Control and Cybernetics

vol. **46** (2017) No. 2

Global existence for strong solutions of viscous Burgers equation. (1) The bounded case*

by

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Abstract: We prove that the viscous Burgers equation $(\partial_t - \Delta)u(t,x) + (u\cdot\nabla)u(t,x) = g(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ $(d \geq 1)$ has a globally defined smooth solution in all dimensions provided the initial condition and the forcing term g are smooth and bounded together with their derivatives. Such solutions may have infinite energy. The proof does not rely on energy estimates, but on a combination of the maximum principle and quantitative Schauder estimates. We obtain precise bounds on the sup norm of the solution and its derivatives, making it plain that there is no exponential increase in time. In particular, these bounds are time-independent if g is zero. To get a classical solution, it suffices to assume that the initial condition and the forcing term have bounded derivatives up to order two.

Keywords: viscous Burgers equation, conservation laws, maximum principle, Schauder estimates

1. Introduction and scheme of proof

1.1. Introduction

The $(1+d)$-dimensional viscous Burgers equation is the following non-linear PDE,

$$(\partial_t - \nu\Delta + u\cdot\nabla)u = g, \qquad u\big|_{t=0} = u_0 \tag{1.1}$$

for a velocity $u = u(t,x) \in \mathbb{R}^d$ $(d \geq 1)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $\nu > 0$ is a viscosity coefficient, Δ the standard Laplacian on $\mathbb{R}^d$, $u\cdot\nabla u = \sum_{i=1}^d u_i \partial_{x_i} u$ the convection term, and g a continuous forcing term. Among other things, this fluid equation, which describes the hydrodynamical limit of interacting particle systems (Spohn, 2016; Kipnis and Landim, 1999), is a simplified version without pression of the incompressible Navier-Stokes equation, and also (assuming g to be random, which turns (1.1) into a so-called stochastic Burgers equation) an interesting model for the study of traffic

*Submitted: September 2016; Accepted: July 2017

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jams, and an interesting toy-model for Navier-Stokes' equation, and, in particular, for the study of turbulence (so-called "Burgulence"), see, e.g., Bec and Khanin (2007), E, Khanin, Mazel and Sinai (2000), Woyczynski (1998). Note that gradient solutions of the stochastic Burgers equation are gradients of solutions of the celebrated Kardar-Parisi-Zhang (KPZ) equation, on which (particularly in one-space dimension, where its highly non-trivial large-scale Tracy-Widom limit appears to arise in many non-equilibrium processes in statistical physics, see, e.g., Mendl and Spohn, 216; Spohn, 1991, for recent perspectives) there is now an immense and quickly evolving literature.

The present study is purely mathematical: we show under the assumptions on u_0 and g, provided further on in this section, that the Cauchy problem

$$(\partial_t - \nu\Delta + u\cdot\nabla)u = g, u\big|_{t=0} = u_0 \tag{1.2}$$

has a unique, globally defined, classical solution in $C^{1,2}$ (i.e. continuously differentiable in the time coordinate and twice continuously differentiable in the space coordinates), and provide explicit bounds for the supremum of u and its derivatives up to second order.

Before we go further, let us mention for the sake of the reader that $u_t, g_t, \cdots$ are the *values* at time t of $u, g, \cdots$ and *not* their time-derivatives (in discrepancy with a notation widely used in particular by specialists of integrable systems, but in accordance with probabilists' notations!).

Assumptions:

(i) (initial condition) $u_0 \in C^2$ *and* $\nabla^2 u_0$ *is* α*-Hölder for every* $\alpha \in (0,1)$*; for* $\kappa = 0,1,2$, $\|\nabla^\kappa u_0\|_\infty := \sup_{x\in\mathbb{R}^d} |\nabla^\kappa u_0(x)| < \infty$;

(ii) (forcing term) on every subset $[0,T]\times\mathbb{R}^d$ *with* $T > 0$ *finite,* g *is bounded and* α*-Hölder continuous for every* $\alpha \in (0,1)$*; furthermore,* g *is* $C^{1,2}$ *and* $t \mapsto \|\nabla^\kappa g_t\|_\infty := \sup_{x\in\mathbb{R}^d}|\nabla^\kappa g_t(x)|$, $t \mapsto \|\partial_t g_t\|_\infty := \sup_{x\in\mathbb{R}^d}|\partial_t g_t(x)|$ *are locally integrable in time.*

For convenience, we redefine $\tilde{t} = \nu t$, $\tilde{u} = \nu^{-1}u$, $\tilde{g} = \nu^{-2}g$. The rescaled equation, $(\partial_{\tilde{t}} - \Delta - \tilde{u}\cdot\nabla)\tilde{u} = \tilde{g}$, has viscosity 1. We skip the tilde in the sequel. Our bounds blow up in the vanishing viscosity limit $\nu \to 0$ (see Remarks after Theorem 1 for a precise statement).

Our approach is the following. We solve inductively the linear transport equations,

$$u^{(-1)} := 0; \tag{1.3}$$

$$(\partial_t - \Delta + u^{(m-1)}\cdot\nabla)u^{(m)} = g, \quad u^{(m)}\big|_{t=0} = u_0 \qquad (m \geq 0). \tag{1.4}$$

If the sequence $(u^{(m)})_m$ converges in appropriate norms, then the limit is a fixed point of (1.4), hence solves the Burgers equation. Let $\|\ \|_\alpha$ denote either the isotropic Hölder semi-norm on $\mathbb{R}^d$,

$$\|u_0\|_\alpha := \sup_{x,y\in\mathbb{R}^d} \frac{|u_0(x)-u_0(y)|}{|x-y|^\alpha},$$

or the parabolic Hölder semi-norm on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$||g||_\alpha := \sup_{(s,x),(t,y)\in\mathbb{R}_+\times\mathbb{R}^d} \frac{|g(s,x) - g(t,y)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

(see Section 4 for more on Hölder norms).

Definition 1 *Let, for $c > 0$,*

$$K_0(t) := ||u_0||_\infty + \int_0^t ds ||g_s||_\infty \tag{1.5}$$

$$K_1(t) := ||\nabla u_0||_\infty + \int_0^t ds ||\nabla g_s||_\infty \tag{1.6}$$

$$K_2(t) := ||\nabla^2 u_0||_\infty + ||u_0||_\infty ||\nabla u_0||_\infty + ||g_0||_\infty + \int_0^t ds \left(||\nabla^2 g_s||_\infty + ||\partial_s g_s||_\infty \right) \tag{1.7}$$

$$K_{2+\alpha}(t) := ||\nabla^2 u_0||_\alpha + ||g_s||_{\alpha,[0,t]\times\mathbb{R}^d}, \qquad \alpha \in (0,1) \tag{1.8}$$

and

$$K(t) := c^2 \left(K_0(t)^2 + K_1(t) + K_2(t)^{2/3} + K_{2+\alpha}(t)^{2/(3+\alpha)} \right). \tag{1.9}$$

Note that $K_0(t), K_1(t), K_2(t), K_{2+\alpha}(t), K(t) < \infty$ for all $t \geq 0$ and $\alpha \in (0,1)$ under the above Assumptions.

Our main result is the following.

Theorem 1 *For every $\beta \in (0, \frac{1}{2})$, there exists an absolute constant $c = c(d,\beta) \geq 1$, depending only on the dimension and on the exponent β, such that the following holds:*

(i) (uniform estimates)

$$||u_t^{(m)}||_\infty \leq K_0(t), \;\; ||\nabla u_t^{(m)}||_\infty \leq K(t); \qquad ||\partial_t u_t^{(m)}||_\infty, ||\nabla^2 u_t^{(m)}||_\infty \leq (cK(t))^{3/2} \tag{1.10}$$

(ii) (short-time estimates) define $v^{(m)} := u^{(m)} - u^{(m-1)}$ for $m \geq 1$. If $0 \leq t \leq T$ and $t \leq m/cK(T)$, then

$$||v_t^{(m)}||_\infty \leq cK_0(T)(cK(T)t/m)^m, \qquad ||\nabla v_t^{(m)}||_\infty \leq cK(T)(cK(T)t/m)^{\beta m}. \tag{1.11}$$

Let us comment on these estimates.

Remarks:

1. The different powers in the expression of $K(t)$ come from the dimension counting dictated by the Burgers equation: the diffusion term Δu, the convection term $u\cdot\nabla u$ and the forcing g are homogeneous if u scales like L^{-1}, where L is a reference space scale, and g like $(LT)^{-1}$, where T is a reference time scale. Assuming parabolic scaling, $K^{-1}(t)$ scales like time and plays the rôle of a reference time scale $T(t)$ at time t, leading to a time-dependent space scale $L = L(t) \sim K^{-\frac{1}{2}}(t)$. The scaling of the other K-parameters is $K_0 \sim T^{-\frac{1}{2}}$; $K_1, K \sim T^{-1}$; $K_2 \sim T^{-3/2}$; $K_{2+\alpha} \sim T^{-(3+\alpha)/2}$.

