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# COUPLED LATERAL-TORSIONAL VIBRATIONS OF A SYMMETRIC ROTOR

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***Abstract***

*The paper presents a cross-sectional analysis of dynamics of rotating systems with different characteristics, starting from the search for solutions in an analytical way, and ending with the numerical attempt to verify the results. There is presented a problem of bending and torsional vibrations in rotational systems, using possibly the simplest model of the phenomenon. It is a heavy disk embedded on the deformable massless shaft. The rotor is symmetric and it is assumed that stiffness can be nonlinear. The analysis of vibrations occurring in rotors is made with the use of this model. This is followed by a preliminary discussion on the problem and the conclusions of the pilot studies. There are presented basic problems related to the phenomenon of lateral-torsional coupling in rotational systems with an emphasis on critical states. Formulation of the problem is the conclusion of preliminary part of the paper. Next, there are discussed the selected methods currently used to solve such problems, together with a discussion of their application in this case. In addition, there is discussed a problem of numerical or analytical character of obtained results and the range of using methods in modelling, designing and experimental studies. The next part of the paper focuses on the solution of the model presented in the introduction, using the following analytical or numerical methods: Krylov-Bogolubov, Adams and Runge-Kutta methods. There are presented assumptions, simplifications of the model, technical details of searching for solutions and the results together with a discussion of their correctness. The results, which were obtained from analysis, are polyharmonic. Due to this fact, the spectra of each solution were presented. These aspects are discussed in the context of the solving methods and motion analysis for rotor systems. The whole paper is summarized with a discussion of the results and their confrontation with reality.*

**Keywords:** *rotational system, nonlinear differential system of equation, perturbation method, critical states, analytical solutions*

## 1. Introduction

The importance of nonlinear problems in the entire engineering and machine construction is increasing. Contemporary industry rarely allows for “constructional compromises” in the form of high values of safety factor or simplified design computation process, which lead to oversizing of a designed product. Complex computing models are used increasingly more often. The aim of this action is to create light, economical objects, which fulfil constructional assumptions in the way possibly nearest to the limitations. The products, which are designed in a way that the operation time exceeds the assumed durability, are inappropriate among the contemporary trends. In many cases, they will be classified as incorrect. At this point, it is important to mention the example of big aircraft constructions, where the decrease of the mass of fuel together with its operation or crack propagation in sheathing and airfoil are taken into account. What is more important, the ready product is allowed for a flight with cracks initiation (if only its length is not smaller than the critical length!). Moreover, there is so much fuel that it is only enough to reach the flight destination [8-10].

Thus, the linear models of many phenomena, in many cases, become too less accurate although there is a clear transparency and a relatively easy way of seeking the solution. A considerable group of constructional problems requires considering the nonlinear aspects, although their influence on the system is not big [1].

That is why; using complex computing models (including nonlinear ones) becomes a norm. Nowadays, seeking solutions of such problems has considerably simplified because of intensive development of computer methods. Many implementations of various algorithms of seeking numerical solutions were developed. A perfect example of this is Finite element method, which in its various applications enables skipping the stage of mathematical modelling. The main problem of solutions found with numerical methods is their "numerical" form. The array of numbers (usually a big and multidimensional table) is the effect of computations. This array represents only one particular solution with the concrete parameters of the system. It means that any constructional change based on the results of the model will require its modification and recalculations. However, it is not certain that the results will be "better" (in a previously defined sense). It is not certain that the change of a given parameter will influence the solution. The results are very often obtained numerically and they are very illegible. What is more, defining the parameters, which are crucial for a discussed problem, requires many iterations. A vast number of people will claim, "there is a method in this craziness". If the equations defining complicated mechanical problem (including nonlinear problems) do not have closed solutions, they must be sought numerically. Many people will be supporting the idea that it is a waste of time to analyse complex equations. They justify it with the fact that the numerical solution can be obtained without arduous transformations on illegible formulas [4-6].

