

## A NOTE ON ON-LINE RAMSEY NUMBERS FOR QUADRILATERALS

Joanna Cyman and Tomasz Dzido

*Communicated by Ingo Schiermeyer*

**Abstract.** We consider on-line Ramsey numbers defined by a game played between two players, Builder and Painter. In each round Builder draws an the edge and Painter colors it either red or blue, as it appears. Builder's goal is to force Painter to create a monochromatic copy of a fixed graph  $H$  in as few rounds as possible. The minimum number of rounds (assuming both players play perfectly) is the *on-line Ramsey number*  $\tilde{r}(H)$  of the graph  $H$ . An asymmetric version of the on-line Ramsey numbers  $\tilde{r}(G, H)$  is defined accordingly. In 2005, Kurek and Ruciński computed  $\tilde{r}(C_3)$ . In this paper, we compute  $\tilde{r}(C_4, C_k)$  for  $3 \leq k \leq 7$ . Most of the results are based on computer algorithms but we obtain the exact value  $\tilde{r}(C_4)$  and do so without the help of computer algorithms.

**Keywords:** Ramsey theory, on-line games.

**Mathematics Subject Classification:** 05C55, 05C57.

### 1. INTRODUCTION

All graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges. Let  $H$  be such a graph.

The following definitions and notations follow that of Grytczuk *et. al* ([4]) and Prałat ([5]).

On-line Ramsey numbers were introduced independently by Beck ([1]) and Kurek and Ruciński ([3]). They are best explained by the following game between Builder and Painter, played on a large set of vertices. In each round Builder joins two nonadjacent vertices by an edge and Painter colors it red or blue. Builder's goal is to force Painter to create a monochromatic copy of a fixed graph  $H$  in as few rounds as possible. Painter will try to resist doing it for as long as possible. The *on-line Ramsey number*  $\tilde{r}(H)$  of a graph  $H$  is the minimum number of rounds in which Builder achieves his goal, assuming both players play perfectly i.e. play as best as they can play at the

moment. An asymmetric version of the on-line Ramsey numbers  $\tilde{r}(G, H)$  is defined accordingly.

We also consider similar version of the on-line Ramsey numbers, when the Builder starts with an empty graph with exactly  $m$  vertices. The *generalized on-line Ramsey number*  $\tilde{r}_m(G, H)$  is defined as the minimum number of rounds in such a game if the Builder wins, otherwise  $\tilde{r}_m(G, H) = \infty$ . Note that, for any two graphs  $G$  and  $H$  and  $m, p \in \mathbb{N}, m < p$ ,

$$\tilde{r}_m(G, H) \geq \tilde{r}_p(G, H) \geq \tilde{r}(G, H), \tag{1.1}$$

since in the generalized version of the game the Builder has more restrictions to follow.

It is difficult to compute the exact value of  $\tilde{r}(G, H)$  unless  $G$  and  $H$  are trivial. In fact, very little has been discovered in the area of small on-line Ramsey numbers up to this point. Up to now, we know only one exact value of on-line Ramsey number for cycles. In the case of triangles  $C_3$ , it has been shown that  $\tilde{r}(C_3) = 8$  ([3]). In this paper we consider the case where  $G$  is a quadrilateral  $C_4$  (cycle of order 4) and  $H$  is a short cycle  $C_k$ , where  $k \in \{3, 4, 5, 6, 7\}$ . In Section 2 we describe the algorithms and computations performed. Section 3 provides a computer-free proof of  $\tilde{r}(C_4) = 10$ .

## 2. ALGORITHMS AND COMPUTATIONS

Clearly,  $\tilde{r}(G, H)$  is at most  $\binom{r(G, H)}{2}$  for every graph  $H$ , where  $r(G, H)$  is the classical Ramsey number (which is the least  $n$  such that there is a red copy of  $G$  or a blue copy of  $H$  in any red-blue coloring of the edges of  $K_n$ ).

In Table 1, we have collected all values of  $r(C_4, C_k)$ . Further detailed references to papers establishing specific values are listed in a regularly updated survey by Radziszowski ([6]).

**Table 1.** Known values of  $r(C_4, C_k)$

|               |   |   |   |          |
|---------------|---|---|---|----------|
| $k$           | 3 | 4 | 5 | $\geq 6$ |
| $r(C_4, C_k)$ | 7 | 6 | 7 | $k + 1$  |

In order to obtain a lower bound for a possible number of vertices  $m$  for a winning strategy by the Builder, one can fix a properly colored labeled copy of  $K_m$ , as follows.