2. The first uniform estimate
$$||u_t^{(m)}||_\infty \le K_0(t) \tag{1.12}$$
follows from a straightforward application of the maximum principle to the transport equation (1.4).
3. (uniform estimates for the gradient). The function $u^{(0)}$ satisfies the linear heat equation $(\partial_t - \Delta)u^{(0)} = g$, whose explicit solution is $u^{(0)}(t) = e^{t\Delta}u_0 + \int_0^t ds e^{(t-s)\Delta} g_s$. Thus
$$||\nabla u_t^{(0)}||_\infty \le ||\nabla u_0||_\infty + \int_0^t ds ||\nabla g_s||_\infty = K_1(t). \tag{1.13}$$
Clearly, $K_1(t) \le K(t)$. Estimates for further iterates $u^{(1)}, u^{(2)}, \ldots$ involve $K(t)$ instead of $K_1(t)$.
4. Fix a time horizon $T > 0$ and consider the series
$$S(t) := \sum_{m=0}^{+\infty} v_t^{(m)} = \sum_{m=0}^{+\infty} (u_t^{(m)} - u_t^{(m-1)})$$
for $t \le T$ (note that, by definition, $v^{(0)} := u^{(0)} - u^{(-1)} = u^{(0)}$). The short-time estimates (1.11) imply that $S(t)$ is absolutely convergent. More precisely, letting $m_0 := \lfloor cK(T)t \rfloor$ and $\gamma := 1$,
$$\begin{aligned} ||u_t^{(n)}||_\infty = \left\| \sum_{m=0}^{n} (u_t^{(m)} - u_t^{(m-1)}) \right\|_\infty &\le ||u_t^{(m_0)}||_\infty + \sum_{m=m_0+1}^{+\infty} ||v_t^{(m)}||_\infty \\ &\le K_0(T) \left\{ 1 + c \sum_{m=m_0+1}^{+\infty} (cK(T)t/m)^{\gamma m} \right\} \end{aligned} \tag{1.14}$$
for all $n \ge m_0$. Let $m > m_0$ and $x = 1 - cK(T)t/m \in [0, 1]$: using $1 - x \le e^{-x}$, one gets $(cK(T)t/m)^{\gamma m} = (1 - x)^{\gamma m} \le e^{\gamma cK(T)t} e^{-\gamma m}$ and
$$\sum_{m=m_0+1}^{+\infty} (cK(T)t/m)^{\gamma m} \le e^{\gamma cK(T)t} \sum_{m=m_0+1}^{+\infty} e^{-\gamma m} \le e^\gamma/(e^\gamma - 1). \tag{1.15}$$
Hence, $||u_t^{(n)}||_\infty \lesssim K_0(T)$. In a similar way, letting $\gamma := \beta$ this time, one shows that
$$||\nabla u_t^{(n)}||_\infty = \left\| \sum_{m=0}^{n} (\nabla u_t^{(m)} - \nabla u_t^{(m-1)}) \right\|_\infty \lesssim K(T). \tag{1.16}$$
These estimates are best when $t = T$; one then retrieves the uniform estimates (1.10) up to some constant.
5. (short-time estimates) Bounds (1.11) are of order $O((Ct)^{\gamma m}/(m!)^\gamma)$, $\gamma = 1$ or β, and are obtained by m successive integrations. For linear equations, or equations

with bounded, uniformly Lipschitz coefficients, successive integrations typically yield $O((Ct)^m/m!)$. The Burgers equation, on the other hand, is strongly non-linear. While using precise Schauder estimates to obtain the gradient bound in (1.11), one stumbles into the condition $\beta < \frac{1}{2}$ at the very end of section 3 which apparently cannot be improved.

6. (blow-up of the above estimates in the vanishing viscosity limit) By undoing the initial rescaling, we obtain ν-dependent estimates,

$$||u_t||_\infty \le K_0(t),\ ||\nabla u_t||_\infty \lesssim \nu^{-1}K(t),\ ||\partial_t u_t||_\infty \lesssim \nu^{-1}K(t)^{3/2},\ ||\nabla^2 u_t||_\infty \lesssim \nu^{-2}K(t)^{3/2} \tag{1.17}$$

with $K_0(t), K_1(t)$ as in (1.5), (1.6),

$$K_2(t) := \nu||\nabla^2 u_0||_\infty + ||u_0||_\infty||\nabla u_0||_\infty + ||g_0||_\infty + \int_0^t ds(\nu||\nabla^2 g_s||_\infty + ||\partial_s g_s||_\infty),$$

$$K_{2+\alpha}(t) := \nu||\nabla^2 u_0||_\alpha + \sup_{\alpha\in[0,t]} ||g_s||_\alpha$$

and

$$K(t) := K_0(t)^2 + \nu K_1(t) + (\nu K_2(t))^{2/3} + (\nu^{1+\alpha}K_{2+\alpha}(t))^{2/(3+\alpha)}.$$

Thus, the derivative bounds $||\nabla^\kappa u_t||_\infty$, $\kappa = 1, 2$ and $||\partial_t u||_\infty$ blow up at different rates when $\nu \to 0$.

From the above theorem, one deduces easily that the solution of the Burgers equation is smooth on $\mathbb{R}_+ \times \mathbb{R}^d$, provided (i) u_0 is smooth and its derivatives are bounded; (ii) g is smooth and its derivatives are bounded on $[0, T] \times \mathbb{R}^d$ for all T:

Corollary 1 *Assume u_0 and g are smooth, and* $||\nabla^\kappa u_0||_\infty < \infty$ $(\kappa = 0, 1, 2, \ldots)$, $||\partial_t^\mu \nabla^\kappa g_t||_\infty < C(\mu, \kappa, T)$, $\mu, \kappa = 0, 1, 2, \ldots$ *for every $t \le T$. Then the Burgers equation (1.1) has a unique smooth solution u such that $||\partial_t^\mu \nabla^\kappa u_t||_\infty < C'(\mu, \kappa, T)$ for every μ, κ and $t \le T$. In particular, $C'(\mu, \kappa, t) = C'(\mu, \kappa)$ is uniform in time if $g = 0$.*

We do not prove this corollary, since it results from standard extension to higher-order derivatives of the initial estimates of Section 2, and an equally standard iterated use of Schauder estimates to derivatives of Burgers equation.

Our results extend without any modification to nonlinearities of the type $F(u) \cdot \nabla u$ with smooth matrix-valued coefficient F if F is sublinear, and even (with different scalings and exponents for the K-constants) to the case when F has polynomial growth at infinity.

Let us compare these with the results available in the literature. The one-dimensional case $d = 1$ or the irrotational d-dimensional case with $g = \nabla f$ of gradient form, is exactly solvable through the Cole-Hopf transformation $u = \nabla \log \phi$, which reduces it to a scalar, linear PDE $\partial_t \phi = \nu\Delta\phi + f\phi$; note also that $\log \phi$ is a solution of the KPZ (Kardar-Parisi-Zhang) equation. In that case the equation is immediately shown to be well-defined for every $t > 0$ under our hypotheses, and estimates similar to ours are easily obtained; specifically, in $d = 1$, an invariant measure is known to exist if g is, e.g., a space-time white noise, see Da Prato, Debussche and Temam (1994). For periodic solutions on the torus in one dimension, the above results extend to the vanishing viscosity limit, see E, Khanin, Mazel and Sinai (2000). The reader may refer, e.g., to Dermoune (1997) for a more extended bibliography.

So, our result is mostly interesting for $d \geq 2$; as mentioned above, our scheme of proof extends to more general non-linearities of the form $F(u) \cdot \nabla u$, for which the equation is not exactly solvable in general. In this setting, the classical result of 1957 is due to Kiselev and Ladyzhenskaya (1957). The authors consider solutions in Sobolev spaces and use repeatedly the energy estimates. They work on a bounded domain Ω with Dirichlet boundary conditions, but their results extend with minor modifications to the case $\Omega = \mathbb{R}^d$. If $u_0 \in \mathcal{H}^s$ with $s > d/2$, then $||u_0||_\infty < \infty$ by Sobolev's imbedding theorem. Then, the maximum principle gives $||u_t||_\infty \leq ||u_0||_\infty$ as long as the solution is classical; this key estimate allows one to bootstrap and get bounds for higher-order Sobolev spaces, which increase exponentially in time, e.g. $||u_t||_{H^1} = O(e^{c||u_0||^2_\infty t})$, as this follows from the proof of Lemma 3 in Kiselev and Ladyzhenskaya (1957). Compared to these estimates, ours present two essential improvements: (i) we do not assume any decrease of the data at spatial infinity, so that they do not necessarily belong to Sobolev spaces; (ii) more importantly, perhaps, our bounds do not increase exponentially in time; in the case the right-hand side g vanishes identically, they are even uniform in time, $K_0(t), K(t) \leq C$, where C is a constant depending only on the initial condition.

The more recent theoretical PDE literature around Burgers' equation is typically concerned with far-reaching generalization of results such as global existence in Sobolev or L^∞ spaces, long-time decay either to zero or to a scaling solution for Burgers-type equations (depending on whether the linear term or the non-linear term is more relevant), etc., for models with viscosity induced by a fractional Laplacian, see e.g. Alibaud, Imbert and Karch (2010), Biler, Funaki and Woyczynski (1998), Chan and Czubak (2010), Constantin, Cordoba and Wu (2001), Droniou, Gallouët and Vovelle (2002), Droniou and Imbert (2006). There, the general interest is to understand, using different techniques (L^p-inequalities à la Schonbek, 1980, De Giorgi theory, maximum principle, viscosity theory, etc.) the smoothing effect induced by the viscosity, understood as a possibly non-local regularization of a hyperbolic equation. Despite the fact that the maximum principle extends to vector-valued parabolic equations, the more sophisticated tools used there are applied only to scalar equations, sometimes only in one spatial dimension. It cannot be excluded that our scheme (1.4), coupled with more refined analytical tools, may help provide alternative (possibly somewhat simpler) proofs for existence of classical solutions for such equations.

Let us finally comment on our choice of approximation scheme (1.4) and provide some perspectives. Simple as it is, its power is best revealed by solving the transport equations by the method of characteristics. For zero or almost zero viscosity, these characteristics are essentially deterministic, and the scheme falls apart. The best way to understand this is to consider a compactly supported initial condition u_0. For $\nu = 0$, $u^{(0)} = u_0$, and it is easy to prove by considering the characteristics that the support of $u^{(1)}$ is included in the support of u_0. By induction, the support of the sequence $(u^{(m)})_{m\geq 0}$ does not increase, thus the solution cannot converge, in general, to the solution of the inviscid Burgers equation. Reasoning the other way round, it appears that the noisy behaviour of characteristics is a central feature of this scheme. In fact, much more can be said about the behaviour of the viscous Burgers equation by studying these, see our articles in preparation, Unterberger (2017a,b). In the closely related framework of the Kardar-Parisi-Zhang (KPZ) equation, the Hamilton-Jacobi-Bellman formalism yields a representation of the solution as the infimum of some functional depending on random characteristics, see Unterberger (2017c), which was actually the starting point for this sequence of articles on Burgers' equation. Drawing on these successes, we plan to investigate numerically the open questions on the large-time behaviour of Burgers' equation using (1.4).

