtained.

The following summary is intended to highlight the most recent results on the oscillation of Eq. (1.1).

Braverman *et al.* [5] defined the sequence $\{\Omega_{m+1}(r,s)\}_{m\geq 0}$ recursively as follows:

$$\Omega_1(r,s) = \prod_{l=r+1}^{s} (1-a(l)), \quad \Omega_{m+1}(r,s) = \prod_{l=r+1}^{s} \left(1 - a(l)\Omega_m^{-1}(l,\xi(l))\right). \tag{1.4}$$

Then the authors proved that Eq. (1.1) is oscillatory if

$$\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j)\Omega_m^{-1}(\rho(r),\xi(j)) > 1, \tag{1.5}$$

or

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\Omega_m^{-1}\left(\rho(r),\xi(j)\right) > 1 - A(\beta). \tag{1.6}$$

Asteris and Chatzarakis [1], Chatzarakis and Jadlovská [8] and Chatzarakis [7] established respectively the conditions

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\prod_{j_1=\rho(r)+1}^{\xi(j)}\frac{1}{1-P_m(j_1)} > 1, \tag{1.7}$$

where

$$P_m(r) = a(r)\left[1+\sum_{j=r+1}^{\rho(r)} a(j)\prod_{j_1=\rho(r)+1}^{\xi(j)}\frac{1}{1-P_{m-1}(j_1)}\right], \quad \text{and} \quad P_0(r)=a(r),$$

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\exp\left(\sum_{j_1=\rho(r)+1}^{\xi(j)} a(j_1)\prod_{j_2=j_1+1}^{\xi(j_1)}\frac{1}{1-Q_m(j_2)}\right) > 1-A(\beta), \tag{1.8}$$

where

$$Q_m(r) = a(r)\left[1+\sum_{j=r+1}^{\xi(r)} a(j)\exp\left(\sum_{j_1=r+1}^{\xi(j)} a(j_1)\prod_{j_2=j_1+1}^{\xi(j_1)}\frac{1}{1-Q_{m-1}(j_2)}\right)\right],$$
$$Q_0(r) = \lambda(\beta)a(r),$$

and

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\exp\left(\sum_{j_1=\rho(r)+1}^{\xi(j)} a(j_1)\ \exp\left(\sum_{j_2=j_1+1}^{\xi(j_1)} a(j_2)\prod_{j_3=j_2+1}^{\xi(j_2)}\frac{1}{1-F_m(j_3)}\right)\right)$$
$$> 1-A(\beta), \tag{1.9}$$

where

$$F_m(r)$$
$$= a(r)\left[1+\sum_{j=r+1}^{\xi(r)} a(j)\exp\left(\sum_{j_1=r+1}^{\xi(j)} a(j_1)\exp\Big(\sum_{j_2=j_1+1}^{\xi(j_1)} a(j_2)\prod_{j_3=j_2+1}^{\xi(j_2)}\frac{1}{1-F_{m-1}(j_3)}\Big)\right)\right],$$

with

$$F_0(r) = a(r)\left[1+\sum_{j=r+1}^{\xi(r)} a(j)\exp\left(\sum_{j_1=r+1}^{\xi(j)} a(j_1)\exp\left(\lambda(\beta)\sum_{j_2=j_1+1}^{\xi(j_1)} a(j_2)\right)\right)\right].$$

Attia and Chatzarakis [4] obtained the oscillation criterion

$$\limsup_{r\to\infty}\left(\frac{W_m(r)}{1-\sum_{j=\sigma(r-1)+1}^{r-1}a(j)} + \sum_{j=r}^{\rho(r)}a(j)\prod_{j_1=\rho(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)Z_m(j_1,\xi(j_1))}\right)>1, \tag{1.10}$$

where

$$W_m(r)=\sum_{j=\sigma(r-1)+1}^{r-1}a(j)\sum_{j_1=r}^{\xi(j)}a(j_1)\prod_{j_2=\rho(r)+1}^{\xi(j_1)}\frac{1}{1-a(j_2)Z_m(j_2,\xi(j_2))},$$

$$Z_m(r,s) = 1+\sum_{j=r+1}^{s}\frac{a(j)}{\left(1-\frac{1}{\xi(j)-r}\sum_{j_1=r+1}^{\xi(j)}a(j_1)\prod_{j_2=j_1+1}^{\xi(j_1)}\frac{1}{1-a(j_2)Z_{m-1}(j_2,\xi(j_2))}\right)^{\xi(j)-r}},$$

$$Z_0(r,s)=1,$$

and

$$\sigma(r)=\max\{s\in\mathbb{N}_0 : s<r,\ \xi(s)<r+1\}. \tag{1.11}$$

2. MAIN RESULTS

2.1. ADVANCED DIFFERENCE EQUATIONS

Let $u(r)$ be an eventually positive solution of Eq. (1.1). The following four lemmas will be needed later.

Lemma 2.1 ([5]). *If $m\in\mathbb{N}$ and $k\geq r$, then*

$$u(r)\leq\Omega_m(r,k)u(k),$$

where $\Omega_m(r,k)$ is defined by (1.4).

Lemma 2.2 ([7, Lemma 2.8]). *If $0<\gamma\leq\frac{1}{e}$, then*

$$\liminf_{n\to\infty}\frac{u(r-1)}{u(\zeta(r))}\geq A(\gamma). \tag{2.1}$$

Lemma 2.3. *If $m\in\mathbb{N}_0$, then $\sum_{j=\sigma(r-1)+1}^{r-1}a(j)<1$ and*

$$\frac{u(r-1)}{u(\zeta(r))}>\frac{\sum_{j=\sigma(r-1)+1}^{r-1}a(j)\sum_{j_1=r}^{\xi(j)}a(j_1)\frac{u(\xi(j_1))}{u(\zeta(j_1))}}{1-\sum_{j=\sigma(r-1)+1}^{r-1}a(j)} \tag{2.2}$$

for all sufficiently large r, where $\sigma(r)$ is defined by (1.11).