At this point, the following question arises: Does the time spent on seeking the numerical solution does not exceed the time necessary to find the analytical solution (e.g. asymptotic or approximate)? Having proper skills and appropriate preparation (environments of symbolic computations appropriately programmed), it is possible to analyse complex systems of equation. What is more, the obtained results have the form of compact formulas Such a solution makes it possible to scale the results for various values of parameters and to assess their importance. In this context, the approximate result (analytical) is better than the precise solution (numerical). Obviously, it is not appropriate to marginalize the meaning of numerical methods. They are a perfect method of checking the accuracy of computations. Moreover, they serve as a perfect tool for finding precise values of sought solutions for previously defined parameters of a product. It is important to be aware of the fact that a complete negligence of using analytical methods for numerical ones in the designing process is not advisable [7, 11].

## 2. Model

The fundamental problem is the choice of the model for the purposes of a given analysis. Unfortunately, in most cases it does not mean that this problem is simple and unequivocal. The model can be treated as a kind of representation of the reality in the language of a given domain (mechanics, electrotechnology, thermodynamics, etc.) That is why; its form depends on the shape of problems. Additionally, the assumed level of accuracy has a big influence on the final form of the model. The level of accuracy can be defined as a collection of phenomena, which were considered to be vital in the modelling process [3].

Obviously, the increase of model accuracy does not lead to proportional increase of accuracy of the results. What is more, it happens that the time spent on improving the model and finding its solutions does not correspond to obtained results. That is why, it is necessary to be careful during this process so as not to start describing the reality too precisely. In this case finding the precise model and its solution could be the problem.

In most cases of the analysis of different type of vibrations, practice shows that it is worth using method of successive approximations. In the beginning, simple models are used so that together with the successive solving success the discussed problem is made more detailed. Such an approach has two basic advantages:

- it allows to check which of the phenomena existing in a given system are basic, and which of them are of less importance,
- it allows for seeking the solutions for a given problem on the given level of accuracy because of the familiarity with the solution of the previous and simplified problem.

Thus, in case of the analysis of bending and torsional oscillations existing in rotors, it is worth to start the discussion with the simplest model in the form presented in Fig. 1.

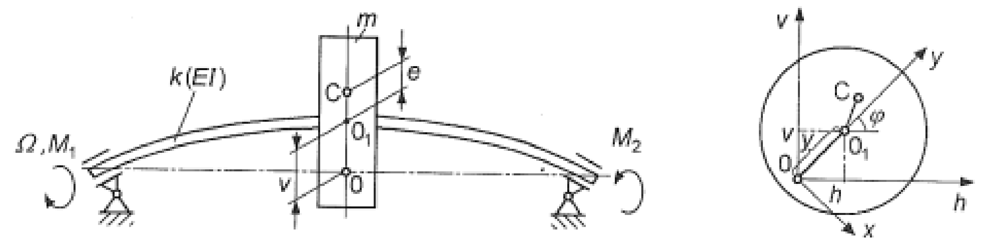


*Fig. 1. Model of the shaft*

This system is represented by the inert disc with a mass of $m$ and a moment of inertia $I$ fixed eccentrically on the elastic shaft within $e$ from the rotation axis. The system presented in figure (1) (physical model) can be described as second order nonlinear system of differential equation (mathematical model) in the form of:

$$m \cdot \ddot{h} - m \cdot e \cdot \sin\varphi \cdot \ddot{\varphi} - m \cdot e \cdot \cos\varphi \cdot \dot{\varphi}^2 + k(h) \cdot h = 0, \tag{1}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos\varphi \cdot \ddot{\varphi} - m \cdot e \cdot \sin\varphi \cdot \dot{\varphi}^2 + k(v) \cdot v = 0, \tag{2}$$

$$(I + m \cdot e^2) \cdot \ddot{\varphi} - m \cdot e \cdot \ddot{h} \cdot \sin\varphi + m \cdot e \cdot \ddot{v} \cdot \cos\varphi = \Delta M(t). \tag{3}$$

where:
$h(t)$ – horizontal displacement of shaft axis at the point of disc mounting,
$v(t)$ – vertical displacement of shaft axis at the point of disc mounting,
$\varphi(t)$ – angular displacement of (rotation) shaft.