1.2. Scheme of proof

Recall that we solve inductively the following linear transport equations, see (1.4),

$$u^{(-1)} := 0; \tag{1.18}$$

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla)u^{(m)} = g, \ \ u^{(m)}\Big|_{t=0} = u_0 \qquad (m \geq 0). \tag{1.19}$$

Under the first set of assumptions, standard results on linear equations show that $u^{(m)}$, $m \geq 0$ is $C^{1,2}$. *Assume* we manage to prove locally the uniform convergence of $u^{(m)}, \nabla u^{(m)}, \nabla^2 u^{(n)}$ when $m \to \infty$. Then, there exists $u \in C^{1,2}$ such that locally uniformly $u^{(m)} \to u$, $\nabla u^{(m)} \to \nabla u$, $\nabla^2 u^{(m)} \to \nabla^2 u$ and $\partial_t u^{(m)} \to \partial_t u$. Hence, $\partial_t u^{(m)} = \Delta u^{(m)} - u^{(m-1)} \cdot \nabla u^{(m)} + g$ converges locally uniformly to $\Delta u - u \cdot \nabla u + g$, and $\partial_t u = \lim_{m\to\infty} \partial_t u^{(m)} = \Delta u - u \cdot \nabla u + g$. In other words, the limit u is a $C^{1,2}$ solution of the Burgers equation.

The key point in our scheme is to prove the locally uniform convergence of $u^{(m)}$ and $\nabla u^{(m)}$, and to show uniform bounds in Hölder norms for second order derivatives $\nabla^2 u^{(m)}$, $\partial_t u^{(m)}$; a simple argument (see below) yields then the convergence of second order derivatives, allowing for application of the above elementary argument. The basic idea is to rewrite u as $\sum_{m=0}^{+\infty} v^{(m)}$, with $v^{(m)} := u^{(m)} - u^{(m-1)}$, and to show that the series is convergent, uniformly in space and locally uniformly in time.

In the sequel we fix a constant $c \geq 1$ such that Theorem 1 holds and let

$$\bar{K}_0(t) := cK_0(t), \quad \bar{K}_1(t) := cK_1(t), \quad \bar{K}(t) := cK(t) \tag{1.20}$$

to simplify notations.

The proof relies on two main ingredients: *a priori estimates* coming from the maximum principle; and *Schauder estimates*. Schauder estimates are difficult to find in a precise form, suitable for the kind of applications we have in view, so the reader will find in the appendix a precise version of these estimates, see Proposition 1, following a multi-scale proof introduced by X.-J. Wang. These imply in particular the following

Lemma 1 *Let* $0 \leq t \leq T$. *Then*

$$||\partial_t u^{(m)}||_{\alpha,[0,T]\times\mathbb{R}^d}, ||\nabla^2 u^{(m)}||_{\alpha,[0,T]\times\mathbb{R}^d} \leq \bar{K}(T)^{(3+\alpha)/2}. \tag{1.21}$$

Lemma 1 is proved in Section 3, alongside with Theorem 1.

We now use a classical result about Hölder spaces: let $C^\alpha(Q)$, with $Q \subset \mathbb{R}\times\mathbb{R}^d$ compact, be the Banach space of α-Hölder functions on Q equipped with the norm $|||u|||_\alpha := ||u||_{\infty,Q} + ||u||_{\alpha,Q}$. Then, the injection $C^{\alpha'}(Q) \subset C^\alpha(Q)$ is compact for every $\alpha' < \alpha$. In particular, Lemma 1 implies the existence of a subsequence $(u^{(n_m)})_m$ such that $\nabla^2 u^{(n_m)} \to_{m\to\infty} v$ in $C^{\alpha'}$-norm. On the other hand, as discussed in Remark 4 above, $u^{(m)} \to u$ and $\nabla u^{(m)} \to \nabla u$ in the sup norm for some $u \in C^{0,1}$. Hence, u is twice continuously differentiable in the space variables, and $\nabla^2 u = v$. Now, every subsequence $(\nabla^2 u^{(n'_m)})_m$ converges to the same limit, $\nabla^2 u$. Hence, $\nabla^2 u^{(n)} \to \nabla^2 u$ in $C^{\alpha'}$. In a similar way, one proves that u is continuously differentiable in the time variable, and $\partial_t u = \lim_{m\to\infty} \partial_t u^{(m)}$ in $C^{\alpha'}$. In particular, $u \in C^{1,2}$, and the arguments given at the very beginning of the present subsection show that u is a classical solution of the Burgers equation. Note that we may reach the same conclusion even if we do not know that the series $||\nabla u^{(m+1)} - \nabla u^{(m)}||_{\infty,Q}$ converges. Actually, the bound on $||\nabla u^{(m+1)} - \nabla u^{(m)}||_{\infty,Q}$ is the trickiest one. We felt, however, that it was one of the most unexpected estimates we had obtained, and thus worth including.

Notations. For $f, g : X \to \mathbb{R}_+$ two positive functions on a set X, we write $f(u) \lesssim g(u)$ if there exists a constant $C = C(d)$ depending only on the dimension such that $f(u) \le Cg(u)$. (If C depends on other parameters, notably on c, then we write explicitly the dependence on them, so that we make it clear that we do not get unwanted extra multiplicative factors $O(c^m)$ in the formulas which would invalidate the proofs).

2. Initial estimates

Initial estimates are different in spirit from those of the next section, since they cannot rely on Schauder estimates. Instead, we use a Gronwall-type lemma based on the maximum principle.

LEMMA 2 (GRONWALL LEMMA) *Let $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, respectively*

$$\bar\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$$

be the solution of the transport equation $(\partial_t - \Delta + b\cdot\nabla - c)\phi = f$, respectively $(\partial_t - \Delta + \bar b\cdot\nabla - \bar c)\bar\phi = \bar f$, with the same initial condition, $\phi\big|_{t=0} = \bar\phi\big|_{t=0}$; the coefficients $c = c(t,x), \bar c = \bar c(t,x) \in M_{d\times d}(\mathbb{R})$ are matrix-valued, and $b, \bar b, c, \bar c$ are assumed to be bounded and continuous, together with their first derivatives. Let $v := \bar\phi - \phi$. Then

$$\begin{aligned}||v_t||_\infty \le &\int_0^t ds\, A(s,t)\, ||\bar b_s - b_s||_\infty\, ||\nabla\phi_s||_\infty + \int_0^t ds\, A(s,t)\, |||\bar c_s - c_s|||_\infty\, ||\phi_s||_\infty \\ &+ \int_0^t ds\, A(s,t)\, ||\bar f_s - f_s||_\infty, \end{aligned} \tag{2.1}$$

where $|||\cdot|||_\infty$ is the supremum over $\mathbb{R}^d$ of the operator norm in $M_{d\times d}(\mathbb{R})$, and $A(s,t) = \exp\int_s^t |||\bar c_r|||_\infty dr$.

PROOF. By subtracting the PDEs, satisfied by ϕ and $\bar{\phi}$, one gets

$$(\partial_t - \Delta + \bar{b}\cdot\nabla - \bar{c})v = -(\bar{b}-b)\cdot\nabla\phi + (\bar{f}-f) + (\bar{c}-c)\phi. \tag{2.2}$$

A simple application of fixed-point theorem in the Banach space

$$\mathcal{W}^{1,\infty}([0,T]) := \{u \in L^\infty([0,T]\times\mathbb{R}^d) \mid |\nabla u| \in L^\infty([0,T]\times\mathbb{R}^d)\}$$

as in Amour and Ben Artzi (1998, Prop. 2.1), for T small enough, implies local-in-time existence for v; since the equation is linear, bounds for $||u_T||_{L^\infty}, ||\nabla u_T||_{L^\infty}$ are linear in terms of $||u_0||_{L^\infty}, ||\nabla u_0||_{L^\infty}$, whence the argument may be repeated over intervals of time $[T,2T]$, $[2T,3T], \ldots$ and yields global existence.

Letting $\tilde{v}_t := A(0,t)^{-1}v_t$ in (2.2), we find

$$(\partial_t - \Delta + \bar{b}\cdot\nabla - \tilde{c})\tilde{v} = A(0,t)^{-1}\left(-(\bar{b}-b)\cdot\nabla\phi + (\bar{f}-f) + (\bar{c}-c)\phi\right) \tag{2.3}$$

where now $\tilde{c}(t,x) := \bar{c}(t,x) - |||\bar{c}_t|||_\infty \le 0$. Hence, as an application of the parabolic comparison principle in unbounded space, e.g. in the form found in Kaplan (1963), in the much wider context of semi-linear parabolic equations,

$$\begin{aligned}||\tilde{v}_t||_\infty \le & \int_0^t ds\, A(0,s)^{-1}\, ||\bar{b}_s - b_s||_\infty\, ||\nabla\phi_s||_\infty + \int_0^t ds\, A(0,s)^{-1}\, |||\bar{c}_s - c_s|||_\infty\, ||\phi_s||_\infty \\ & + \int_0^t ds\, A(0,s)^{-1}\, ||\bar{f}_s - f_s||_\infty, \end{aligned} \tag{2.4}$$

implying (2.1). ∎

DEFINITION 2 *Let* $t_{init} := \inf\left\{t > 0;\ t\bar{K}(t) = 1\right\}$.