Proof. Summing up Eq. (1.1) from $\sigma(r-1)+1$ to $r-1$, we get

$$u(r-1)-u(\sigma(r-1))-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)u(\xi(j))=0. \tag{2.3}$$

In view of $\xi(j)\geq r$ for $\sigma(r-1)+1\leq j\leq r-1$, we conclude that

$$u(\xi(j))=u(r-1)+\sum_{j_1=r}^{\xi(j)} a(j_1)u(\xi(j_1)).$$

Substituting into (2.3), we obtain

$$u(r-1)-u(\sigma(r-1))-u(r-1)\sum_{j=\sigma(r-1)+1}^{r-1} a(j)$$
$$-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\sum_{j_1=r}^{\xi(j)} a(j_1)u(\xi(j_1))=0.$$

Since $\zeta(j_1)\geq\zeta(r)$ for $j_1\geq r$, then the nondecreasing nature of $u(r)$ implies that

$$u(r-1)-u(\sigma(r-1))-u(r-1)\sum_{j=\sigma(r-1)+1}^{r-1} a(j)$$
$$-u(\zeta(r))\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\sum_{j_1=r}^{\xi(j)} a(j_1)\frac{u(\xi(j_1))}{u(\zeta(j_1))}\geq 0.$$

From this and the positivity of $u(\sigma(r-1))$, it follows that

$$u(r-1)\left(1-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\right)>u(\zeta(r))\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\sum_{j_1=r}^{\xi(j)} a(j_1)\frac{u(\xi(j_1))}{u(\zeta(j_1))}\geq 0.$$

Consequently, $\sum_{j=\sigma(r-1)+1}^{r-1} a(j)<1$ and

$$\frac{u(r-1)}{u(\zeta(r))}>\frac{\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\sum_{j_1=r}^{\xi(j)} a(j_1)\frac{u(\xi(j_1))}{u(\zeta(j_1))}}{1-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)}.$$

The proof is complete. □

Define the sequence $\{S_m(r)\}_{m=0}^{\infty}$ as follows:

$$S_m(r)=\frac{\prod_{j=\zeta(r)+1}^{\xi(r)}\frac{1}{1-a(j)S_{m-1}(j)}}{1-G_m(r)}\quad \text{for } m=1,2,\ldots,$$

where $S_0(r) = 1$ and

$$G_m(r) = \sum_{k=1}^{m-1} \prod_{j=2}^{k} \frac{1}{1 - G_{m-1}(\zeta^{j-1}(r))} \cdot \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_k=\zeta^{k-1}(r)+1}^{\xi(j_{k-1})} a(j_k) + \prod_{j=2}^{m} \frac{1}{1 - G_{m-1}(\zeta^{j-1}(r))} \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_m=\zeta^{m-1}(r)+1}^{\xi(j_{m-1})} a(j_m) \prod_{j_{m+1}=\zeta^m(r)+1}^{\xi(j_m)} \frac{1}{1 - a(j_{m+1}) S_{m-1}(j_{m+1})}$$

for $m = 1, 2, \ldots$

Lemma 2.4. *If $m \in \mathbb{N}_0$, then $a(r)S_m(r) < 1$ and*

$$\frac{u(\xi(r))}{u(r)} \geq S_m(r) \tag{2.4}$$

for all sufficiently large r.

Proof. Since $u(r)$ is an eventually positive solution of Eq. (1.1), then $u(r)$ is an eventually non-decreasing sequence. Hence, for all sufficiently large r, we have

$$\frac{u(\xi(r))}{u(r)} \geq S_0(r). \tag{2.5}$$

Dividing Eq. (1.1) by $u(r)$, we obtain

$$0 < \frac{u(r-1)}{u(r)} = 1 - a(r)\frac{u(\xi(r))}{u(r)}. \tag{2.6}$$

Taking the product on both sides, from $k+1$ to r, we get

$$\frac{u(k)}{u(r)} = \prod_{j=k+1}^{r} \frac{u(j-1)}{u(j)} = \prod_{j=k+1}^{r} \left(1 - a(j)\frac{u(\xi(j))}{u(j)}\right),$$

that is,

$$u(r) = u(k) \prod_{j=k+1}^{r} \frac{1}{1 - a(j)\frac{u(\xi(j))}{u(j)}} \quad \text{for all } r \geq k. \tag{2.7}$$

Summing up Eq. (1.1) from $r+1$ to $\zeta(r)$, we have

$$u(\zeta(r)) - u(r) - \sum_{j_1=r+1}^{\zeta(r)} a(j_1)u(\xi(j_1)) = 0. \tag{2.8}$$

Taking into account the fact that $\xi(j_1) \geq \zeta(j_1) \geq \zeta(r)$ for $j_1 \geq r+1$, it follows from (2.7) that

$$u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \prod_{j_2=\zeta(r)+1}^{\xi(j_1)} \frac{1}{1 - a(j_2)\frac{u(\xi(j_2))}{u(j_2)}} = 0,$$

which in view of (2.5) implies that

$$u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \prod_{j_2=\zeta(r)+1}^{\xi(j_1)} \frac{1}{1 - a(j_2)S_0(j_2)} \geq 0.$$

That is,

$$u(\zeta(r)) \left(1 - \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \prod_{j_2=\zeta(r)+1}^{\xi(j_1)} \frac{1}{1 - a(j_2)S_0(j_2)} \right) \geq u(r) > 0,$$

which in turn leads to

$$\frac{u(\zeta(r))}{u(r)} \geq \frac{1}{1 - \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \prod_{j_2=\zeta(r)+1}^{\xi(j_1)} \frac{1}{1-a(j_2)S_0(j_2)}} = \frac{1}{1 - G_1(r)}. \tag{2.9}$$

This together with (2.5) and (2.7), implies that

$$\begin{aligned} \frac{u(\xi(r))}{u(r)} &= \frac{u(\xi(r))}{u(\zeta(r))} \frac{u(\zeta(r))}{u(r)} \\ &\geq \frac{\prod_{j=\zeta(r)+1}^{\xi(r)} \frac{1}{1-a(j)S_0(j)}}{1 - \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \prod_{j_2=\zeta(r)+1}^{\xi(j_1)} \frac{1}{1-a(j_2)S_0(j_2)}} = S_1(r). \end{aligned} \tag{2.10}$$