Finding the equation system (1-3) is possible with a direct implementation of the second Newton's law of dynamics or any formalism of analytical mechanics e.g. Lagrange equations of II kind or D'Alembert's principle.

Before solving the dynamic system (1-3) it is worth to think about possible simplifications. Presentation of angular shaft displacement in the following form $\varphi(t)$ is justified because the vibrations in rotor systems should be treated as side effects:

$$\varphi(t) = \psi(t) + \theta(t), \tag{4}$$

where:
$\psi(t)$ – basic motion,
$\theta(t)$ – motion disturbance.

Linear-elastic model of deformable crankshaft.

Particular elements of expression (4) can be interpreted as follows. Function $\psi(t)$ presents the general form of rotor motion and it should be treated as unknown (expected motion of rotor). Whereas mapping $\psi(t)$ represents the deviation of rotor motion from the expected value $\psi(t)$.

Furthermore, it was assumed that the function of disturbance is periodic. However, this assumption does not have to be obligatory. That is why; the function $\theta(t)$ presents the torsional vibrations of shaft. These vibrations are at the same time the disturbances of the basic motion $\psi(t)$.

It is unacceptable that the values of disturbance and its velocity were too high. It is connected with the operation. The system where the changes of velocity would be big, in most technical cases would not be able to fulfill the assumed expectations. That is why, oscillation of angular and velocity displacement of shaft should fulfill the following conditions:

$$\theta_0 \le \theta_{\text{dop}} = \frac{\pi}{m_{\text{dop}}}, \tag{5}$$

$$\Omega_\theta \le \kappa_{\text{dop}} \cdot \Omega, \tag{6}$$

where:

$\theta_{\text{dop}}$ – maximum acceptable amplitude of disturbance of angular displacement,
$m_{\text{dop}}$ – multiplicity of angle $\theta_{\text{dop}}$ in angle $\pi$, $m_{\text{dop}} = \pi/\theta_{\text{dop}}$,
$\kappa_{\text{dop}}$ – maximum acceptable quotient of amplitude of disturbance velocity and angular velocity of the shaft,
$\theta_0 = \sup|\theta(t)|$ – maximum amplitude of disturbance of angular displacement,
$\Omega = \sup|\dot{\psi}(t)|$ – maximum velocity of basic motion of the shaft,
$\Omega_0 = \sup|\dot{\theta}(t)|$ – maximum amplitude of velocity disturbance of angular displacement.

In addition, it can be assumed that the periodic function $\theta(t)$ fulfills the Dirichlet's conditions. That means it can be expanded to Fourier series. All mechanical systems fulfil this assumption, and only some untypical cases, which do not qualify for this group, are beyond the area of objects, which are the subject of these considerations. No matter, what the character of function is $\psi(t)$ and $\theta(t)$, angular velocity and acceleration in the analyzed case are as follows:

$$\dot{\varphi}(t) = \dot{\psi}(t) + \dot{\theta}(t), \tag{7}$$

$$\ddot{\varphi}(t) = \ddot{\psi}(t) + \ddot{\theta}(t). \tag{8}$$

Using the assumption of small vibrations, (formula (5)) further simplifications can be done. The expressions including terms proportional to $\theta(t)$ (angular displacement) can be neglected. It gives governing equations as follows:

$$m \cdot \ddot{h} - m \cdot e \cdot \sin\psi(t) \cdot \left(\ddot{\psi}(t) + \ddot{\theta}(t)\right) - m \cdot e \cdot \cos\psi(t) \cdot \dot{\psi}^2(t) + k(h) \cdot h = 0, \tag{9}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos\psi(t) \cdot \left(\ddot{\psi}(t) + \ddot{\theta}(t)\right) - m \cdot e \cdot \sin\psi(t) \cdot \dot{\psi}^2(t) + k(v) \cdot v = 0, \tag{10}$$

$$(I + m \cdot e^2) \cdot \left(\ddot{\psi}(t) + \ddot{\theta}(t)\right) - m \cdot e \cdot \ddot{h} \cdot \sin\psi(t) + m \cdot e \cdot \ddot{v} \cdot \cos\psi(t) = \Delta M(t). \tag{11}$$

## 3. Solution

The solution of the preliminary problem for system of equations (9-11) is possible with the use of a few methods. The methods can be divided into two groups:

- analytical methods,
- numerical methods.