By hypothesis, $t_{init} > 0$. If $u_0 \equiv 0$ and $g \equiv 0$, then $t_{init} = +\infty$ and the solution of Burgers' equation is simply 0. The case of u_0=Cst, $\nabla g = 0$ reduces to the previous one by the generalized Galilean transformation $x \mapsto x + \int_0^t a(s)ds$, $u \mapsto u - a$ with $a(t) = u_0 + \int_0^t g_s ds$. We henceforth exclude this trivial case, so that $t_{init} \in (0,+\infty)$.

THEOREM 2 (INITIAL ESTIMATES) *Let $t \le t_{init}$. Then the following estimates hold:*

(i)

$$||u_t^{(m)}||_\infty \le K_0(t_{init}), \quad ||\nabla u_t^{(m)}||_\infty \le K(t_{init}); \qquad ||\partial_t u_t^{(m)}||_\infty, ||\nabla^2 u_t^{(m)}||_\infty \le \bar{K}(t_{init})^{3/2}. \tag{2.5}$$

Furthermore,

$$||\partial_t u_t^{(m)}||_\alpha, ||\nabla^2 u_t^{(m)}||_\alpha \le C\bar{K}(t_{init})^{(3+\alpha)/2} \tag{2.6}$$

with $C = C(d,\alpha)$.

(ii) let $m \ge 1$, then

$$||v_t^{(m)}||_\infty \le \bar{K}_0(t_{init})(\bar{K}(t_{init})t/m)^m, \qquad ||\nabla v_t^{(m)}||_\infty \le \bar{K}(t_{init})(\bar{K}(t_{init})t/m)^m. \tag{2.7}$$

Remarks.

1. Let $T \leq t_{init}$, then (2.5), (2.6) and (2.7) remain true for $t \leq T$ if one replaces $K_0(t_{init})$, $\bar{K}_0(t_{init})$, $K(t_{init})$, $\bar{K}(t_{init})$ by $K_0(T)$, $\bar{K}_0(T)$, $K(T)$, $\bar{K}(T)$. Hence Theorem 1 is proved for $t \leq t_{init}$ (actually with $\beta = 1$).
2. The value of t_{init} depends on the choice of c. We provide in the course of the proof a rather explicit minimal value of c, for which (2.5), (2.6), (2.7) hold. Further estimates in the next section may require a larger value of c.
3. From Hölder interpolation estimates (see Lemma 3), one also has a bound for lower-order Hölder norms,

$$||u^{(m)}||_\alpha \lesssim K_0(t_{init})^{1-\alpha}\bar{K}(t_{init})^\alpha + K_0^{1-\alpha/2}(t_{init})\bar{K}^{3\alpha/4}(t_{init}), \tag{2.8}$$

and, for fixed $s \leq t_{init}$,

$$||\nabla u_s^{(m)}||_\alpha \lesssim K^{1-\alpha}(t_{init})\bar{K}(t_{init})^{3\alpha/2}. \tag{2.9}$$

Proof. Let us abbreviate $K_0(t_{init})$, $\bar{K}_0(t_{init})$, $K_1(t_{init})$, $\bar{K}_1(t_{init})$, $K(t_{init})$, $\bar{K}(t_{init})$ to K_0, $\bar{K}_0$, K_1, $\bar{K}_1$, K, $\bar{K}$.

(i) We first prove estimates (i) by induction, assuming them to be proved for $m-1$. Note first that (2.5) holds true for $m = 0$ with $c = 1$, see eq. (1.13); as for (2.6),

$$\begin{aligned} ||\nabla^2 u_t^{(0)}||_\gamma &\lesssim ||\nabla^2 u_0||_\gamma + \int_0^t ds\, ||\nabla^2 e^{s\Delta} g_{t-s}||_\gamma \\ &\leq K_2^{1-\gamma/\alpha}(t_{init})K_{2+\alpha}^{\gamma/\alpha}(t_{init}) + t_{init}^{(\alpha-\gamma)/2} K_{2+\alpha}(t_{init}) \\ &\leq C(d,\alpha,\gamma)\bar{K}^{(3+\gamma)/2}, \qquad \gamma < \alpha \end{aligned} \tag{2.10}$$

as follows from Hölder interpolation inequalities (see Lemma 3) and Corollary 2. Time variations of $\nabla^2 u_t^{(0)}$ scale similarly, yielding

$$||\nabla^2 u^{(0)}||_{\gamma,[0,t_{init}]\times\mathbb{R}^d} \lesssim \bar{K}^{(3+\gamma)/2}$$

(see Lemma 4, eq. (4.8), and Corollary 2). Note that, similarly,

$$||\nabla u^{(0)}||_{\gamma,[0,t_{init}]\times\mathbb{R}^d} \lesssim \bar{K}^{(2+\gamma)/2}.$$

The estimate for $||u_t^{(m)}||_\infty$ is a direct consequence of the maximum principle. Then, $\nabla u^{(m)}$ satisfies the gradient equation

$$(\partial_t - \Delta + u^{(m-1)}\cdot\nabla + \nabla u^{(m-1)})\nabla u^{(m)} = \nabla g, \tag{2.11}$$

where $\nabla u^{(m-1)}(t,x)$ is viewed as the $d\times d$ matrix $(\partial_j u_k(t,x))_{jk}$ acting on the vector $(\partial_k u_i)_k$. Note that

$$|||\nabla u^{(m-1)}(t,x)||| \leq \sqrt{\mathrm{Tr}(\nabla u^{(m-1)}(t,x))(\nabla u^{(m-1)}(t,x))^*} = |\nabla u^{(m-1)}(t,x)|. \tag{2.12}$$

By the maximum principle,

$$||\nabla u_t^{(m)}||_\infty \leq A(0,t)\,||\nabla u_0||_\infty + \int_0^t ds\, A(s,t)\,||\nabla g_s||_\infty, \tag{2.13}$$

where $A(s,t) := \exp \int_s^t ||\nabla u_r^{(m-1)}||_\infty dr$ is the exponential amplification factor of Lemma 2. By induction hypothesis and Definition 2, $A(s,t) \le A(0,t_{init}) \le e^{t_{init}K} \le e$, hence (provided $c^2 \ge e$)

$$||\nabla u_t^{(m)}||_\infty \le eK_1 \le K. \tag{2.14}$$

To bound $\nabla^2 u_t^{(m)}$, we differentiate once more,

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla + \nabla u^{(m-1)})\nabla^2 u^{(m)} = \nabla^2 g - \nabla^2 u^{(m-1)} \nabla u^{(m)}, \tag{2.15}$$

where $\nabla u^{(m-1)}$ is viewed this time as the $d^2 \times d^2$ matrix

$$\left(\partial_{j'} u_k^{(m-1)} \delta_{k',j} + \partial_j u_k^{(m-1)} \delta_{k',j'}\right)_{(jj'),(kk')}$$

acting on the vector $(\partial^2_{kk'} u_i)_{kk'} \in \mathbb{R}^{d^2}$, and has matrix norm

$$|||\nabla u^{(m-1)}(t,x)|||_{M_{d^2\times d^2}(\mathbb{R})} \le C_d |\nabla u^{(m-1)}(t,x)|,$$

yielding an amplification factor

$$\tilde{A}(s,t) := \exp \int_s^t ||\ |||\nabla u_r^{(m-1)}(t,x)|||_{M_{d^2\times d^2}(\mathbb{R})}\ ||_\infty dr \le C'_d.$$

By the maximum principle,

$$\begin{aligned} ||\nabla^2 u_t^{(m)}||_\infty &\le C'_d \left(||\nabla^2 u_0||_\infty + \int_0^t ds \left(||\nabla^2 g_s||_\infty + ||\nabla^2 u_s^{(m-1)}||_\infty ||\nabla u_s^{(m)}||_\infty \right) \right) \\ &\le C'_d \left(||\nabla^2 u_0||_\infty + \int_0^t ds ||\nabla^2 g_s||_\infty + t_{init} \bar{K}^{3/2} K \right) \\ &\le C'_d (K_2(t_{init}) + \bar{K}^{\frac{1}{2}} K) \le C'_d (c^{-3} + c^{-1}) \bar{K}^{3/2} \le \bar{K}^{3/2}, \end{aligned} \tag{2.16}$$

provided $c \ge 2 \max(1, C'_d)$.

Similarly, $\partial_t u^{(m)}$ satisfies the transport equation

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla)\partial_t u^{(m)} = \partial_t g - \partial_t u^{(m-1)} \cdot \nabla u^{(m)}, \tag{2.17}$$

hence

$$\begin{aligned} ||\partial_t u_t^{(m)}||_\infty &\le ||\nabla^2 u_0||_\infty + ||u_0||_\infty ||\nabla u_0||_\infty + ||g_0||_\infty + \int_0^t ds ||\partial_s g_s||_\infty + t_{init} \bar{K}^{3/2} K \\ &\le K_2(t_{init}) + \bar{K}^{\frac{1}{2}} K \le (c^{-3} + c^{-1}) \bar{K}^{3/2} \le \bar{K}^{3/2}, \end{aligned} \tag{2.18}$$

provided $c \ge 2$.