For $r+1 \leq j_1 \leq \zeta(r)$, we have

$$u(\xi(j_1)) = u(\zeta(r)) + \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2)u(\xi(j_2)).$$

Substituting into (2.8), we get

$$u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) - \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2)u(\xi(j_2)) = 0.$$

Since $\xi(j_2) \geq \zeta^2(r)$ for $\xi(j_1) \geq j_2 \geq \zeta(r)$, it follows from (2.7) that

$$\begin{aligned} &u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \\ &- u(\zeta^2(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \prod_{j_3=\zeta^2(r)+1}^{\xi(j_2)} \frac{1}{1 - a(j_3)\frac{u(\xi(j_3))}{u(j_3)}} = 0. \end{aligned}$$

From (2.9) and (2.10) we have

$$u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) - u(\zeta(r)) \frac{1}{1-G_1(\zeta(r))} \cdot \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \prod_{j_3=\zeta^2(r)+1}^{\xi(j_2)} \frac{1}{1-a(j)S_1(j_3)} \geq 0,$$

which in turn leads to

$$\frac{u(\zeta(r))}{u(r)} \geq \frac{1}{1-G_2(r)}.$$

Then

$$\frac{u(\xi(r))}{u(r)} \geq S_2(r).$$

Continuing in this way m times, we get

$$\begin{aligned} & u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) - u(\zeta^2(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \\ & - \ldots - u(\zeta^{m-1}(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_{m-1}=\zeta^{m-2}(r)+1}^{\xi(j_{m-2})} a(j_{m-1}) \\ & - u(\zeta^m(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_m=\zeta^{m-1}(r)+1}^{\xi(j_{m-1})} a(j_m) \\ & \cdot \prod_{j_{m+1}=\zeta^m(r)+1}^{\xi(j_m)} \frac{1}{1-a(j_{m+1})\frac{u(\xi(j_{m+1}))}{u(j_{m+1})}} = 0. \end{aligned} \tag{2.11}$$

It is clear for $k = 2, 3, \ldots, m$ that

$$u(\zeta^k(r)) = \frac{u(\zeta^k(r))}{u(\zeta^{k-1}(r))} \cdots \frac{u(\zeta^2(r))}{u(\zeta(r))} u(\zeta(r)) = u(\zeta(r)) \prod_{j=2}^{k} \frac{u(\zeta^j(r))}{u(\zeta^{j-1}(r))}.$$

On the other hand, if we assume that

$$\frac{u(\zeta(r))}{u(r)} \geq \frac{1}{1-G_{m-1}(r)} \quad \text{and} \quad \frac{u(\xi(r))}{u(r)} \geq S_{m-1}(r),$$

then

$$u(\zeta^k(r)) \geq u(\zeta(r)) \prod_{j=2}^{k} \frac{1}{1-G_{m-1}(\zeta^{j-1}(r))}, \quad k = 1, 2, \ldots, m.$$

Substituting into (2.11), we get

$$\begin{aligned}
&u(\zeta(r)) - u(r) - u(\zeta(r)) \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \\
&- u(\zeta(r)) \frac{1}{1 - G_{m-1}(\zeta(r))} \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \\
&- \ldots - u(\zeta(r)) \prod_{j=2}^{m-1} \frac{1}{1 - G_{m-1}(\zeta^{j-1}(r))} \\
&\quad \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_{m-1}=\zeta^{m-2}(r)+1}^{\xi(j_{m-2})} a(j_{m-1}) \\
&- \ldots - u(\zeta(r)) \prod_{j=2}^{m} \frac{1}{1 - G_{m-1}(\zeta^{j-1}(r))} \\
&\quad \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \ldots \sum_{j_m=\zeta^{m-1}(r)+1}^{\xi(j_{m-1})} a(j_m) \\
&\quad \prod_{j_{m+1}=\zeta^m(r)+1}^{\xi(j_{m+1})} \frac{1}{1 - a(j_{m+1}) S_{m-1}(j_{m+1})} \geq 0,
\end{aligned}$$

that is,

$$\frac{u(\zeta(r))}{u(r)} \geq \frac{1}{1 - G_m(r)}. \tag{2.12}$$

Therefore,

$$\frac{u(\xi(r))}{u(r)} \geq S_m(r).$$

By virtue of (2.6) and the above inequality, we get $a(r)S_m(r) < 1$. The proof is complete. □

Theorem 2.5. *Let $m \in \mathbb{N}_0$. Then each of the following conditions is sufficient for the oscillation of Eq.* (1.1)*:*

(i)
$$a(r_j)S_m(r_j) \geq 1 \qquad \text{for all } j \in \mathbb{N}, \tag{2.13}$$

where $\{r_j\}_{j\geq 1}$ is an unbounded sequence of positive integers,

(ii)
$$\limsup_{r\to\infty} \sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)S_m(j_1)} > 1 - A(\gamma). \tag{2.14}$$

Proof. If not, without loss of generality we can assume that there exists an eventually positive solution $u(r)$ of Eq. (1.1). It follows from Lemma 2.4 that $a(r)S_m(r) < 1$ for all $m \in \mathbb{N}_0$ and all sufficiently large r. This contradicts (2.13) and completes the proof of (i). Summing up Eq. (1.1) from r to $\zeta(r)$, we get

$$u(\zeta(r)) - u(r-1) - \sum_{j=r}^{\zeta(r)} a(j)u(\xi(j)) = 0.$$

By (2.7), we obtain

$$u(\zeta(r)) - u(r-1) - u(\zeta(r)) \sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)\frac{u(\xi(j_1))}{u(j_1)}} = 0.$$

From this and (2.4) we get

$$u(\zeta(r)) - u(r-1) - u(\zeta(r)) \sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)S_m(j_1)} \geq 0,$$

that is,

$$\sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)S_m(j_1)} \leq 1 - \frac{u(r-1)}{u(\zeta(r))}.$$

Consequently,

$$\limsup_{r\to\infty} \sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)S_m(j_1)} \leq 1 - \liminf_{n\to\infty} \frac{u(r-1)}{u(\zeta(r))}.$$