1. For  $k = 3$  or  $k = 5$  let us consider the complete graph  $K_6$  containing two isolated red triangles, the remaining edges of this graph are blue.
2. For  $k = 4$  consider the complete graph on 5 vertices containing a red and a blue cycle  $C_5$ .
3. For the general case, consider the complete graph on  $m \leq k$  ( $k \geq 3$ ) vertices containing a red copy of  $S_{m-1}$  and all other edges are colored blue.

All described above complete graphs do not contain a red  $C_4$  and a blue  $C_k$ . Now every time the Builder presents an edge and the Painter follows the predetermined coloring. Thus the Painter can avoid a red  $C_4$  or a blue  $C_k$ , so we have the following observation.

**Observation 2.1.**

$$\widetilde{r}_m(C_4, C_k) = \infty, \text{ where } \begin{cases} m \leq 6 & \text{for } k = 3 \text{ or } k = 5, \\ m \leq 5 & \text{for } k = 4, \\ m \leq k & \text{for } k \geq 6. \end{cases}$$

Using Observation 2.1, inequalities (1.1) and computer algorithms, it is enough to show the exact values of the on-line Ramsey numbers as we present in Table 2.

**Table 2.** Values of  $\widetilde{r}(C_4, C_k)$

|                           |  |  |
|---------------------------|--|--|
| $k$                       | 3  | 4  |
|                           | $\widetilde{r}_m(C_4, C_3) = \infty, m \leq 6$ | $\widetilde{r}_m(C_4, C_4) = \infty, m \leq 5$ |
|                           | $\widetilde{r}_m(C_4, C_3) = 11, m \geq 7$     | $\widetilde{r}_m(C_4, C_4) = 10, m \geq 6$     |
| $\widetilde{r}(C_4, C_k)$ | 11   | 10   |

|                           |  |  |
|---------------------------|--|--|
| $k$                       | 5  | 6  |
|                           | $\widetilde{r}_m(C_4, C_5) = \infty, m \leq 6$ | $\widetilde{r}_m(C_4, C_6) = \infty, m \leq 6$ |
|                           | $\widetilde{r}_m(C_4, C_5) = 12, m \geq 7$     | $\widetilde{r}_m(C_4, C_6) = 12, m \geq 7$     |
| $\widetilde{r}(C_4, C_k)$ | 12   | 12   |

|                           |  |
|---------------------------|--|
| $k$                       | 7  |
|                           | $\widetilde{r}_m(C_4, C_7) = \infty, m \leq 7$ |
|                           | $\widetilde{r}_m(C_4, C_7) = 13, m \geq 8$     |
| $\widetilde{r}(C_4, C_k)$ | 13   |

The algorithm we use is based on a very popular method, the backtracking method. We wrote the program in C similar to the one written by Pralat ([5]). The program was tested by using it to calculate known values  $\widetilde{r}_m(C_3) = 8$  and  $\widetilde{r}_m(C_4) = 10$  for  $m \geq 6$  (see the proof of this value in Section 3) and it agreed. In addition, two separate modifications of the algorithm were prepared by the two authors, the results compared, and no discrepancies were found. We were able to run the program starting from two different initial graphs consisting of one edge colored with a red and blue color.

Now we present the following upper bound.

**Theorem 2.2.** *Let  $k \geq 3$ . Then  $\widetilde{r}(C_4, C_k) \leq 4k + 11$ .*

*Proof.* Since  $\widetilde{r}(C_4, P_k) \leq 4k - 4$  ([2]), then after  $4k - 4$  rounds we certainly obtain a blue path on  $k$  vertices  $P_k = (v_1, v_2, \dots, v_k)$ . Then Builder draws the edges from vertex  $v_i$  for  $i = 1$  and for  $i = k - 1$  to five new vertices  $w_1, w_2, \dots, w_5$ . To avoid a blue  $C_k$ , there are at least three red edges from vertex  $v_i$  for some  $i \in \{1, k - 1\}$  to vertices  $w_1, w_2, \dots, w_5$ . Assume that Painter colors the edges  $\{v_i, w_1\}$ ,  $\{v_i, w_2\}$  and  $\{v_i, w_3\}$  red. Then Builder draws the edges from vertex  $v_k$  to  $w_1, w_2$  and  $w_3$ . To avoid a red  $C_4$ , Painter colors at least two of them, say  $\{v_k, w_1\}$ ,  $\{v_k, w_2\}$ , blue. In the last step, Builder introduces 2 new edges  $\{v_2, w_1\}$  and  $\{v_2, w_2\}$ , forcing Painter to create a red  $C_4$  or a blue  $C_k$ . □

### 3. COMPUTER-FREE PROOF OF $\tilde{R}(C_4) = 10$

Now, we present the exact computer-free determination of the on-line Ramsey number for quadrilaterals.