Finally, we must prove the Hölder estimate (2.6): for that, we use the integral representation

$$\nabla^2 u_t^{(m)} = \nabla^2 u_t^{(0)} - \int_0^t \nabla^2 e^{(t-s)\Delta} \left((u_s^{(m-1)} \cdot \nabla) u_s^{(m)} \right) ds \tag{2.19}$$

obtained by differentiating twice the explicit solution

$$e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}((u_s^{(m-1)}\cdot\nabla)u_s^{(m)})\,ds$$

of (1.4). By Lemma 3, considering α-Hölder norms on $[0,t_{init}]\times\mathbb{R}^d$,

$$\begin{aligned}||(u_s^{(m-1)}\cdot\nabla)u_s^{(m)}||_\gamma &\le ||u_s^{(m-1)}||_\infty\,||\nabla u_s^{(m)}||_\gamma + ||\nabla u_s^{(m)}||_\infty\,||u_s^{(m-1)}||_\gamma \\ &\lesssim K_0K^{1-\gamma}\bar{K}^{3\gamma/2} + KK_0^{1-\gamma}\bar{K}^\gamma \lesssim \bar{K}^{(3+\gamma)/2}. \qquad (2.20)\end{aligned}$$

Thus, by Lemma 4,

$$\begin{aligned}||\nabla^2u_t^{(m)} - \nabla^2u_{t'}^{(m)}||_\infty &\lesssim ||\nabla^2u_t^{(0)} - \nabla^2u_{t'}^{(0)}||_\infty + \int_{t'}^t (t-s)^{\frac{\alpha}{2}-1}||(u_s^{(m-1)}\cdot\nabla)u_s^{(m)}||_\alpha ds \\ &\lesssim (t-t')^{\alpha/2}\bar{K}^{(3+\alpha)/2} \qquad (2.21)\end{aligned}$$

for $t' < t$, and (choosing any $\gamma\in(\alpha,1)$)

$$||\nabla^2u_t^{(m)}||_\alpha \lesssim ||\nabla^2u_t^{(0)}||_\alpha + C'(d,\alpha,\gamma)\bar{K}^{(3+\gamma)/2}\int_0^{t_{init}}(t-s)^{-1+(\gamma-\alpha)/2}ds \lesssim \bar{K}^{(3+\alpha)/2}, \qquad (2.22)$$

hence the result for $||\nabla^2u^{(m)}||_\alpha$. Similarly,

$$\begin{aligned}||\nabla u_t^{(m)} - \nabla u_{t'}^{(m)}||_\alpha &\lesssim ||\nabla u_t^{(0)} - \nabla u_{t'}^{(0)}||_\alpha + \int_{t'}^t (t-s)^{(\alpha-1)/2}||(u_s^{(m-1)}\cdot\nabla)u_s^{(m)}||_\alpha ds \\ &\lesssim (t-t')^{\alpha/2}\bar{K}^{(2+\alpha)/2} + (t-t')^{(\alpha+1)/2}\bar{K}^{(3+\alpha)/2} \\ &\lesssim (t-t')^{\alpha/2}\bar{K}^{(2+\alpha)/2} + (t-t')^{\alpha/2}t_{init}^{\frac{1}{2}}\bar{K}^{(3+\alpha)/2} \lesssim (t-t')^{\alpha/2}\bar{K}^{(2+\alpha)/2}, \qquad (2.23)\end{aligned}$$

hence (using Hölder interpolation inequalities once more) $||\nabla u^{(m)}||_\alpha \lesssim \bar{K}^{(2+\alpha)/2}$. From the previous bounds immediately follows

$$||\partial_t u^{(m)}||_\alpha \lesssim ||\nabla^2u^{(m)}||_\alpha + ||(u^{(m-1)}\cdot\nabla)u^{(m)}||_\alpha \lesssim \bar{K}^{(3+\alpha)/2}.$$

(ii) Apply Lemma 2 with $\phi = \bar{b} = u^{(m-1)}$, $b = u^{(m-2)}$, $\bar{\phi} = u^{(m)}$, $f = \bar{f} = g$ and $c = \bar{c} = 0$. This yields

$$||v_t^{(m)}||_\infty \le \int_0^t ds||v_s^{(m-1)}||_\infty||\nabla u_s^{(m-1)}||_\infty. \qquad (2.24)$$

Thus, using the induction hypothesis,

$$||v_t^{(m)}||_\infty \le \int_0^t ds\bar{K}_0(\bar{K}s/(m-1))^{m-1}K \le \bar{K}_0(\bar{K}t/m)^m(1-\frac{1}{m})^{-(m-1)}(K/\bar{K}) \le \bar{K}_0(\bar{K}t/m)^m \qquad (2.25)$$

($m \geq 2$) for c large enough, and

$$||v_t^{(1)}||_\infty \leq \int_0^t ds ||u_s^{(0)}||_\infty ||\nabla u_s^{(0)}||_\infty \leq K_0 K t \leq \bar{K}_0(\bar{K}t). \tag{2.26}$$

Consider now as in (i) the gradient of the transport equations of index $m-1, m$,

$$(\partial_t - \Delta + u^{(n-1)} \cdot \nabla + \nabla u^{(n-1)})\nabla u^{(n)} = \nabla g, \qquad n = m-1, m \tag{2.27}$$

and apply Lemma 2 with $\phi = \nabla u^{(m-1)}$, $\bar{\phi} = \nabla u^{(m)}$, $b = u^{(m-2)}$, $\bar{b} = u^{(m-1)}$ and $c = \nabla u^{(m-2)}$, $\bar{c} = \nabla u^{(m-1)}$. Using the induction hypothesis, one gets

$$\begin{aligned}
||\nabla v_t^{(m)}||_\infty &\leq \int_0^t ds A(s,t) ||v_s^{(m-1)}||_\infty ||\nabla^2 u_s^{(m-1)}||_\infty + \int_0^t ds\, A(s,t)\, ||\nabla v_s^{(m-1)}||_\infty ||\nabla u_s^{(m-1)}||_\infty \\
&\leq e \int_0^t ds (\bar{K}_0 \bar{K}^{3/2} + \bar{K}K)(\bar{K}s/(m-1))^{m-1} \\
&\leq e(1-\frac{1}{m})^{-(m-1)} (\bar{K}t/m)^m (\bar{K}_0 \bar{K}^{\frac{1}{2}} + K) \\
&\leq e(1-\frac{1}{m})^{-(m-1)} (c^{-\frac{1}{2}} + c^{-1}) \bar{K} (\bar{K}t/m)^m \\
&\leq \bar{K}(\bar{K}t/m)^m, \qquad m \geq 2
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
||\nabla v_t^{(1)}||_\infty &\leq \int_0^t ds \left(||u_s^{(0)}||_\infty ||\nabla^2 u_s^{(0)}||_\infty + ||\nabla u_s^{(0)}||_\infty^2 \right) \\
&\leq e(K_0 \bar{K}^{3/2} + K^2)t \leq \bar{K}(\bar{K}t)
\end{aligned} \tag{2.29}$$

for c large enough.

■

3. Proof of main theorem

By Remark 1, following Theorem 2, we may now restrict considerations to times larger than t_{init}. We fix a time horizon $T > t_{init}$ and distinguish two regimes: a *short-time regime*, $t \leq m/\bar{K}(T)$; and a *long-time regime*, $t > m/\bar{K}(T)$. Clearly, the short-time regime does not exist for $m = 0$; as already noted before (see comments after Theorem 1), this case is trivial and estimates (1.10), proven in the course of Theorem 2 in the initial regime, extend without any modification to arbitrary time. So we assume henceforth that $m \geq 1$.

Theorem 1 follows immediately from an estimate for $u^{(m)}, \nabla u^{(m)}$, valid over the whole region $t \in [t_{init}, T]$ and another estimate for $v^{(m)}, \nabla v^{(m)}$, valid only in the short-time regime. These are proved by induction.

Theorem 3 (estimates for $u^{(m)}$ and $\nabla u^{(m)}$) *Let $m \geq 1$ and $t \in [t_{init}, T]$. Then*

$$||u_t^{(m)}||_\infty \leq K_0(T),\ ||\nabla u_t^{(m)}||_\infty \leq K(T); \qquad ||\partial_t u_t^{(m)}||_\infty, ||\nabla^2 u_t^{(m)}||_\infty \leq \bar{K}(T)^{3/2}. \tag{3.1}$$

Furthermore,

$$||\partial_t u_t^{(m)}||_\alpha, ||\nabla^2 u_t^{(m)}||_\alpha \lesssim \bar{K}(T)^{(3+\alpha)/2}. \tag{3.2}$$

PROOF. As already noted, the inequality $||u_t^{(m)}||_\infty \leq K_0(T)$ follows immediately from the maximum principle, so we consider only the bound for the gradient and higher-order derivatives in (3.1). We prove it by induction on m, assuming it to be true for $m-1$. We abbreviate $K_0(T), K(T), \bar{K}(T)$ to $K_0, K, \bar{K}$.

We apply Proposition 1 on the parabolic ball $Q^{(j)} = [t - M^j, t] \times \bar{B}(x, M^{j/2})$, with $M^j := \frac{1}{2}\bar{K}(T)^{-1}$. Note that, by definition, $t - M^j \geq t_{init} - \frac{1}{2}\bar{K}(t_{init})^{-1} \geq \frac{1}{2}t_{init} > 0$. We consider first the bound (4.17) for the gradient,

$$||\nabla u^{(m)}||_{\infty,Q^{(j-1)}} \lesssim R_b^{-1}\bar{K}^{-(\alpha+1)/2}||g||_{\alpha,Q^{(j)}} + R_b^{-1}K_0\left(\bar{K}^{-(\alpha+\frac{1}{2})}R_b^{-1}||u^{(m-1)}||^2_{\alpha,Q^{(j)}} + \bar{K}^{\frac{1}{2}}\right). \quad (3.3)$$

The multiplicative factor R_b^{-1} is bounded by

$$1 + (2\bar{K})^{-\frac{1}{2}}||u^{(m-1)}||_{\infty,Q^{(j)}} \leq 1 + \bar{K}^{-\frac{1}{2}}K_0 \leq 2.$$

On the other hand, by Hölder interpolation inequalities (see Lemma 3),

$$\begin{aligned} ||u^{(m-1)}||_{\alpha,Q^{(j)}} &\lesssim K^\alpha K_0^{1-\alpha} + \bar{K}^{3\alpha/4}K_0^{1-\alpha/2} \\ &\leq (1 + c^{3\alpha/4}(K_0^2/K)^{\alpha/4})K^\alpha K_0^{1-\alpha} \\ &\leq (1 + c^{\alpha/4})K^\alpha K_0^{1-\alpha} \leq (1 + c^{\alpha/4})c^{2\alpha-2}K^{(1+\alpha)/2}. \end{aligned} \quad (3.4)$$

Hence

$$\begin{aligned} ||\nabla u^{(m)}||_{\infty,Q^{(j-1)}} &\lesssim \bar{K}^{-\alpha-1/2}K_{2+\alpha}(T) + K_0\bar{K}^{-\alpha-\frac{1}{2}} \cdot c^{\alpha/2}K^{2\alpha}K_0^{2-2\alpha} + \bar{K}^{\frac{1}{2}}K_0 \\ &\leq c^{-\alpha-1/2}K + c^{-(1+\alpha)/2}K^{\alpha-\frac{1}{2}}K_0^{3-2\alpha} + c^{-\frac{1}{2}}K, \end{aligned} \quad (3.5)$$

which is $\leq K$ for c large enough.

