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ON STRONG AUTOMORPHISMS OF DIRECT PRODUCTS OF WITT RINGS (II)

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Abstract. In the paper strong automorphisms of finitely generated Witt rings are considered. Every finitely generated Witt ring can be expressed in terms of $\mathbb{Z}/2\mathbb{Z}$ and basic indecomposable Witt rings using the operations of group ring formation and direct product. Groups of strong automorphisms of basic indecomposables and their direct products and the description of all strong automorphisms of any group Witt ring are known. In this paper the strong automorphisms of direct products of group Witt rings are considered. Presented are two wide classes of Witt rings where the group of strong automorphisms is isomorphic to the direct product of groups of strong automorphisms of Witt rings which are factors in the direct product.

Keywords: *Witt rings, strong automorphisms, group rings*

1. Introduction

We consider abstract Witt rings in terminology of M. Marshall [1]. We shall describe the groups of strong automorphisms of some direct products of group Witt rings. The results presented here are continuation of what is included in [2]. Therefore we omit some definitions (in particular the definition of abstract Witt ring) and facts and limit preliminaries to the minimum necessary in the paper. For further information the reader can go to [1] or [2].

Let the pair $W = (R, G)$ be a Witt ring, which is finitely generated by the group G. As we know there is one-to-one correspondence between Witt rings and quaternionic structures ([1, Theorem 4.5]). The correspondence extends to their automorphisms and the group of strong automorphisms of Witt ring W is isomorphic to the group of automorphisms of the quaternionic structure (G, Q, q) associated to W (compare [3]). Since we shall use the properties of quaternionic structures, we recall the definition.

Definition 1.1. Let G be a group of exponent 2, i.e. $a^2 = 1$ for all $a \in G$ with distinguished element $-1 \in G$ and let us denote $-a = -1 \cdot a$. Let Q be the set with

distinguished element θ and let $q\colon G \times G \to Q$ be a surjective map. The triplet (G, Q, q) is called a *quaternionic structure* if for every $a, b, c, d \in G$ the map q fulfills:
Q_1: $q(a, b) = q(b, a)$
Q_2: $q(a, -a) = \theta$
Q_3: $q(a, b) = q(a, c) \Rightarrow q(a, bc) = \theta$
Q_4: If $q(a, b) = q(c, d)$, then there exists such $x \in G$ that $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$.

Let (G, Q, q) be a quaternionic structure. A *(quadratic) form of dimension* $n \geq 1$ over G is n-tuple $f = (a_1, \dots, a_n)$, where $a_1, \dots, a_n \in G$. Two forms of dimension n are called *equivalent* if:
(1) $n = 1$, $(a) \cong (b) \Leftrightarrow a = b$
(2) $n = 2$, $(a, b) \cong (c, d) \Leftrightarrow ab = cd$ and $q(a, b) = q(c, d)$
(3) $n > 2$, $(a_1, \dots, a_n) \cong (b_1, \dots, b_n) \Leftrightarrow \exists a, b, c_3, \dots, c_n \in G$ such that $(a_2, \dots, a_n) \cong (a, c_3, \dots, c_n)$, $(a_1, a) \cong (b_1, b)$ and $(b_2, \dots, b_n) \cong$ $\cong (b, c_3, \dots, c_n)$.
The form $(1, a_1) \otimes \cdots \otimes (1, a_n)$, where $a_1, \dots, a_n \in G$, $n > 0$ is named *n-fold Pfister form*. We say that form *f represents element* $a \in G$ if there exist $a_2, \dots, a_n \in$ $\in G$, such that $f \cong (a, a_2, \dots a_n)$. The set of all elements represented by form f (*value set of the form f*) is denoted by $D(f)$.

Let $W = (R, G)$ be a Witt ring and let (G, Q, q) be corresponding quaternionic structure. By *strong automorphism* of W we understand any ring automorphism $\varphi \in Aut(R)$ such that $\varphi(G) = G$. By [3, Theorem 2.1] there exists a corresponding automorphism of quaternionic structure σ, i.e. a group automorphism $\sigma \in Aut(G)$ such that $\sigma(-1) = -1$ and $q(a, b) = \theta \Leftrightarrow q(\sigma(a), \sigma(b)) = \theta$ for all $a, b \in G$.