The positivity of $u(r)$ and (2.1) implies that

$$\limsup_{r\to\infty} \sum_{j=r}^{\zeta(r)} a(j) \prod_{j_1=\zeta(r)+1}^{\xi(j)} \frac{1}{1 - a(j_1)S_m(j_1)} \leq 1 - A(\gamma),$$

which contradicts with (2.14). The proof is complete. □

Theorem 2.6. *Let $m \in \mathbb{N}_0$. Then each of the following conditions is sufficient to imply the oscillation of Eq.* (1.1)*:*

(i)

$$\sum_{j=\sigma(r'_k-1)+1}^{r'_k-1} a(j) \geq 1 \qquad \textit{for all } k \geq \mathbb{N}, \tag{2.15}$$

where $\{r'_k\}_{k\geq 1}$ is an unbounded sequence of positive integers,

(ii)

$$\limsup_{r\to\infty}\left(R(r,m)+\sum_{j=r}^{\zeta(r)}a(j)\prod_{j_1=\zeta(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)S_m(j_1)}\right)>1, \tag{2.16}$$

where

$$R(r,m)=\frac{\sum_{j=\sigma(r-1)+1}^{r-1}a(j)\sum_{j_1=r}^{\xi(j)}a(j_1)\prod_{j_2=\zeta(j_1)+1}^{\xi(j_1)}\frac{1}{1-a(j_2)S_m(j_2)}}{1-\sum_{j=\sigma(r-1)+1}^{r-1}a(j)},$$

where $\sigma(r)$ *is defined by* (1.11).

Proof. As before, assume that $u(r)$ is an eventually positive solution of Eq. (1.1). Then Lemma 2.3 implies that

$$\sum_{j=\sigma(r-1)+1}^{r-1}a(j)<1 \quad \text{for all sufficiently large } r.$$

This inequality contradicts (2.15) and hence completes the proof of (i).

Using the same argument that is given in the proof of Theorem 2.5, we obtain

$$\frac{u(r-1)}{u(\zeta(r))}+\sum_{j=r}^{\zeta(r)}a(j)\prod_{j_1=\zeta(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)\frac{u(\xi(j_1))}{u(j_1)}}=1.$$

By (2.2) and (2.4), we have

$$\frac{\sum_{j=\sigma(r-1)+1}^{r-1}a(j)\sum_{j_1=r}^{\xi(j)}a(j_1)\frac{u(\xi(j_1))}{u(\zeta(j_1))}}{1-\sum_{j=\sigma(r-1)+1}^{r-1}a(j)}+\sum_{j=r}^{\zeta(r)}a(j)\prod_{j_1=\zeta(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)S_m(j_1)}<1. \tag{2.17}$$

From (2.4) and (2.7), we obtain

$$\frac{u(\xi(j_1))}{u(\zeta(j_1))}\geq\prod_{j_2=\zeta(j_1)+1}^{\xi(j_1)}\frac{1}{1-a(j_2)S_m(j_2)}.$$

Substituting into (2.17), we have

$$R(r,m)+\sum_{j=r}^{\zeta(r)}a(j)\prod_{j_1=\zeta(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)S_m(j_1)}<1.$$

Then

$$\limsup_{r\to\infty}\left(R(r,m)+\sum_{j=r}^{\zeta(r)}a(j)\prod_{j_1=\zeta(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)S_m(j_1)}\right)\leq 1.$$

This contradiction completes the proof of theorem. □

Using Lemma 2.1 instead of Lemma 2.4 in the proof of the preceding theorem, we obtain the following result:

Theorem 2.7. *If* $m \in \mathbb{N}$ *and*

$$\limsup_{r\to\infty}\left(D(r,m)+\sum_{j=r}^{\zeta(r)} a(j)\Omega_m^{-1}\left(\zeta(r),\xi(j)\right)\right)>1, \tag{2.18}$$

where

$$D(r,m)=\frac{\sum_{j=\sigma(r-1)+1}^{r-1} a(j)\sum_{j_1=r}^{\xi(j)} a(j_1)\Omega_m^{-1}(\zeta(j_1),\xi(j_1))}{1-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)},$$

where $\sigma(r)$ *and* $\Omega_m(r,k)$, $k\ge r$ *are defined respectively by* (1.11) *and* (1.4)*, then every solution of* (1.1) *is oscillatory.*

2.2. RETARDED DIFFERENCE EQUATIONS

In this section, we obtain many oscillation criteria for the (dual) retarded difference equation (1.2). The proofs of these results are quite similar to those for the advanced difference equation (1.1), and hence they will be omitted.

Let $u(r)$ be a positive solution of Eq. (1.2). The following results are crucial in establishing our main results.

Lemma 2.8 ([5])**.** *If* $m \in \mathbb{N}$ *and* $r\ge k$*, then*

$$u(r)\le \overline{\Omega}_m(r,k)u(k),$$

where $\overline{\Omega}_m(r,k)$ *is defined by*

$$\overline{\Omega}_1(r,s)=\prod_{l=s}^{r-1}\left(1-\overline{a}(l)\right),\quad \overline{\Omega}_{m+1}(r,s)=\prod_{l=s}^{r-1}\left(1-\overline{a}(l)\overline{\Omega}_m^{-1}\left(l,\overline{\xi}(l)\right)\right).$$

Lemma 2.9 ([7, Lemma 2.2])**.** *If* $0<\overline{\gamma}\le\frac{1}{e}$*, then*

$$\liminf_{n\to\infty}\frac{u(r+1)}{u(\overline{\zeta}(r))}\ge A(\overline{\gamma}).$$

Lemma 2.10. *Let*

$$\overline{\sigma}(r)=\min\{s\in\mathbb{N}: s>r,\overline{\xi}(s)>r-1\}.$$

If $m\in\mathbb{N}_0$*, then* $\sum_{j=r+1}^{\overline{\sigma}(r+1)-1}\overline{a}(j)<1$ *and*

$$\frac{u(r+1)}{u(\overline{\zeta}(r))}\ge\frac{\sum_{j=r+1}^{\overline{\sigma}(r+1)-1}\overline{a}(j)\sum_{j_1=\overline{\overline{\xi}}(j)}^{r}\overline{a}(j_1)\frac{u(\overline{\xi}(j_1))}{u(\overline{\zeta}(j_1))}}{1-\sum_{j=r+1}^{\overline{\sigma}(r+1)-1}\overline{a}(j)}$$

for all sufficiently large r.