**Theorem 3.1.**  $\tilde{r}(C_4) = 10$ .

*Proof.* For the upper bound  $\tilde{r}(C_4) \leq 10$ , let us consider the following strategy. Builder first draws 4 edges originating from a single vertex  $v_0$  to vertices from the set  $V_1 = \{v_1, v_2, v_3, v_4\}$  of the neighbors of  $v_0$ . We have to consider 3 possible cases depending on the coloring of these edges produced by Painter. We prove that in at most 6 consecutive moves Builder will force Painter to create a monochromatic  $C_4$ .

*Case 1.* All edges  $\{v_0, v_i\}$ , where  $1 \leq i \leq 4$ , are monochromatic.

Without loss of generality we can assume that all edges of a star are red. Since  $R(P_3, C_4) = 4$  and  $\binom{4}{2} = 6$ , then Builder can force Painter to draw either a red  $P_3$  or a blue  $C_4$  in at most 6 moves.

*Case 2.* The edges  $\{v_0, v_i\}$ , where  $1 \leq i \leq 3$  are blue and  $\{v_0, v_4\}$  is red.

Builder draws the edges from vertices  $v_1, v_2, v_3$ , to a new vertex  $v_5$ . To avoid a blue  $C_4$ , Painter cannot color two of these edges blue. So, we have two subcases.

*Subcase 2.1.* All edges  $\{v_i, v_5\}$ , where  $1 \leq i \leq 3$  are red.

The way for Builder to finish the game is to create a monochromatic  $C_4$  in the next three rounds. This is possible by using the edges  $\{v_i, v_4\}$ , where  $i \in \{1, 2, 3\}$ . At least two of them have the same color and we are done.

*Subcase 2.2.* The edges  $\{v_i, v_5\}$  for  $i \in \{2, 3\}$  are red and  $\{v_1, v_5\}$  is blue.

Builder draws an edge from vertex  $v_0$  to  $v_5$ . Let us consider the possibility of Painter playing this edge. If Painter uses a blue color, then Builder draws the edges  $\{v_1, v_2\}$  and  $\{v_1, v_3\}$ . To avoid a blue  $C_4$ , Painter will color these edges red and she is forced to create a red  $C_4$  on vertices  $\{v_1, v_2, v_5, v_3\}$ . It means that Painter must color the edge  $\{v_0, v_5\}$  red. Then Builder draws the edges  $\{v_2, v_4\}$  and  $\{v_3, v_4\}$ . In this situation, to avoid a red  $C_4$ , Painter will color these edges blue and she is forced to create a blue  $C_4$  on vertices  $\{v_0, v_2, v_4, v_3\}$ .

*Case 3.* The edges  $\{v_0, v_1\}$ ,  $\{v_0, v_2\}$  are blue and  $\{v_0, v_3\}$ ,  $\{v_0, v_4\}$  are red.

Builder draws an edge from vertex  $v_0$  to a new vertex  $v_5$ . Without loss of generality, we may assume that Painter colors the edge  $\{v_0, v_5\}$  red. Then Builder draws the edges  $\{v_1, v_i\}$ , where  $i \in \{3, 4, 5\}$ . To avoid a red  $C_4$ , Painter colors at least two of them, say  $\{v_1, v_3\}$ ,  $\{v_1, v_4\}$ , blue. In the last two steps, Builder introduces 2 new edges  $\{v_2, v_3\}$  and  $\{v_2, v_4\}$ , forcing Painter to create a monochromatic  $C_4$ .

Notice that the total number of edges drawn by Builder in all above cases is 10, so we have  $\tilde{r}(C_4) \leq 10$ .

In order to show that  $\tilde{r}(C_4) > 9$ , we will describe the strategy of Painter to be followed for the first eight moves which guarantees that the ninth edge is safe. Consider the following strategy for Painter: color an edge blue if it does not create a blue copy of  $C_4$  or a blue copy of one of the graphs shown in Figure 1, otherwise use the red color.