We say that a Witt ring $W = (R, G)$ is *indecomposable* if firstly $W \neq \mathbb{Z}/2\mathbb{Z}$, and secondly, if $(R, G) \cong (R_1, G_1) \sqcap (R_2, G_2)$ implies either $R_1 \cong \mathbb{Z}/2\mathbb{Z}$ or $R_2 \cong \mathbb{Z}/2\mathbb{Z}$ ([1]). An element $a \in G$ is *rigid* if $b \in D(1, a) \Rightarrow b = 1$ or $b = a$ or the value set of the form $(1, a)$ is just $\{1, a\}$. If $G \neq \{1, -1\}$ we will say that $a \in G$ is *basic* if either a or $-a$ is not rigid. In case $G = \{1, -1\}$ we consider -1 and 1 both to be basic. The elements in the group G, which are not basic we will call *birigid*. A Witt ring $W = (R, G)$ is said to be *basic Witt ring* if all elements of G are basic. A Witt ring, which is both basic and indecomposable is, called *basic indecomposable Witt ring* [compare [1]).

M. Marshall proved that any finitely generated Witt ring can be expressed in terms of $\mathbb{Z}/2\mathbb{Z}$ and basic indecomposable Witt rings using the operations of group ring formation and direct product ([1, Theorem 5.23]). M. Marshall in [1] presented the all known basic indecomposables as $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}$, and Witt rings of local type. Therefore, if we want to find strong automorphisms of finitely generated Witt rings we should know how to describe the groups of automorphisms of Witt rings of local type, group Witt rings and their direct products. We know the description of strong automorphisms of Witt rings of local type ([4]), their direct products ([5])

and group Witt rings ([6]). In this paper we show that in many cases the group of strong automorphisms of direct product of group Witt rings can be precisely described.

2. Automorphisms of direct products of group Witt rings

Let $W = (R, G)$ be finite direct product of Witt rings $W_1 = (R_1, G_1), \dots, W_n = (R_n, G_n)$ which are not pairwise (strongly) isomorphic. Let (G, Q, q) be the quaternionic structure associated to W and assume that $(G, Q, q) = \prod_{i=1}^{n}(G_i, Q_i, q_i)$. Denote the subgroup $\{1\} \times \dots \times \{1\} \times G_k \times \{1\} \times \dots \times \{1\}$ of the group $G = G_1 \times \dots \times G_n$ by G'_k, where $1 \le k \le n$. There was shown in [2] that the necessary condition for $Aut(G, Q, q) \cong \prod_{i=1}^{n} Aut(G_i, Q_i, q_i)$ (and consequently $Aut(W) \cong \prod_{i=1}^{n} Aut(W_i)$) is that any automorphism of the group G *preserves the factors of the product* i.e. if for all $k \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $\sigma(G'_k) = G'_j$.

Below we present two broad classes of direct products of group Witt rings where the condition described above is fulfilled.

Proposition 2.1. *Let* Δ_n *be a group of exponent* 2 *with cardinality* 2^n, $n \in \mathbb{N}$. *Let* (G_1, Q_1, q_1) *be a quaternionic structure associated to the Witt ring* $W_1 = (S[\Delta_1], G_1)$, *where* S *is a basic Witt ring and let* (G_2, Q_2, q_2) *be the quaternionic structure associated to Witt ring* $W_2 \cong W(\mathbb{Q}_5)[\Delta_n]$. *If* $|G_1| = 2^m$, *then for* $n \ge m - 1$ *and for* $\sigma \in Aut(G, Q, q) = Aut(G_1 \times G_2, Q_1 \times Q_2, q_1 \times q_2)$ *holds:*

(i) $\sigma(G_1 \times \{1_2\}) = G_1 \times \{1_2\}$ *and*

(ii) $\sigma(\{1_2\} \times G_2) = \{1_2\} \times G_2$.

Proof. Let us calculate the possible cardinalities of value sets of binary Pfister forms $(1_i, b)$, where elements b are from G_i, $i = 1,2$.

At least a half of elements of the group G_1 are birigid, hence we have $|D_1(1_1, b)| = 2$ if b is birigid. The remaining elements of G_1 fulfill $|D_1(1_1, b')| \le 2^m$ (in particular $|D_1(1_1, -1_1)| = 2^m$).