Benefits and problems resulting from using the above methods were generally explained in the introduction. The problem of choice of one of the group comes down to answers to the following questions: What kind of solution is necessary for a given problem? What could be the results used for? In most design processes using closed formulas has more benefits than using numerical data. Therefore, using numerical methods is less important in relation to the analytical approach.

In the group of analytical methods, the following division can be made:
- precise methods,
- approximate methods:
- perturbation (e.g. Krylov–Bogolubov,)
- variational (e.g. Galerkin).

Unfortunately, the precise methods should be excluded from the methods, which are possible to use in the engineering. It should be done because of considerable difficulties connected with algorithmisation of the process of solving the problem. That is why Krylov–Bogolubov method was proposed to be used. This method is one of the perturbation methods (methods of small parameter) and it is based on the assumption of a special form of functions, which are expected to be a desired solution. This method assumes that [1]:

$$x_i(t) = a_i(t) \cdot \cos(\psi_i(t)) + \sum_{k=1}^{\infty} \varepsilon^k \cdot X_{i,k}, \tag{12}$$

$$\dot{a}_i(t) = \sum_{k=1}^{\infty} \varepsilon^k \cdot A_{i,k}, \tag{13}$$

$$\dot{\psi}_i(t) = \omega_{0i} + \sum_{k=1}^{\infty} \varepsilon^k \cdot B_{i,k}, \tag{14}$$

where:
$\varepsilon$ – small parameter,
$a_i(t)$ – instantaneous amplitude,
$\psi_i(t)$ – instantaneous phase,
$X_{i,k}$ – function of approximation of $k$ order of expected solution,
$A_{i,k}$ – function of approximation of $k$ order of instantaneous amplitude.
$B_{i,k}$ – function of approximation of $k$ order of instantaneous frequency.

While substituting dependencies (12-14) into system of equation (9-11) the recurrence system of equation is obtained. Based on this system it is possible to appoint unknown quantity, and as a result finding the approximate solution in the form of formula (12).

The applied method occurred to be quite complex in the perspective of calculus. That is why, the software application − Computer Algebra System, which performs symbolic computation, was used. Software Maxima was used mainly because of great computing capabilities. Another reason for using this software was the fact that it is free (this software is distributed under GNU/GPL license). In order to solve the problem the whole set of procedures was created. These procedures were creating, transforming and solving recurrence system of equation, which result from application of Krylov-Bogolubov method. The procedures were created in such a way so that they could be used in case of other kinds of problems. The solution found with the help of software MAXIMA has the following form:

$$h(t) = a_h(t) \cdot \cos(\psi_h(t)) - \frac{\Omega^2}{\Omega^2 - \omega_h^2} \cdot \cos(\psi_w(t)) + \\ + e \cdot \frac{\omega_\theta^2 \cdot a_\theta}{2 \cdot ((\Omega - \omega_\theta)^2 - \omega_h^2)} \cdot \sin(\psi_w(t) - \psi_\theta(t)), \tag{15}$$

$$v(t) = a_v(t) \cdot \cos(\psi_v(t)) - \frac{\Omega^2}{\Omega^2 - \omega_v^2} \cdot \sin(\psi_w(t)) + \\ - e \cdot \frac{\omega_\theta^2 \cdot a_\theta}{2 \cdot ((\Omega - \omega_\theta)^2 - \omega_v^2)} \cdot \sin(\psi_w(t) - \psi_\theta(t)), \tag{16}$$

$$\theta(t) = a_\theta(t) \cdot \cos(\psi_\theta(t)) + e \cdot \frac{m \cdot a_v}{2 \cdot I \cdot ((\Omega - \omega_v)^2 - \omega_\theta^2)} \cdot \cos(\psi_w(t) - \psi_v(t)) + \\ - e \cdot \frac{m \cdot a_h}{2 \cdot I \cdot ((\Omega - \omega_h)^2 - \omega_\theta^2)} \cdot \sin(\psi_w(t) - \psi_h(t)). \tag{17}$$