Bounds for higher-order derivatives $||\partial_t u_t^{(m)}||_\infty, ||\nabla^2 u_t^{(m)}||_\infty$ follow from (4.19) instead, contributing an extra $M^{-j/2} \approx \bar{K}^{\frac{1}{2}}$ multiplicative factor. They hold true for c large enough. Finally, (4.20) yields

$$\begin{aligned} ||\partial_t u^{(m)}||_{\alpha,Q^{(j-1)}}, ||\nabla^2 u^{(m)}||_{\alpha,Q^{(j-1)}} &\lesssim ||g||_{\alpha,Q^{(j)}} + K_0\left(||u^{(m-1)}||^{(2+\alpha)/(1+\alpha)}_{\alpha,Q^{(j)}} + \bar{K}^{1+\alpha/2}\right) \\ &\lesssim K_{2+\alpha}(T) + K_0 \cdot c^{(\frac{\alpha}{4}+2\alpha-2)(2+\alpha)/(1+\alpha)}K^{1+\alpha/2} + c^{-1}\bar{K}^{(3+\alpha)/2} \\ &\lesssim \bar{K}^{(3+\alpha)/2}, \end{aligned} \quad (3.6)$$

from which

$$\begin{aligned} &||\nabla^2 u^{(m)}||_{\alpha,[t_{init},T]\times\mathbb{R}^d} \lesssim \\ &\quad \sup_{(t,x)\in[t_{init},T]\times\mathbb{R}^d} ||\nabla^2 u^{(m)}||_{\alpha,Q^{(j-1)}(t,x)} + M^{-j\alpha/2}||\nabla^2 u^{(m)}||_{\infty,[t_{init},T]\times\mathbb{R}^d} \lesssim \bar{K}^{(3+\alpha)/2}, \end{aligned} \quad (3.7)$$

and similarly for $||\partial_t u^{(m)}||_{\alpha,[t_{init},T]\times\mathbb{R}^d}$.

We take the opportunity to derive from (4.18) a bound for $||\nabla u^{(m)}||_{\alpha,Q^{(j-1)}}$ (also valid for $||\nabla u^{(m)}||_{\alpha,[t_{init},T]\times\mathbb{R}^d}$) that will be helpful in the next theorem,

$$\begin{aligned}||\nabla u^{(m)}||_{\alpha,Q^{(j-1)}} &\lesssim \bar{K}^{-1/2}(1+\bar{K}^{-(1+\alpha)/2}||u^{(m-1)}||_{\alpha,Q^{(j)}})||g||_\alpha + \\ &\qquad K_0\bar{K}^{(1+\alpha)/2}\left(1+\bar{K}^{-(1+\alpha)/2}||u^{(m-1)}||_{\alpha,Q^{(j)}}+(\bar{K}^{-(1+\alpha)/2}||u^{(m-1)}||_{\alpha,Q^{(j)}})^3\right) \\ &\lesssim \bar{K}^{1+\alpha/2} \end{aligned} \tag{3.8}$$

since (from (3.4)) $||u^{(m-1)}||_{\alpha,Q^{(j)}} \lesssim \bar{K}^{(1+\alpha)/2}$. ■

Theorem 4 (short-time estimates for $v^{(m)}$ and $\nabla v^{(m)}$) *Let $m \geq 1$ and $t \in [t_{init}, \min(T, m/\bar{K}(T))]$. Then*

$$||v_t^{(m)}||_\infty \leq \bar{K}_0(T)(\bar{K}(T)t/m)^m, \qquad ||\nabla v_t^{(m)}||_\infty \leq \bar{K}(T)(\bar{K}(T)t/m)^{\beta m}. \tag{3.9}$$

Proof. We abbreviate, as before, $K_0(T), \bar{K}_0(T), K(T), \bar{K}(T)$ to K_0, $\bar{K}_0$, K, $\bar{K}$ and prove simultaneously the bounds on $||v^{(m)}||_\infty$ and $||\nabla v^{(m)}||_\infty$, assuming them to be true for $m-1$.

(i) (bound for $v_t^{(m)}$) As in the proof of Theorem 2 (ii), the case of $m = 1$ is essentially trivial: namely, using Lemma 2, we have for $t \leq \bar{K}^{-1}$

$$||v_t^{(1)}||_\infty \leq \int_0^t ds\, ||u_s^{(0)}||_\infty\, ||\nabla u_s^{(0)}||_\infty \leq K_0Kt \leq \bar{K}_0(\bar{K}t). \tag{3.10}$$

So, we now restrict the considerations to $m \geq 2$.

Assume first that $t \leq (m-1)/\bar{K}$, so that t is in the short-time regime for $u^{(m-1)}$. By Lemma 2 (see proof of Theorem 2 (ii)),

$$\begin{aligned}||v_t^{(m)}||_\infty &\leq \int_0^t ds\, ||v_s^{(m-1)}||_\infty\, ||\nabla u_s^{(m-1)}||_\infty \\ &\leq \int_0^t ds\, \bar{K}_0(\bar{K}s/(m-1))^{m-1}K \leq (\bar{K}t/(m-1))^m\bar{K}_0(K/\bar{K}) \\ &\leq c^{-1}\bar{K}_0(\bar{K}t/(m-1))^m \leq \frac{1}{2}\bar{K}_0(\bar{K}t/m)^m \end{aligned} \tag{3.11}$$

for c large enough.

For $s, t \in [(m-1)/\bar{K}, m/\bar{K}]$, one uses, instead,

$$||v_s^{(m-1)}||_\infty \leq ||u_s^{(m-1)}||_\infty + ||u_s^{(m-2)}||_\infty \leq 2K_0$$

and obtains

$$\begin{aligned}||v_t^{(m)}||_\infty &\leq \int_0^{(m-1)/\bar{K}} ds\, ||v_s^{(m-1)}||_\infty\, ||\nabla u_s^{(m-1)}||_\infty + \int_{(m-1)/\bar{K}}^{m/\bar{K}} ds\, ||v_s^{(m-1)}||_\infty\, ||\nabla u_s^{(m-1)}||_\infty \\ &\leq \frac{1}{2}\bar{K}_0(\bar{K}t/m)^m + \bar{K}^{-1}\cdot 2K_0K \\ &\leq \bar{K}_0(\bar{K}t/m)^m \end{aligned} \tag{3.12}$$

for c large enough.

(ii) (bound for $\nabla v_t^{(m)}$) We start from the observation (see (2.2)) that $v^{(m)}$ satisfies the transport equation $(\partial_t - \Delta + u^{(m-1)} \cdot \nabla)(v^{(m)}) = -v^{(m-1)} \cdot \nabla u^{(m-1)}$ and apply Schauder estimates on $Q^{(j)} = Q^{(j)}(t_0, x_0)$ as in the proof of Theorem 3, with $M^j \approx \bar{K}(T)^{-1}$, $b = u^{(m-1)}$ and $f := v^{(m-1)} \cdot \nabla u^{(m-1)}$. In the course of the proof of Theorem 3, and in (i), we obtained $||u^{(m-1)}||_{\infty,Q^{(j)}} \le K_0$ and

$$||u^{(m-1)}||_{\alpha,Q^{(j)}} \lesssim \bar{K}^{(1+\alpha)/2}, \qquad ||v^{(m)}||_{\infty,Q^{(j)}} \le \bar{K}_0(\bar{K}t/m)^m, \qquad ||\nabla u^{(m-1)}||_{\alpha,Q^{(j)}} \lesssim \bar{K}^{1+\alpha/2}. \tag{3.13}$$

Furthermore, from Hölder interpolation inequalities (see Lemma 3) and induction hypothesis,

$$||v^{(m-1)}||_{\alpha,Q^{(j)}} \lesssim \bar{K}_0^{1-\alpha}\bar{K}^{\alpha}(\bar{K}t/(m-1))^{\beta(m-1)}. \tag{3.14}$$

Hence (using once again the induction hypothesis)

$$\begin{aligned} ||f||_{\alpha,Q^{(j)}} &\lesssim ||v^{(m-1)}||_{\alpha,Q^{(j)}} ||\nabla u^{(m-1)}||_{\infty,Q^{(j)}} + ||v^{(m-1)}||_{\infty,Q^{(j)}} ||\nabla u^{(m-1)}||_{\alpha,Q^{(j)}} \\ &\lesssim (\bar{K}t/(m-1))^{\beta(m-1)}(\bar{K}_0^{1-\alpha}\bar{K}^{\alpha}K + \bar{K}_0\bar{K}^{1+\alpha/2}) \\ &\lesssim c^{-1}\bar{K}^{(3+\alpha)/2}(\bar{K}t/(m-1))^{\beta(m-1)}. \end{aligned} \tag{3.15}$$