In G_2 we have $1_2 = -1_2$, then $|D_2(1_2, -1_2)| = 2^{n+2}$ and $|D_2(1_2, b)| = 2$ if $b \ne 1_2$ (since the group $G_{\mathbb{Q}_5}$ has 4 elements).

Notice first that $|D(\mathbf{1}, -\mathbf{1})| = |D([1_1, 1_2], [-1_1, 1_2])| = |D_1(1_1, -1_1)| \cdot |D_2(1_2, 1_2)| = 2^m \cdot 2^{n+2}$.

Now let us consider an element $\boldsymbol{c}$ such that $[1_1, 1_2] \ne \boldsymbol{c} = [c_1, c_2] \in G_1 \times G_2$. We have $|D(\mathbf{1}, \boldsymbol{c})| = |D([1_1, 1_2], [c_1, c_2])| = |D_1(1_1, c_1) \times D_2(1_2, c_2)| = |D_1(1_1, c_1)| \cdot |D_2(1_2, c_2)|$. Then

$$|D(\mathbf{1}, \boldsymbol{c})| \le 2^m \cdot 2 = 2^{m+1} \text{if } c_2 \ne 1_2 \tag{2.1}$$

$$|D(\mathbf{1}, \boldsymbol{c})| \ge 2 \cdot 2^{n+2} = 2^{n+3} \ge 2^{m+2} \text{if } c_2 = 1_2 \tag{2.2}$$

Let $\boldsymbol{a} = [x, 1_2] \in G_1 \times G_2$. Then by (2.2) we get $|D(\mathbf{1}, \boldsymbol{a})| \geq 2^{m+2}$. Since σ is an automorphism of quaternionic structure (G, Q, q), then we also have $|D(\mathbf{1}, \sigma(\boldsymbol{a}))| \geq 2^{m+2}$.

Suppose that $\sigma(\boldsymbol{a}) = \sigma([x, 1_2]) = [x', y'] \in G_1 \times G_2$ and $y' \neq 1_2$. Then $|D_2(1_2, y')| = 2$ and from $y' \neq 1_2$ it follows that $|D(\mathbf{1}, \sigma(\boldsymbol{a}))| = |D(\mathbf{1}, [x', y'])| = |D_1(1_1, x')| \cdot |D_2(1_2, y')| \leq 2^m \cdot 2 = 2^{m+1}$ - a contradiction. Therefore we have $\sigma(\boldsymbol{a}) = \sigma([x, 1_2]) = [x', 1_2]$ for some $x' \in G_1$ and since $\boldsymbol{a} = [x, 1_2]$ is any element in $G_1 \times G_2$ which means that the condition (i) is fulfilled.

Let us now take an element $\boldsymbol{b} = [1_1, y]$ such that $y \neq 1_2$. Then $|D_2(1_2, y)| = 2$ and by (2.1) it follows that $|D(\mathbf{1}, \boldsymbol{b})| \leq 2^{m+1}$. Consider the opposite element $-\boldsymbol{b} = [-1_1, -y]$, where $-y \neq 1_2$. Then $|D(\mathbf{1}, -\boldsymbol{b})| = |D([1_1, 1_2], [-1_1, -y])| = |D_1(1_1, -1_1)| \cdot |D_2(1_2, -y)| = 2^m \cdot 2$, hence also $|D(\mathbf{1}, \sigma(-\boldsymbol{b}))| = 2^{m+1}$.

Assume that $\sigma(\boldsymbol{b}) = [x'', y'']$. By $\left|D\left(\mathbf{1}, \sigma(-\boldsymbol{b})\right)\right| = 2^{m+1}$ and by (2.1) it follows that $y'' \neq 1_2$ (thus we have $|D_2(1_2, y'')| = 2$). On the other hand $\sigma(-\boldsymbol{b}) = [-x'', -y'']$ and $y'' \neq 1_2$, hence also $-y'' \neq 1_2$, thus $|D_2(1_2, -y'')| = 2$. If $x'' \neq 1_1$, then $-x'' \neq -1$ and then $\left|D\left(\mathbf{1}, \sigma(-\boldsymbol{b})\right)\right| = |D([1_1, 1_2], [-x'', -y''])| = |D_1(1_1, -x'')| \cdot |D_2(1_2, -y'')| < 2^m \cdot 2 = 2^{m+1}$ – a contradiction. Therefore $x'' = 1_1$, hence $\sigma(\boldsymbol{b}) = \sigma([1, y]) = [1, y'']$, which means that the condition (ii) is fulfilled.■

The following proposition concerns another class of direct products of group Witt rings and is similar to Proposition 2.1.