Obviously instantaneous phases are (with appropriately chosen initial conditions):

$$\psi_w(t) = \Omega \cdot t, \psi_h(t) = \omega_h \cdot t, \psi_v(t) = \omega_v \cdot t, \psi_\theta(t) = \omega_\theta \cdot t. \tag{18}$$

Finally, it can be written that solution (15-17) has the following form:

$$h(t) = a_h(t) \cdot \cos(\omega_h \cdot t) - \frac{\Omega^2}{\Omega^2 - \omega_h^2} \cdot \cos(\Omega \cdot t) + e \cdot \frac{\omega_\theta^2 \cdot a_\theta}{2 \cdot ((\Omega - \omega_\theta)^2 - \omega_h^2)} \cdot \sin(\Omega \cdot t - \omega_\theta \cdot t), \tag{19}$$

$$v(t) = a_v(t) \cdot \cos(\omega_v \cdot t) - \frac{\Omega^2}{\Omega^2 - \omega_v^2} \cdot \sin(\Omega \cdot t) - e \cdot \frac{\omega_\theta^2 \cdot a_\theta}{2 \cdot ((\Omega - \omega_\theta)^2 - \omega_v^2)} \cdot \sin(\Omega \cdot t - \omega_\theta \cdot t), \tag{20}$$

$$\theta(t) = a_\theta(t) \cdot \cos(\omega_\theta \cdot t) + e \cdot \frac{m \cdot a_v}{2 \cdot I \cdot ((\Omega - \omega_v)^2 - \omega_\theta^2)} \cdot \cos(\Omega \cdot t - \omega_v \cdot t) + \\ - e \cdot \frac{m \cdot a_h}{2 \cdot I \cdot ((\Omega - \omega_h)^2 - \omega_\theta^2)} \cdot \sin(\Omega \cdot t - \omega_h \cdot t). \tag{21}$$

As dependencies (19-21) show, transverse vibrations of the analysed rotor can have the following frequencies:

$$\omega_v = \omega_v = \omega_o, \Omega, \Omega - \omega_\theta, \Omega + \omega_\theta. \tag{22}$$

That is why the enumerated frequencies should be present in the spectrum of transverse vibrations.

## 4. Numerical solution

In order to check the accuracy of solution (19-21) the numerical solution was additionally found. Nowadays, numerical solution can be found with any accuracy. Therefore, the confrontation of the results obtained numerically and theoretically could give a lot of information about the correctness of assumptions and the relevance of the assumed approximation etc. Numerical computations were done in the environment of Scilab.

It is expected that frequencies (22) will occur in the system. The presence of these frequencies is the evidence of accuracy of the implemented model. Fig. 2 presents the amplitude spectra of vibrations obtained simultaneously for different frequencies.

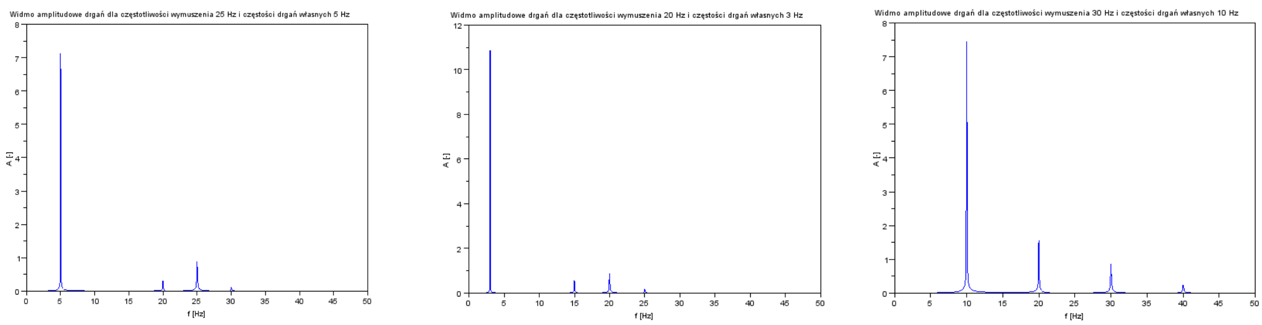


*Fig. 2. Selected amplitude spectra of transverse vibrations of shaft*

Figure 2 shows excitation frequencies, free vibrations and the difference of excitation frequency and torsional vibrations. The second spectrum in figure (2) presents the frequency of free vibration of the system 3 Hz (the highest spectrum bar) and the frequency of excitation 20 Hz and frequency 15 Hz and 25 Hz which are the sum and difference of excitation frequency and the frequency of torsional vibrations of the system (this frequency was 5 Hz) [2, 3].