Let the sequence $\{\overline{S}_m(r)\}_{m=0}^{\infty}$ be defined by

$$\overline{S}_m(r) = \frac{\prod_{j=\overline{\xi}(r)}^{\overline{\zeta}(r)-1} \frac{1}{1-\overline{a}(j)\overline{S}_{m-1}(j)}}{1-\overline{G}_m(r)} \quad \text{for } m = 1, 2, \ldots,$$

where $\overline{S}_0(r) = 1$ and

$$\overline{G}_m(r) = \sum_{k=1}^{m-1} \prod_{j=2}^{k} \frac{1}{1-\overline{G}_{m-1}(\overline{\zeta}^{j-1}(r))} \sum_{j_1=\overline{\zeta}(r)}^{r-1} \overline{a}(j_1) \sum_{j_2=\overline{\xi}(j_1)}^{\overline{\zeta}(r)-1} \overline{a}(j_k) \ldots \sum_{j_k=\overline{\xi}(j_{k-1})}^{\overline{\zeta}^{k-1}(r)-1} \overline{a}(j_r)$$
$$+ \prod_{j=2}^{m} \frac{1}{1-G_{m-1}(\overline{\zeta}^{j-1}(r))} \sum_{j_1=\overline{\zeta}(r)}^{r-1} \overline{a}(j_1) \sum_{j_2=\overline{\xi}(j_1)}^{\overline{\zeta}(r)-1} \overline{a}(j_2) \ldots \sum_{j_m=\overline{\xi}(j_{m-1})}^{\overline{\zeta}^{m-1}(r)-1} \overline{a}(j_m)$$
$$\cdot \prod_{j_{m+1}=\overline{\xi}(j_m)}^{\overline{\zeta}^{m}(r)-1} \frac{1}{1-\overline{a}(j_{m+1})\overline{S}_{m-1}(j_{m+1})}$$

for $m = 1, 2, \ldots$

Lemma 2.11. *If $m \in \mathbb{N}_0$, then $\overline{a}(r)\overline{S}_m(r) < 1$ and*

$$\frac{u(\overline{\xi}(r))}{u(r)} \geq \overline{S}_m(r)$$

for all sufficiently large r.

Theorem 2.12. *Let $m \in \mathbb{N}_0$. Then each of the following conditions is sufficient for the oscillation of Eq.* (1.2)*:*

(i) *there exists an unbounded sequence of positive integers $\{r_j\}_{j\geq 1}$ such that*

$$\overline{a}(r_j)\overline{S}_m(r_j) \geq 1 \quad \textit{for all } j \in \mathbb{N},$$

(ii)

$$\limsup_{r\to\infty} \sum_{j=\overline{\zeta}(r)}^{r} \overline{a}(j) \prod_{j_1=\overline{\xi}(j)}^{\overline{\zeta}(r)-1} \frac{1}{1-\overline{a}(j_1)\overline{S}_m(j_1)} > 1 - A(\overline{\gamma}).$$

Theorem 2.13. *Assume that $m \in \mathbb{N}_0$. Then Eq.* (1.2) *is oscillatory if either one of the following holds:*

(i) *there exists an unbounded sequence of positive integers $\{r'_k\}_{k\geq 1}$ such that*

$$\sum_{j=r'_k+1}^{\overline{\sigma}(r'_k+1)-1} \overline{a}(j) \geq 1 \qquad \textit{for all } k \geq \mathbb{N},$$

(ii)
$$\limsup_{r\to\infty}\left(\overline{R}(r,m)+\sum_{j=\overline{\zeta}(r)}^{r}\overline{a}(j)\prod_{j_1=\overline{\xi}(j)}^{\overline{\zeta}(r)-1}\frac{1}{1-\overline{a}(j_1)\overline{S}_m(j_1)}\right)>1,$$

where
$$\overline{R}(r,m)=\frac{\sum_{j=r+1}^{\overline{\sigma}(r+1)-1}\overline{a}(j)\sum_{j_1=\overline{\xi}(j)}^{r}\overline{a}(j_1)\prod_{j_2=\overline{\xi}(j_1)}^{\overline{\zeta}(j_1)-1}\frac{1}{1-\overline{a}(j)\overline{S}_m(j_2)}}{1-\sum_{j=r+1}^{\overline{\sigma}(r+1)-1}\overline{a}(j)}.$$

Theorem 2.14. *If $m\in\mathbb{N}$ and*
$$\limsup_{r\to\infty}\left(\overline{D}(r,m)+\sum_{j=r}^{\overline{\zeta}(r)}\overline{a}(j)\overline{\Omega}_m^{-1}\left(\overline{\zeta}(r),\overline{\xi}(j)\right)\right)>1,\tag{2.19}$$

where
$$\overline{D}(r,m)=\frac{\sum_{j=\sigma(r-1)+1}^{r-1}\overline{a}(j)\sum_{j_1=r}^{\overline{\xi}(j)}\overline{a}(j_1)\overline{\Omega}_m^{-1}(\overline{\zeta}(j_2),\overline{\xi}(j_1))}{1-\sum_{j=\sigma(r-1)+1}^{r-1}\overline{a}(j)},$$

then every solution of Eq. (1.2) *is oscillatory.*

Remark 2.15.

(i) Condition (2.18) improves condition (1.5).
(ii) Many previous works can be improved by using Lemmas 2.3 and 2.10.

3. NUMERICAL EXAMPLES

Using some numerical examples, we clarify the strength of some of our results. All the following calculations are performed by the Maple software.