Proposition 2.2. *Let* Δ_n *be a group of exponent* 2 *with cardinality* 2^n, $n \in \mathbb{N}$. *Let* (G_1, Q_1, q_1) *be a quaternionic structure associated to the Witt ring* $W_1 = \left(W(\mathbb{Q}_3) \sqcap W(\mathbb{Q}_5)\right)[\Delta_1]$. *and let* (G_2, Q_2, q_2) *be the quaternionic structure associated to Witt ring* $W_2 \cong W(\mathbb{Q}_5)[\Delta_n]$.
Then for $\sigma \in Aut(G, Q, q) = Aut(G_1 \times G_2, Q_1 \times Q_2, q_1 \times q_2)$ *holds:*
(i) $\sigma(G_1 \times \{1_2\}) = G_1 \times \{1_2\}$ *and*
(ii) $\sigma(\{1_2\} \times G_2) = \{1_2\} \times G_2$.

Proof. Let us use notation $W_1 = (R_1, G_1)$ and $W_2 = (R_2, G_2)$. Let us first describe the groups G_1 and G_2 and value sets of 1-fold Pfister forms in quaternionic structures associated to W_1 and W_2. We shall use this information in our proof.

The ring W_1 is a group Witt ring of the group $\Delta_1 = \{1, x\}$ with coefficients in Witt ring $W(\mathbb{Q}_3) \sqcap W(\mathbb{Q}_5)$ being a direct product of Witt rings of local type. As we know the groups $G_{\mathbb{Q}_3}$ and $G_{\mathbb{Q}_5}$can be represented in the form: $G_{\mathbb{Q}_3} = \{1, -1, p, -p\}$ for some $p \in \mathbb{Q}_3$, $p \neq \pm 1_{\mathbb{Q}_3}$ and $G_{\mathbb{Q}_5} = \{1, p, u, up\}$ for some $u, p \in \mathbb{Q}_5$, $u, p \neq 1_{\mathbb{Q}_5} = -1_{\mathbb{Q}_5}$ (compare [7, Theorem 2.2, p. 152] or [8, Corollary on p. 18]). Using a definition of elements represented by forms in the direct products of Witt rings and in group Witt rings we can describe value sets of 1-fold Pfister forms in W_1 in the following way.

1) $|D_1(1_1,-1_1)| = 32$.
2) There are 16 birigid elements, i.e. such that $|D_1(1_1,d)| = 2$, where $\pm 1_1 \neq d \in G_1$ is of the form $d = [d',x]$, $x \in \Delta_1$, $d' \in G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3}$, $x \in \Delta_1$ (then $D_1(1_1,d) = \{1_1,d\}$).
3) There are 9 elements of the form $\pm 1_1 \neq d \neq [d',1] \in (G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_5}) \times \Delta_1$, such that $|D_1(1_1,d)| = 4$.
4) There are 6 elements of the form $-1_1 \neq d = [b,1] \in (G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_5}) \times \Delta_1$, such that $|D_1(1_1,d)| = 8$ (in particular $|D_1(1_1,1_1)| = 8$).

The ring W_2 is the group Witt ring of a group Δ_n of exponent 2 (for $n \geq 1$) with coefficients in Witt ring of local type $W(\mathbb{Q}_5)$. We have $G_2 = G_{\mathbb{Q}_5} \times \Delta_n$, hence $|G_2| = 2^2 \cdot 2^n = 2^{n+2}$. Therefore we can describe value sets of 1-fold Pfister forms as follows.

1) $D_2(1_2,1_2) = D_2(1_2,-1_2) = G_2$.
2) There are 3 elements of the form $1_2 \neq d = [d',1] \in G_{\mathbb{Q}_5} \times \Delta_n$ such that $|D_2(1_2,d)| = 2$.
3) There are $2^{n+2} - 4$ elements of the form $d = [d',d''] \in G_{\mathbb{Q}_5} \times \Delta_n$, $d' \neq 1_2$ in $G_{\mathbb{Q}_5}$, $d'' \neq 1$ in Δ_n such that $|D_2(1_2,d)| = 2$.