The results of solutions (19-21) are compatible with the presented numerical solution, which increases the trust in the chosen analytical method. It is also promising as for the future application of this method of seeking the solutions.

## 5. Conclusion

During the work over the design project the series of theoretical and experimental research was done. Its aim was to extend the knowledge about the dynamics of rotors and working out the analytical models, which include more dynamic effects than usually used model in the form of harmonic oscillator (it is a very big simplification). In relation to this research, it was possible to find less radically simplified model. The article presents the simplifying assumption and necessary substitutions, which are necessary in order to obtain the analysed mathematical model.

Based on mathematical model, the analytical and numerical solutions were found. The compatibility of the results of Krylov-Bogolubov method and numerical simulations were presented in the adequate chapter. It allows using this method effectively in order to analyse the dynamics of certain class of rotor systems.

## References

[1] Batko, W., Dabrowski, Z., Kiciński, J., *Nonlinear Effects in Technical Diagnostics*, Radom ITE-PIB, 2008.

[2] Burdzik, R., *Monitoring system of vibration propagation in vehicles and method of analyzing vibration modes*, in: J. Mikulski (Ed.), Telematics in the Transport Environment, CCIS 329, pp. 406-413, Springer, Heidelberg, 2012.

[3] Charles, P., Sinha, J., K., Gu, F., Lidstone, L., Ball, A., D., *Detecting the crankshaft torsional vibration of diesel engines for combustion related diagnosis*, Journal of Sound and Vibration Vol. 321, pp. 1171-1185, 2009.

[4] Chiliński, B., *Analysis of disturbance torque influence on critical state in rotational systems*, Transportation Problems, Vol. 8, Is. 4, pp. 137-146, 2013.

[5] Chiliński, B., Pakowski, R., *Analysis of bending and torsinal vibroations of rotors with using perturbation methods*, in: Kleiber M., et al., (ed.), 3rd Polish Congress of Mechanics and 21st International Conference on Computer Methods in Mechanics – PCM-CMM, Polish Society of Theoretical and Applied Mechanics, Short papers, Vol. 1, 2, pp. 15-17, 2015.

[6] Chiliński, B., Zawisza, M., *Modelling of lateral-torsional vibrations of the crank system with a damper of vibrations*, Vibroengineering Procedia, Vol. 6 (6), pp. 61-65, 2015.

[7] Dąbrowski, Z., Chiliński, B., *Identification of a model of crank shaft with a damper of torsional vibrations*, Vibroengineering Procedia, Vol. 6 (6), pp. 50-54, 2015.

[8] Desbazeille, M., Randall, R.,B., Guillet, F., El Badaoui, M., Hoisnard, C., *Model-based diagnosis of large diesel engines based on angular speed variations of the crankshaft*, Mechanical Systems and Signal Processing, Vol. 24, pp. 1529-1541, 2010.

[9] Deuszkiewicz, P., Pankiewicz, J., Dziurdź, J. Zawisza, M., *Modeling of powertrain system dynamic behavior with torsional vibration damper*, Advanced Materials Research, Vol. 1036, pp. 586-591, 2014.

[10] Homik, W., Pankiewicz, J., *Examinations of torsional vibration dampers used in reciprocating internal combustion engines*, Polish Journal of Environmental Studies, Vol. 20, No. 5A, pp. 108-111, Olsztyn 2011.

[11] Klekot, G., *Application of vibro-acoustic energy propagation measures to monitor status of the object and as a tool in the manage of noise*, ITE, Radom2012.