Example 3.1. Consider the equation
$$\nabla u(r)-a(r)u(\xi(r))=0,\quad r\in\mathbb{N},\tag{3.1}$$

where
$$a(r)=\begin{cases}\mu & \text{if } r\in\{2r_k+9,2r_k+7,\dots,2r_k-1\},\quad k\in\mathbb{N},\\ 0 & \text{otherwise,}\end{cases}$$

where $r_k\in\mathbb{N}$, $2r_{k+1}>2r_k+13$ and
$$\xi(r)=\begin{cases}r+3 & \text{if } r=2k,\\ r+1 & \text{if } r=2k+1,\end{cases}\quad k\in\mathbb{N}.$$

In view of (1.3) and (1.11), we have respectively
$$\rho(r)=\begin{cases}r+2 & \text{if } r=2k,\\ r+1 & \text{if } r=2k+1,\end{cases}\quad k\in\mathbb{N},$$

and

$$\sigma(r) = \begin{cases} r-1 & \text{if } r = 2k, \\ r-2 & \text{if } r = 2k+1, \end{cases} \qquad k \in \mathbb{N}.$$

Since

$$0 \le \liminf_{r\to\infty} \sum_{i=r+1}^{\xi(r)} a(i) \le \lim_{k\to\infty} \sum_{i=2r_k+10}^{\xi(2r_k+9)} a(i) = \lim_{k\to\infty} \sum_{i=2r_k+10}^{2r_k+10} a(i) = 0,$$

then $\beta = 0$, and so condition (1.6) is not satisfied. Also,

$$\begin{aligned} &\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j)\Omega_3^{-1}\left(\rho(r), \xi(j)\right) \\ &= \lim_{k\to\infty} \sum_{j=2r_k}^{\rho(2r_k)} a(j)\Omega_3^{-1}\left(\rho(2r_k), \xi(j)\right) \\ &= \frac{\mu}{1-\mu W(\mu)} + \mu + \frac{\mu}{(1-\mu W(\mu))^2 \left(1 - \frac{\mu W(\mu)}{\left(1-\frac{\mu}{1-\mu}\right)^2}\right)}, \end{aligned}$$

where $W(\mu) = \frac{1}{1-\frac{\mu}{(1-\mu)^3}}$. Therefore,

$$\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j)\Omega_3^{-1}\left(\rho(r), \xi(j)\right) < 0.998 < 1$$

for all $\mu \in [0.1705, 0.1785]$, and hence condition (1.5) can not be applied for all $\mu \in [0.1705, 0.1785]$. Also,

$$\begin{aligned} &\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j) \prod_{j_1=\rho(r)+1}^{\xi(j)} \frac{1}{1-P_1(j_1)} \\ &= \lim_{k\to\infty} \sum_{j=2r_k}^{\rho(2r_k)} a(j) \prod_{j_1=\rho(2r_k)+1}^{\xi(j)} \frac{1}{1-P_1(j_1)} \\ &= \frac{\mu}{1-\mu\left(1+B(\mu)\right)} + \mu \\ &\quad + \frac{\mu}{\left(1-\mu\left(1+B(\mu)\right)\right)^2 \left(1-\mu\left(\mu+2+B(\mu)+\frac{\mu}{(-1+\mu)^2}\right)\right)}, \end{aligned}$$

where $B(\mu) = \frac{\mu}{(1-\mu)^3}$. Therefore,

$$\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j) \prod_{j_1=\rho(r)+1}^{\xi(j)} \frac{1}{1-P_1(j_1)} < 0.9987 < 1$$

for all $\mu \in [0.1705,\ 0.192]$. Then condition (1.7) with $m = 1$ can not be applied for all $\mu \in [0.1705,\ 0.192]$. Finally, condition (1.8) with $m = 1$ is not satisfied for $\mu = 0.1705$, since

$$\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j) \exp\left(\sum_{j_1=\rho(r)+1}^{\xi(j)} a(j_1) \prod_{j_2=j_1+1}^{\xi(j_1)} \frac{1}{1-Q_1(j_2)} \right) < 0.893 < 1.$$

On the contrary, we show how Theorem 2.7 can be used to prove the oscillation of Eq. (3.1) for all $\mu \in [0.1705,\ 0.23]$.

Let $\zeta(r) = \rho(r)$ (that is, defined by (1.3)) and

$$\Lambda(r) = D(r,3) + \sum_{j=r}^{\zeta(r)} a(j)\Omega_3^{-1}\left(\zeta(r), \xi(j)\right),$$

where

$$D(r,m) = \frac{\sum_{j=\sigma(r-1)+1}^{r-1} a(j) \sum_{j_1=r}^{\xi(j)} a(j_1)\Omega_m^{-1}(\zeta(r), \xi(j_1))}{1 - \sum_{j=\sigma(r-1)+1}^{r-1} a(j)}.$$

Then

$$\Lambda(2r_k) = \frac{\mu^2 \sum_{j=2r_k-2}^{2r_k-1} \sum_{j_1=2r_k}^{\xi(j)} \Omega_3^{-1}(2r_k+3, \xi(j_1))}{1 - \sum_{j=2r_k-2}^{2r_k-1} a(j)} + \sum_{j=2r_k}^{2r_k+2} a(j)\Omega_3^{-1}\left(2r_k+3, \xi(j)\right).$$

By using Maple, we obtain

$$\Lambda(2n_k) = \frac{1}{1-2\,\mu}\left(\mu^2 + \frac{2\mu^2}{1-\mu\,W(\mu)}\right) + \frac{\mu}{1-\mu\,W(\mu)} + \mu + \frac{\mu}{(1-\mu\,W(\mu))^2\left(1 - \frac{\mu\,W(\mu)}{1-\frac{\mu}{1-\mu}}\right)},$$

where $W(\mu)$ is defined as above. Consequently,

$$\limsup_{r\to\infty}\left(D(r,3) + \sum_{j=r}^{\zeta(r)} a(j)\Omega_3^{-1}\left(\zeta(r), \xi(j)\right)\right) = \lim_{k\to\infty} \Lambda(2r_k) > 1$$

for all $\mu \in [0.1705,\ 0.23]$. Therefore, condition (2.18) with $m = 3$ is satisfied, and hence Eq. (3.1) is oscillatory for all $\mu \in [0.1705,\ 0.23]$.