Notice that in the case $n = 0$ (or for the Witt ring $W(\mathbb{Q}_5)$) such elements do not exist!

Now we are going to the main part of our proof.

Notice first that $1_2 = -1_2 \in G_2$, hence $[1_1,-1_2] = [1_1,1_2] = \mathbf{1}$ and $[-1_1,1_2] = [-1_1,-1_2] = -\mathbf{1} \in G_1 \times G_2$, thus for any automorphism $\sigma \in Aut(G,Q,q)$ we get $\sigma([1_1,-1_2]) = [1_1,-1_2]$ and $\sigma([-1_1,1_2]) = [-1_1,1_2]$.

Let us consider an element $\boldsymbol{a} = [1_1,y] \in G_1 \times G_2$ such that $y \neq 1_2$. Then $|D(\mathbf{1},\boldsymbol{a})| = |D_1(1_1,1_1) \times D_2(1_2,y)| = 8 \cdot 2 = 16$ and for the opposite element $-a = [-1_1,-y]$ we have $|D(\mathbf{1},-\boldsymbol{a})| = |D_1(1_1,-1_1) \times D_2(1_2,-y)| = 32 \cdot 2 = 64$. Since σ is an automorphism of quaternionic structure and it preserves value sets of forms, then we get also $|D(\mathbf{1},\sigma(\boldsymbol{a}))| = 16$ and $|D(\mathbf{1},\sigma(-\boldsymbol{a}))| = 64$. Assume that $\sigma(\boldsymbol{a}) = \sigma([1_1,y]) = [x',y'] \in G_1 \times G_2$. We have $16 = |D(1,\sigma(\boldsymbol{a}))| = |D_1(1_1,x')| \cdot |D_2(1_2,y')|$. As we know there is impossible $|D_1(1_1,x')| = 1$ and $|D_2(1_2,y')| = 1$ in $G_1 \times G_2$, therefore either of the three cases occur:

(1) $|D_1(1_1,x')| = 2$ and $|D_2(1_2,y')| = 8$ or
(2) $|D_1(1_1,x')| = 4$ and $|D_2(1_2,y')| = 4$ or
(3) $|D_1(1_1,x')| = 8$ and $|D_2(1_2,y')| = 2$.

We shall show, that the cases (1) and (2) are impossible and only case (3) holds.
Let us consider all above cases one after the other.

Ad (1). By the second condition $|D_2(1_2,y')| = 8$ and by properties of the group G_2 it follows that $y' = -1_2 = 1_2 \in G_2$, which means that $y' = -y' = 1_2$ and $|G_2| = 8$ (or $n = 1$). Therefore we have $\sigma(\boldsymbol{a}) = [x',1_2]$ for some $x' \in G_1$, thus $\sigma(-\boldsymbol{a}) = [-x',1_2]$. Since $64 = |D(\mathbf{1},\sigma(-\boldsymbol{a}))| = |D([1_1,1_2],[-x',1_2])| =$

$= |D_1(1_1, -x')| \cdot |D_2(1_2, 1_2)|$ and $|D_2(1_2, 1_2)| = 8$, then $|D_1(1_1, -x')| = 8$. But from properties of G_1 we know that for any $x' \in G_1$ such that $|D_1(1_1, -x')| = 8$ we have $|D_1(1_1, x')| \neq 2$ – a contradiction to (1).

Ad (2). By similar argumentation as in the previous case we get $|D_2(1_2, y')| = 4$ implies $y' = -y' = 1_2 \in G_2$ and $|G_2| = 8$ and $n = 0$. Therefore we have $64 = \left|D\big(\mathbf{1}, \sigma(-\boldsymbol{a})\big)\right| = |D([1_1, 1_2], [-x', 1_2])| = |D_1(1_1, -x')| \cdot |D_2(1_2, 1_2)|$ and $|D_2(1_2, 1_2)| = 4$, then $|D_1(1_1, -x')| = 16$ – a contradiction, because for any $d \in G_1$ we have $|D_1(1_1, d)| \neq 16$.