A priori we should now use the Schauder estimate (4.18) to bound $||\nabla v^{(m)}||_{\alpha,Q^{(j-1)}}$; as in the proof of Theorem 3, $R_b^{-1} \le 2$, so

$$\begin{aligned} ||\nabla v^{(m)}||_{\infty,Q^{(j-1)}} &\lesssim \bar{K}^{-(1+\alpha)/2}||f||_{\alpha} + \bar{K}^{1/2}\bar{K}_0\left(1 + (\bar{K}^{-1-\alpha/2}||u^{(m-1)}||_{\alpha})^2\right)(\bar{K}t/m)^{\beta m} \\ &\lesssim \bar{K}^{-(1+\alpha)/2}||f||_{\alpha} + \bar{K}^{1/2}\bar{K}_0(\bar{K}t/m)^{\beta m}. \end{aligned} \tag{3.16}$$

The second term in (3.16) is bounded by $c^{-1}\bar{K}(\bar{K}t/m)^{\beta m}$, in agreement with the desired bound (3.9), but not the first one, which is bounded by

$$c^{-1}\bar{K}(\bar{K}t/(m-1))^{\beta(m-1)}.$$

In order to get an integrated bound of order $(\bar{K}t/m)^{\beta m}$ for the first term, we need a refinement of Proposition 1. Fix $(t_1, x_1) \in Q^{(j)}$. We let (for $k \ge 0$ large enough so that $Q^{(j-k)}(t_1, x_1) \subset Q^{(j)}$)

$$\tilde{v}^{(m)}(t,x) := v^{(m)}(t,x) + \int_t^{t_1} f(s, x_1)ds, \qquad (t,x) \in Q^{(j-k)}(t_1, x_1) \tag{3.17}$$

so that $\tilde{v}^{(m)}$ satisfies the modified transport equation

$$(\partial_{t'} - \Delta + v^{(m-1)} \cdot \nabla)\tilde{v}^{(m)}(t,x) = \tilde{f}(t,x) \tag{3.18}$$

with

$$\tilde{f}(t,x) := f(t,x) - f(t,x_1). \tag{3.19}$$

Note that $\nabla\tilde{v}^{(m)} = \nabla v^{(m)}, \nabla^2\tilde{v}^{(m)} = \nabla v^{(m)}$. This introduces the following modifications. First, letting $\bar{B}_1^{(j-k)} := \bar{B}(x_1, M^{(j-k)/2})$,

$$||\tilde{v}^{(m)} - v^{(m)}||_{\infty,Q^{(j-k)}(t_1,x_1)} \le \int_{t_1-M^j}^{t_1} ds||f(s)||_{\infty,\bar{B}_1^{(j-k)}} \le \bar{K}_0(\bar{K}t/m)^{\beta m} \tag{3.20}$$

as follows from (3.11), (3.12). Thus, $||\tilde{v}^{(m)}||_{\infty,Q^{(j-k)}(t_1,x_1)} \lesssim \bar{K}_0(\bar{K}t/m)^{\beta m}$ is bounded like $||v^{(m)}||_{\infty,Q^{(j-1)}}$. Second (see (4.26)), $\tilde{f}(t,x) - \tilde{f}(t_1,x_1) = f(t,x) - f(t,x_1)$ involves values of f *only* at time t. (Eventually this spares us having to bound inductively $\partial_t v^{(m)}$).

We now go through the proof of Proposition 1, writing $\tilde{v}^{(m)}(t_1,x_1)$ as the sum of a series $\tilde{v}^{(m)}_{k_1+1}(t_1,x_1) + \sum_{k=k_1+1}^{\infty}(\tilde{v}^{(m)}_{k+1} - \tilde{v}^{(m)}_k)(t_1,x_1)$, and binding only $||\nabla\tilde{v}||_\infty = ||\nabla v||_\infty$ and $||\nabla^2\tilde{v}||_\infty = ||\nabla^2 v||_\infty$. Instead of (4.27), we get from the maximum principle

$$\sup_{Q_1^{(j-1-k)}} |\tilde{v}^{(m)}_{k+1} - \tilde{v}^{(m)}_k| \lesssim M^{(j-k)(1+\alpha/2)} \left(\fint_{t_1-M^{j-1-k}}^{t_1} ...ds||f(s)||_{\alpha,\bar{B}_1^{(j-1-k)}} + ||u^{(m-1)}||_\alpha \sup_{Q_1^{(j-1-k)}} \nabla\tilde{v}^{(m)} \right), \tag{3.21}$$

where

$$\fint_{t-M^{j-1-k}}^{t} (\,\cdot\,)ds := M^{-(j-1-k)} \int_{t_1-M^{j-1-k}}^{t_1} (\,\cdot\,)ds$$

is the average over the time interval $[t_1 - M^{j-1-k}, t_1]$. We have proved above that

$$||f(s)||_{\alpha,\bar{B}_1^{(j)}} \lesssim c^{-1}\bar{K}^{(3+\alpha)/2}(\bar{K}s/(m-1))^{\beta(m-1)};$$

thus (by explicit computation)

$$\begin{aligned}\fint_{t_1-M^{j-1-k}}^{t_1} ds||f(s)||_{\alpha,\bar{B}_1^{(j-1-k)}} &\lesssim c^{-1}\bar{K}^{(3+\alpha)/2} \fint_{t-M^{j-1-k}}^{t} ds\,(\bar{K}s/(m-1))^{\beta(m-1)} \\ &\equiv c^{-1}\bar{K}^{(3+\alpha)/2}(\bar{K}t/(m-1))^{\beta(m-1)}a_k,\end{aligned} \tag{3.22}$$

with

$$a_k := M^{k-j}t^{-\beta(m-1)}\frac{1}{\beta(m-1)+1}\left(t^{\beta(m-1)+1} - (t - M^{j-1-k})^{\beta(m-1)+1}\right).$$

Let $k_0 := \inf\{k \ge 0; M^{j-1-k} < t/m\}$; since $M^{j-1} \gtrsim t/m$ by hypothesis, $M^{j-1-k_0} \approx t/m$. For $k > k_0$, $a_k \approx 1$, as follows from Taylor's formula; bounding all $a_k, k \ge 0$ by 1 would yield the estimate (3.16). However, for $k \le k_0$, $a_k \lesssim M^{k-j}\frac{t}{m}$, which is a much better bound for $k_0 - k$ large. Summarizing, the only change in the right-hand side of (4.34) is that $||f||_\alpha$ may be replaced by

$$\sum_k M^{-k\alpha/2} \fint_{t_1-M^{j-1-k}}^{t_1} ds\,||f(s)||_{\alpha,\bar{B}_1^{(j-1-k)}} \lesssim c^{-1}\bar{K}^{(3+\alpha)/2}(\bar{K}t/(m-1))^{\beta(m-1)}(A_1 + A_2),$$

$$(3.23)$$

where

$$A_1 := \sum_{k\geq k_0} M^{-k\alpha/2} \lesssim (\bar{K}t/m)^{\alpha/2} \tag{3.24}$$

and similarly

$$A_2 := \sum_{k=0}^{k_0-1} M^{-k\alpha/2} M^{k-j}\frac{t}{m} \lesssim M^{k_0(1-\alpha/2)}(\bar{K}t/m) \approx (\bar{K}t/m)^{\alpha/2}. \tag{3.25}$$

Altogether, with respect to the rougher bound (3.16), we have gained a small multiplicative factor of order $A_1 + A_2 \lesssim (\bar{K}t/m)^{\beta}$, with $\beta := \alpha/2$. Thus

$$\begin{aligned} \|\nabla v^{(m)}\|_{\infty,Q^{(j-1)}} &\lesssim c^{-1}\bar{K}(\bar{K}t/(m-1))^{\beta(m-1)} \cdot (\bar{K}t/m)^{\beta} + c^{-1}\bar{K}(\bar{K}t/m)^{\beta m} \\ &\lesssim c^{-1}\bar{K}(\bar{K}t/m)^{\beta m}. \end{aligned} \tag{3.26}$$

■

4. Hölder estimates

We prove in this section the elementary Hölder estimates, together with a precise form of the Schauder estimates, which is crucial in the proof of Theorem 1, provided in Section 3.

Definition 3 (Hölder semi-norms) *Let $\gamma \in (0,1)$.*

1. *$f_0 : \mathbb{R}^d \to \mathbb{R}$ is γ-Hölder continuous if*

$$||f_0||_{\gamma} := \sup_{x,x'\in\mathbb{R}^d} \frac{|f_0(x) - f_0(x')|}{|x-x'|^{\gamma}} < \infty.$$

2. *$f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is γ-Hölder continuous if*

$$||f||_{\gamma} := \sup_{(t,x),(t',x')\in\mathbb{R}_+\times\mathbb{R}^d} \frac{|f(t,x) - f(t',x')|}{|x-x'|^{\gamma} + |t-t'|^{\gamma/2}} < \infty.$$

In the denominator appearing in the definition of $||f||_{\gamma}$, we find a power of the *parabolic distance*, $d_{par}((t,x),(t',x')) = |x-x'| + \sqrt{|t-t'|}$. Note that $||\ ||_{\gamma}$ is only a semi-norm, since $||1||_{\gamma} = 0$. We also define Hölder semi-norms for functions restricted to $Q_0 \subset \mathbb{R}_+ \times \mathbb{R}^d$ or $Q \subset \mathbb{R}^d$ compact, with the obvious definitions,

$$||f_0||_{\gamma,Q_0} := \sup_{x,x'\in Q_0} \frac{|f_0(x) - f_0(x')|}{|x-x'|^{\gamma}}, \qquad ||f||_{\gamma,Q} := \sup_{(t,x),(t',x')\in Q} \frac{|f(t,x) - f(t',x')|}{|x-x'|^{\gamma} + |t-t'|^{\gamma/2}}. \tag{4.1}$$

Remark. For $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, we use in this article either the parabolic Hölder semi-norm $||f||_{\alpha,Q}$ or the isotropic Hölder semi-norm $||f(t)||_{\alpha,Q_0}$ for $t \in \mathbb{R}_+$ fixed. The distinction is really important in the proof of Theorem 4 (ii). Clearly, $||f(t)||_{\alpha,Q_0} \leq ||f||_{\alpha,I\times Q_0}$ if I is some time interval containing t.