Example 3.2. Consider the equation

$$\nabla u(r) - a(r)u(\xi(r)) = 0, \quad r \in \mathbb{N}, \tag{3.2}$$

where

$$a(r) = \begin{cases} 0.177 & \text{if } r \in \{3r_k, 3r_k+1, \ldots, 3r_k+16\}, \\ 0.0001 & \text{otherwise}, \end{cases} \quad k \in \mathbb{N},$$

where $r_k \in \mathbb{N}$, $3r_{k+1} > 3r_k + 18$ and

$$\xi(r) = \begin{cases} r+1 & \text{if } r = 3k, \\ r+3 & \text{if } r = 3k+1, \\ r+1 & \text{if } r = 3k+2, \end{cases} \quad k \in \mathbb{N}.$$

As before, (1.3) and (1.11) lead respectively to

$$\rho(r) = \begin{cases} r+1 & \text{if } r = 3k, \\ r+2 & \text{if } r = 3k+1, \\ r+1 & \text{if } r = 3k+2, \end{cases} \quad k \in \mathbb{N},$$

and

$$\sigma(r) = \begin{cases} r-1 & \text{if } r = 3k, \\ r-1 & \text{if } r = 3k+1, \\ r-2 & \text{if } r = 3k+2, \end{cases} \quad k \in \mathbb{N}.$$

Let $\zeta(r) = \rho(r)$ (that is, defined by (1.3)) and

$$I(r) = \frac{\sum_{j=\sigma(r-1)+1}^{r-1} a(j) \sum_{j_1=r}^{\xi(j)} a(j_1) \prod_{j_2=\rho(j_1)+1}^{\xi(j_1)} \frac{1}{1-a(j_2)S_2(j_2)}}{1 - \sum_{j=\sigma(r-1)+1}^{r-1} a(j)} + \sum_{j=r}^{\rho(r)} a(j) \prod_{j_1=\rho(j_1)+1}^{\xi(j)} \frac{1}{1-a(j_1)S_2(j_1)},$$

where

$$S_2(r) = \frac{\prod_{j=\zeta(r)+1}^{\xi(r)} \frac{1}{1-a(j)S_1(j)}}{1 - G_2(r)},$$

and

$$G_2(r) = \sum_{j_1=r+1}^{\zeta(r)} a(j_1) + \frac{1}{1-G_1(\zeta(r))} \sum_{j_1=r+1}^{\zeta(r)} a(j_1) \sum_{j_2=\zeta(r)+1}^{\xi(j_1)} a(j_2) \prod_{j_3=\zeta^2(r)+1}^{\xi(j_2)} \frac{1}{1-a(j_3)S_1(j_3)}.$$

Then

$$I(3r_k+1) = \frac{\sum_{j=\sigma(3r_k)+1}^{3r_k} 0.177 \sum_{j_1=3r_k+1}^{\xi(j)} 0.177 \prod_{j_2=\rho(j_1)+1}^{\xi(j_2)} \frac{1}{1-0.177S_2(j_2)}}{1-\sum_{j_1=\sigma(3n_k)+1}^{3n_k} 0.177} + \sum_{j=3r_k+1}^{\rho(3r_k+1)} 0.177 \prod_{j_1=\rho(3r_k+1)+1}^{\xi(j)} \frac{1}{1-0.177S_2(j_1)}.$$

Consequently, $I(3r_k+1) = 1.001595294 > 1$, and so $\lim_{k\to\infty} I(3r_k+1) > 1$, it follows that condition (2.16) with $m=2$ is satisfied, and hence Eq. (3.2) oscillates.

However, as we will show, many previous oscillation conditions fail to do so. It is clear that

$$0.0001 \leq \liminf_{r\to\infty} \sum_{j=r+1}^{\xi(r)} a(j) \leq \lim_{k\to\infty} \sum_{j=3r_k+18}^{\xi(3r_k+17)} a(j) = \lim_{k\to\infty} \sum_{j=3r_k+18}^{3r_k+18} a(j) = 0.0001,$$

it follows that $\beta = \gamma = 0.0001$, $\lambda(\beta) = 1.010152720$ and $1 - A(\beta) > 0.9999$. Observe that

$$\sum_{j=3r_k+1}^{\rho(3r_k+1)} a(j)\Omega_4^{-1}\left(\rho(3r_k+1), \xi(j)\right) = 0.177 \sum_{j=3r_k+1}^{3r_k+3} \Omega_4^{-1}\left(3r_k+3, \xi(i)\right) < 0.9349.$$

Consequently,

$$\limsup_{r\to\infty} \sum_{j=r}^{\rho(r)} a(j)\Omega_4^{-1}\left(\rho(r), \xi(j)\right) = 0.177 \lim_{k\to\infty} \sum_{j=3r_k+1}^{\rho(3r_k+1)} \Omega_4^{-1}\left(\rho(3r_k+1), \xi(j)\right) < 0.9349 < 1 - A(\beta).$$

That is, condition (1.6) with $m=4$ can not be applied. Finally, observe that

$$\sum_{j=3r_k+1}^{\rho(3r_k+1)} a(j) \exp\left(\sum_{j_1=\rho(3r_k+1)+1}^{\xi(j)} a(j_1) \prod_{j_2=j_1+1}^{\xi(j_1)} \frac{1}{1-Q_1(j_2)} \right) < 0.9165785 < 1 - A(\beta),$$

$$\sum_{j=3r_k+1}^{\rho(3r_k+1)} a(j) \exp\left(\sum_{j_1=\rho(3r_k+1)+1}^{\xi(j)} a(j_1) \exp\left(\sum_{j_2=j_1+1}^{\xi(j_1)} a(j_2) \prod_{j_3=j_2+1}^{\xi(j_2)} \frac{1}{1-F_1(j_3)} \right) \right) < 0.99490 < 1 - A(\beta),$$

and

$$\frac{W_m(3r_k+1)}{1-\sum_{j=\sigma(3r_k)+1}^{3r_k} a(j)} + \sum_{j=3r_k+1}^{\rho(3r_k+1)} a(j) \prod_{j_1=\rho(3r_k+1)+1}^{\xi(j)} \frac{1}{1-a(j_1)Z_1(j_1, \xi(j_1))} < 0.835 < 1.$$