Therefore we have shown that the case (3) holds. Assuming that $\sigma([1_1, y]) =$ $= [x', y']$ for some $x' \in G_1$, $y' \in G_2$ we shall show that $x' = 1_1$.
We have $\sigma(-\boldsymbol{a}) = [-x', -y']$ and $\left|D\big(\mathbf{1}, \sigma(-\boldsymbol{a})\big)\right| = 64$. If $y' = 1_2$, then by condition (3) it follows that $|D_2(1_2, y')| = |D_2(1_2, 1_2)| = 2$, hence $|G_2| = 2$, a contradiction (since $|G_2| \geq 4$). Therefore $y' \neq 1_2$ in G_2. Then $-y' \neq 1_2$ and by properties of the group G_2 it follows that $|D_2(1_2, -y')| = 2$. Since $64 = \left|D\big(\mathbf{1}, \sigma(-\boldsymbol{a})\big)\right| = |D_1(1_1, -x')| \cdot |D_2(1_2, -y')|$, then $|D_1(1_1, -x')| = 32$ and $-x' = -1_1 \in G_1$ and consequently $x' = 1_1$. It means that for every $y \in G_2$ we have $\sigma(\boldsymbol{a}) = \sigma([1_1, y]) = [1_1, y'] \in G_1 \times G_2$, which finishes the proof (ii).

Let $1_2 \neq d, e \in G_2$ and $d \neq e$. Then by (ii) we can write $\sigma([1_1, d]) = [1_1, d']$ and $\sigma([1_1, e]) = [1_1, e']$ for some $d', e' \in G_2$. In order to prove (i) we shall calculate $D([1_1, 1_2], [-1_1, d]) \cap D([1_1, 1_2], [-1_1, d'])$.
Let us first notice that

$$D([1_1, 1_2], [-1_1, d]) \cap D([1_1, 1_2], [-1_1, e]) =$$
$$= \big(D_1(1_1, -1_1) \times D_2(1_2, d)\big) \cap \big(D_1(1_1, -1_1) \times D_2(1_2, e)\big) =$$
$$= \big(D_1(1_1, -1_1) \cap D_1(1_1, -1_1)\big) \times \big(D_2(1_2, d) \cap D_2(1_2, e)\big) = G_1 \times \{1_2\}.$$

Therefore for any $\sigma \in Aut(G, Q, q)$ we get

$$\sigma(G_1 \times \{1_2\}) =$$
$$= \sigma(D\left([1_1, 1_2], [[-1_1, d]\right) \cap D([1_1, 1_2], [-1_1, e])\big) =$$
$$= D([1_1, 1_2]), \sigma([-1_1, d])) \cap D([1_1, 1_2], \sigma([-1_1, e])) =$$
$$= D([1_1, 1_2], [-1_1, d']) \cap D([1_1, 1_2], [-1_1, e']) = G_1 \times \{1_2\}$$

which finishes the proof of (i).■

Propositions 2.1 and 2.2 show that a lot of examples of direct products of group Witt rings which preserves their factors exists. In consequence we can calculate their groups of strong automorphisms as the direct product groups of automorphisms of factors of direct product. Unfortunately not every direct product of group Witt rings preserves the factors of direct product as it shows the following example.

Example 2.3. Let us consider a Witt ring $W = \mathbb{Z}/2\mathbb{Z}[\Delta_1] \sqcap \mathbb{Z}/2\mathbb{Z}[\Delta_2]$. As we know, the only strong automorphism of $\mathbb{Z}/2\mathbb{Z}[\Delta_1]$ is identity (compare [3, §3]).

In [2] we have mentioned that $|Aut(\mathbb{Z}/2\mathbb{Z}[\Delta_2])| = 6$. It follows that $|Aut(\mathbb{Z}/2\mathbb{Z}[\Delta_1]\,)| \cdot |Aut(\mathbb{Z}/2\mathbb{Z}[\Delta_2])| = 6$. It turns out that in this case the formula $Aut(W) = Aut(\mathbb{Z}/2\mathbb{Z}[\Delta_1]) \times Aut(\mathbb{Z}/2\mathbb{Z}[\Delta_2])$ is not true. In fact we cannot describe the group $Aut(W)$. However, we have shown that $|Aut(W)| = 24$ with the help of complex computer program (see algorithm of our program in [7]).

Unfortunately we have no tool for describing the group of strong automorphisms of Witt rings which are direct products of any group Witt rings.

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