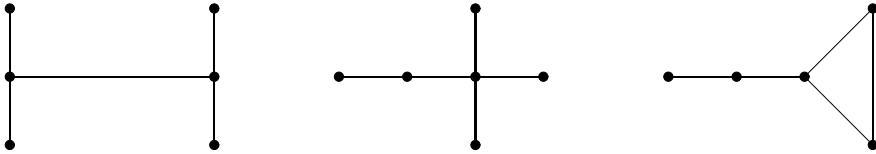


Fig. 1. Undesired blue subgraphs

Note that Painter is forced to complete a monochromatic  $C_4$  if and only if the current edge to be colored is the additional edge drawn by a dotted line of one among the graphs shown in Figure 2. Such graphs, where edges drawn by a dotted line are non-existent edges, will be called *fatal*.

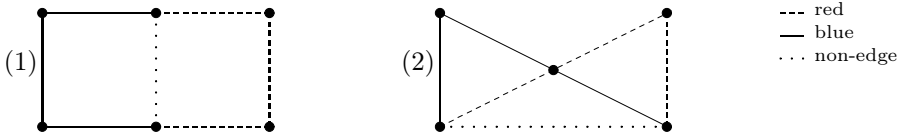


Fig. 2. Fatal graphs

For as long as she can, Painter will try to avoid creating a fatal graph. Now, let us see that the strategy as discribed is successful. We claim that Painter can avoid making a fatal graph in 8 moves. Suppose to the contrary, and consider the situation when Painter was forced to create a fatal graph in the eighth move. How come a fatal graph has been created? All fatal graphs have exactly 6 edges, therefore Builder, except for edges of fatal graph, can use only two additional edges.

We have to consider two cases with respect to all possible fatal graphs.

*Case A.* Painter was forced to create a copy of the first fatal graph.

It is possible in the eighth move if there were at most 2 additional rounds in which Painter colored edges blue. These rounds were necessary to force red edges and were possible in 3 situations as shown below in Figure 3.

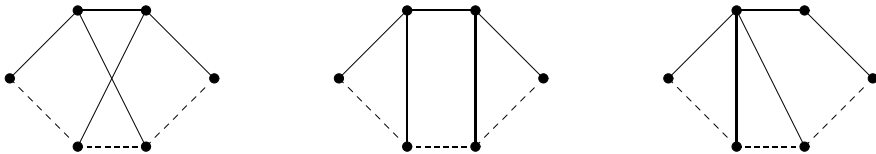
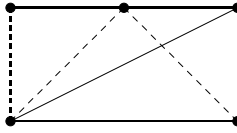


Fig. 3. Possible situations in case A

All these options lead us to one of the graphs shown in Figure 1 which are impossible according to the strategy chosen by Painter.

*Case B.* Painter was forced to create a copy of the second fatal graph.

Similarly to the previous case, it is possible in the situation as shown in Figure 4.



**Fig. 4.** Possible situation in case B

We also obtain one of undesired graphs which is impossible and we are done. The two above inequalities  $\tilde{r}(C_4) \leq 10$  and  $\tilde{r}(C_4) > 9$  finish the proof.  $\square$

## REFERENCES

- [1] J. Beck, *Achievement games and the probabilistic method* [in:] *Combinatorics, Paul Erdős is Eighty*, Bolyai Soc. Math. Stud. **1** (1993), 51–78.
- [2] J. Cyman, T. Dzido, J. Lapinskas, A. Lo, *On-line Ramsey numbers of paths and cycles*, submitted.
- [3] A. Kurek, A. Ruciński, *Two variants of the size Ramsey number*, *Discussiones Mathematicae Graph Theory* **25** (2005), 141–149.
- [4] J.A. Grytczuk, H.A. Kierstead, P. Prałat, *On-line Ramsey numbers for paths and stars*, *Discrete Mathematics and Theoretical Computer Science* **10** (2008), 63–74.
- [5] P. Prałat,  $\tilde{r}(3, 4) = 17$ , *Electronic Journal of Combinatorics* **15** (2008) #R67.
- [6] S.P. Radziszowski, *Small Ramsey Numbers*, *Electronic Journal of Combinatorics, Dynamic Survey 1*, <http://www.combinatorics.org>.

Joanna Cyman  
joana@mif.pg.gda.pl

Gdańsk University of Technology  
Department of Technical Physics and Applied Mathematics  
Narutowicza 11/12, 80-952 Gdańsk, Poland

Tomasz Dzido  
tdz@inf.ug.edu.pl

University of Gdańsk  
Institute of Informatics  
Wita Stwosza 57, 80-952 Gdańsk, Poland

*Received: July 18, 2013.*

*Revised: February 2, 2014.*

*Accepted: February 13, 2014.*