Then

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\exp\left(\sum_{j_1=\rho(r)+1}^{\xi(j_1)} a(j_1)\prod_{j_2=j_1+1}^{\xi(j_1)}\frac{1}{1-Q_1(j_2)}\right) < 1-A(\beta),$$

$$\limsup_{r\to\infty}\sum_{j=r}^{\rho(r)} a(j)\exp\left(\sum_{j_1=\rho(r)+1}^{\xi(j)} a(j_1)\right.$$
$$\left.\exp\left(\sum_{j_2=j_1+1}^{\xi(j_1)} a(j_2)\prod_{j_3=j_2+1}^{\xi(j_2)}\frac{1}{1-F_1(j_3)}\right)\right) < 1-A(\beta),$$

and

$$\limsup_{r\to\infty}\left(\frac{W_m(r)}{1-\sum_{j=\sigma(r-1)+1}^{r-1} a(j)}\right.$$
$$\left.+\sum_{j=r}^{\rho(r)} a(j)\prod_{j_1=\rho(r)+1}^{\xi(j)}\frac{1}{1-a(j_1)Z_1(j_1,\xi(j_1))}\right) < 0.836 < 1.$$

Therefore none of the conditions (1.8), (1.9) and (1.10) with $m=1$ can be applied to Eq. (3.2).

4. CONCLUSION

In this work we studied the oscillation criteria for first-order difference equations with deviating arguments. We obtained new oscillation criteria that improve many previous ones. The application as well as the strength of some of our results have been shown using two examples. Some of our results could support the development of the oscillation theory for difference equations with deviating arguments, for example, Lemmas 2.4 and 2.11. Using the techniques given in this paper, several new oscillation criteria for first-order difference equations with several nonmonotone deviating arguments, as well as delay difference equations with oscillating coefficients, can be obtained.

Acknowledgements

This publication was supported by the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.

REFERENCES

[1] P.G. Asteris, G.E. Chatzarakis, *Oscillation tests for difference equations with non-monotone arguments*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **24** (2017), no. 4, 287–302.

[2] E.R. Attia, *Oscillation tests for first-order linear differential equations with non-monotone delays*, Adv. Differ. Equ. **2021** (2021), Article no. 41.

[3] E.R. Attia, G.E. Chatzarakis, *Oscillation tests for difference equations with non-monotone retarded arguments*, Appl. Math. Lett. **123** (2022), 107551.

[4] E.R. Attia, G.E. Chatzarakis, *Iterative oscillation criteria for first-order difference equations with non-monotone advanced arguments*, J. Appl. Math. Comput. (2021), 1–17.

[5] E. Braverman, G.E. Chatzarakis, I.P. Stavroulakis, *Iterative oscillation tests for difference equations with several non-monotone arguments*, J. Difference Equ. Appl. **21** (2015), 854–874.

[6] E. Braverman, B. Karpuz, *On oscillation of differential and difference equations with non-monotone delays*, Appl. Math. Comput. **218** (2011), 3880–3887.

[7] G.E. Chatzarakis, *Sufficient oscillation conditions for deviating difference equations*, Filomat **33** (2019), 3291–3305.

[8] G.E. Chatzarakis, I. Jadlovská, *Oscillations in deviating difference equations using an iterative technique*, J. Inequal. Appl. **173** (2017), 1–24.

[9] G.E. Chatzarakis, I. Jadlovská, *Improved iterative oscillation tests for first-order deviating difference equations*, Int. J. Difference Equ. **12** (2017), 185–210.

[10] G.E. Chatzarakis, I. Jadlovská, *Oscillations of deviating difference equations using an iterative method*, Mediterr. J. Math. **16** (2019), 1–20.

[11] G.E. Chatzarakis, L. Shaikhet, *Oscillation criteria for difference equations with non-monotone arguments*, Adv. Differ. Equ. **62** (2017), 1–16.

[12] G.E. Chatzarakis, I.P. Stavroulakis, *Oscillation of difference equations with general advanced argument*, Cent. Eur. J. Math. **10** (2012), 807–823.

[13] G.E. Chatzarakis, S.R. Grace, I. Jadlovská, *Oscillation tests for linear difference equations with non-monotone arguments*, Tatra Mt. Math. Publ. **79** (2021), 81–100.

[14] G.E. Chatzarakis, R. Koplatadze, I.P. Stavroulakis, *Oscillation criteria of first order linear difference equations with delay argument*, Nonlinear Anal. **68** (2008), 994–1005.

[15] G.E. Chatzarakis, R. Koplatadze, I.P. Stavroulakis, *Optimal oscillation criteria of first order linear difference equations with delay argument*, Pacific J. Math. **235** (2008), 15–33.

[16] G.E. Chatzarakis, Ch.G. Philos, I.P. Stavroulakis, *An oscillation criterion for linear difference equations with general delay argument*, Port. Math. **66** (2009), 513–533.

[17] G.E. Chatzarakis, I.K. Purnaras, I.P. Stavroulakis, *Oscillation of retarded difference equations with a non-monotone argument*, J. Difference Equ. Appl. **23** (2017), 1354–1377.

[18] L.H. Erbe, Q.K. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[19] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[20] B. Karpuz, *Sharp oscillation and nonoscillation tests for linear difference equations*, J. Difference Equ. Appl. **23** (2017), 1929–1942.

[21] J. Shen, I.P. Stavroulakis, *Oscillation criteria for delay difference equations*, Electron. J. Differential Equations **2001** (2001), 1–15.

[22] I.P. Stavroulakis, *Oscillations of delay difference equations*, Comput. Math. Appl. **29** (1995), 83–88.

[23] I.P. Stavroulakis, *Oscillation criteria for first order delay difference equations*, Mediterr. J. Math. **1** (2004), 231–240.

[24] I.P. Stavroulakis, *Oscillation criteria for delay and difference equations with non-monotone arguments*, Appl. Math. Comput. **226** (2014), 661–672.

[25] B.G. Zhang, C.J. Tian, *Nonexistence and existence of positive solutions for difference equations with unbounded delay*, Comput. Math. Appl. **36** (1998), 1–8.

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Received: November 25, 2021.
Revised: March 28, 2022.
Accepted: March 31, 2022.