Lemma 3 (Hölder interpolation estimates)

1. *(on $\mathbb{R}^d$) Let $Q_0 \subset \mathbb{R}$ be a convex set, and $u_0 : Q_0 \to \mathbb{R}$ such that $||u_0||_{\infty,Q_0}$, $||\nabla u_0||_{\infty,Q_0} < \infty$. Then*

$$||u_0||_{\alpha,Q_0} \le ||u_0||^{1-\alpha}_{\infty,Q_0} ||\nabla u_0||^{\alpha}_{\infty,Q_0}, \qquad \alpha \in (0,1). \tag{4.2}$$

2. *(on $\mathbb{R}_+ \times \mathbb{R}^d$) Let $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$ be a convex set, and $u : \mathbb{R}^d \to \mathbb{R}$ such that $||u||_{\infty,Q}, ||\nabla u_0||_{\infty,Q}, ||\partial_t u_0||_{\infty,Q} < \infty$. Then*

$$||u||_{\alpha,Q} \le 2\left(||u||^{1-\alpha}_{\infty,Q} ||\nabla u||^{\alpha}_{\infty,Q} + ||u||^{1-\alpha/2}_{\infty,Q} ||\partial_t u||^{\alpha/2}_{\infty,Q} \right), \qquad \alpha \in (0,1). \tag{4.3}$$

Proof. (See Lieberman, 1996) We prove (ii). Let $X = (t, x)$ and $X' = (t', x')$ in Q, then

$$\begin{aligned} |u(X) - u(X')| &= \left| \int_0^1 \frac{d}{d\tau} u((1-\tau)X + \tau X') d\tau \right| \\ &\le |t - t'| \, ||\partial_t u||_{\infty,Q} + |x - x'| \, ||\nabla u||_{\infty,Q} \\ &\le 2 \max \left(|t - t'| \, ||\partial_t u||_{\infty,Q}, |x - x'| \, ||\nabla u||_{\infty,Q} \right). \end{aligned} \tag{4.4}$$

On the other hand, $|u(X) - u(X')| \le 2||u||_\infty$. Hence

$$|u(X) - u(X')| \le 2 \max \left(||u||^{1-\alpha/2}_{\infty,Q} ||\partial_t u||^{\alpha/2}_{\infty,Q}, ||u||^{1-\alpha}_{\infty,Q} ||\nabla u||^{\alpha}_{\infty,Q} \right). \tag{4.5}$$

■

Lemma 4 *Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be α-Hölder. Then*

$$||\nabla^\kappa (e^{t\Delta} u_0)||_\infty \le C(d, \kappa, \alpha) t^{(\alpha-\kappa)/2} \, ||u_0||_\alpha \qquad (\kappa \ge 1); \tag{4.6}$$

$$||\nabla^2 (e^{t\Delta} u_0)||_\gamma \le C'(d, \gamma, \alpha) t^{-1+(\alpha-\gamma)/2} \, ||u_0||_\alpha \qquad (\gamma \in (0,1)); \tag{4.7}$$

$$||e^{t\Delta} u_0 - e^{t'\Delta} u_0||_\infty \le C''(d, \alpha)(t - t')^{\alpha/2} \, ||u_0||_\alpha \qquad (\alpha \in (0,1), \ t > t' > 0). \tag{4.8}$$

Proof. (4.7) follows by Lemma 3 from the bounds (4.6) with $\kappa = 2, 3$. Thus, let us first prove (4.6). The regularizing operator $e^{t\Delta}$ is defined by convolution with respect to the heat kernel p_t. By translation invariance, it is enough to bound the quantity

$$I(\varepsilon) := \nabla^{\kappa-1}(e^{t\Delta} u_0)(0) - \nabla^{\kappa-1}(e^{t\Delta} u_0)(\varepsilon)$$

in the limit $\varepsilon \to 0$. The quantities in (4.6) are invariant through the substitution $u_0 \to u_0 - u_0(0)$, so we assume that $u_0(0) = 0$. We may also assume $|\varepsilon| \ll \sqrt{t}$. Let $A := \varepsilon^\beta t^{(1-\beta)/2}$ with $\beta = (1 - \alpha)/d$; note that $|\varepsilon| \ll A \ll \sqrt{t}$. We split the integral into three parts, $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$, with

$$I_1(\varepsilon) := \int_{|x|<A} dx \, \nabla^{\kappa-1} p_t(x)(u_0(x) - u_0(x + \varepsilon)),$$

$$I_2(\varepsilon) := \int_{|x|>A} dx \, (\nabla^{\kappa-1} p_t(x) - \nabla^{\kappa-1} p_t(x + \varepsilon))(u_0(x) - u_0(0)) \tag{4.9}$$

$$I_3(\varepsilon) = \left(\int_{|x|>A} dx - \int_{|x-\varepsilon|>A} dx\right) \nabla^{\kappa-1} p_t(x+\varepsilon)(u_0(x) - u_0(0)). \tag{4.10}$$

We use $|u_0(x) - u_0(x+\varepsilon)| \le ||u_0||_\alpha \, |\varepsilon|^\alpha$ in the first integral, and get

$$I_1(\varepsilon) \lesssim ||u_0||_\alpha A^d t^{-(\kappa+d-1)/2} |\varepsilon|^\alpha = ||u_0||_\alpha t^{(\alpha-\kappa)/2} |\varepsilon|. \tag{4.11}$$

For the second integral, we use

$$|\nabla^{\kappa-1} p_t(x) - \nabla^{\kappa-1} p_t(x+\varepsilon)| \lesssim \frac{|\varepsilon|}{t^{\kappa/2}} p_t(x) \quad \text{ and } \quad |u_0(x) - u_0(0)| \le ||u_0||_\alpha |x|^\alpha,$$

yielding the same estimate. Finally, the integration volume in the third integral is $O(A^{d-1}|\varepsilon|)$, hence

$$I_3(\varepsilon) \lesssim ||u_0||_\alpha A^{d-1} |\varepsilon| t^{-(\kappa-1)/2} A^\alpha \lesssim ||u_0||_\alpha A^d t^{-(\kappa+d-1)/2} |\varepsilon|^\alpha \;\cdot\; (|\varepsilon|/A)^{1-\alpha}$$

is negligible with respect to the first integral (compare with (4.11)). Taking $\varepsilon \to 0$, this gives the desired bound for $||\nabla^\kappa(e^{t\Delta} u_0)||_\infty$.

Finally, (4.8) may be obtained through the use of the fractional derivative

$$|\nabla|^\alpha : u_0 \mapsto \left(|\nabla|^\alpha u_0 : x \mapsto \int d\xi dy |\xi|^\alpha e^{\mathrm{i}(x-y)\xi} u_0(y) \right),$$

namely,

$$\begin{aligned} |(e^{t\Delta} u_0 - e^{t'\Delta} u_0)(x)| &= \left| \int_{t'}^t ds \int dy \, \partial_s p_s(x-y) u_0(y) \right| = \left| \int_{t'}^t ds \int dy \, \Delta p_s(x-y) u_0(y) \right| \\ &\lesssim \int_{t'}^t ds \int dy \, | \, |\nabla|^{2-\alpha/2} p_s(x-y)| \;\; | \, |\nabla|^\alpha u_0(y)| \lesssim (t^{\alpha/2} - (t')^{\alpha/2}) \, ||u_0||_\alpha \\ &\lesssim (t-t')^{\alpha/2} ||u_0||_\alpha. \end{aligned} \tag{4.12}$$

■

Corollary 2 *Let $g : [0,t] \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $(g_s)_{s\in[0,t]}$ are uniformly α-Hölder, and $\gamma < \alpha$. Then $s \mapsto ||\nabla^2(e^{(t-s)\Delta} g_s)||_\gamma$ is L^1_{loc} and, for $0 < t' < t$,*

$$\int_{t'}^t ds \, ||\nabla^2(e^{(t-s)\Delta} g_s)||_\gamma \le C''(d,\gamma,\alpha)(t-t')^{(\alpha-\gamma)/2} \sup_{s\in[t',t]} ||g_s||_\alpha. \tag{4.13}$$

We now turn to our Schauder estimates. The multi-scale proof of the Proposition below is inspired by Wang (2006). We fix a constant $M > 1$, e.g. $M = 2$, for a dyadic scale decomposition.

Definition 4 (parabolic balls) *Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ and $j \in \mathbb{Z}$. Then, the scale j parabolic ball issued from (t_0, x_0) is the closed subset $Q^{(j)}(t_0, x_0) := \{(t,x) \in \mathbb{R} \times \mathbb{R}^d;\ t_0 - M^j \le t \le t_0, x \in \bar{B}(x_0, M^{j/2})\}$.*

The set $\{(t,x)\mid t\le t_0, d_{par}((t,x),(t_0,x_0))\le M^{j/2}\}$ is comparable to $Q^{(j)}(t_0,x_0)$, in the sense that there exist $\delta k_0,\delta k_1\ge 0$ such that

$$Q^{(j)}(t_0,x_0)\subset\{(t,x)\mid t\le t_0, d_{par}((t,x),(t_0,x_0))\le M^{(j+\delta k_0)/2}\}\subset Q^{(j+\delta k_0+\delta k_1)/2}(t_0,x_0)$$

(one may actually choose $\delta k_1=0$), which is why $Q^{(j)}(t_0,x_0)$ is called a 'ball'; but mind the causality condition $t\le t_0$. In the sequel we let $\delta k=\delta k(M)$ be some large enough integer, depending only on M, used in several occasions to make different parabolic balls fit exactly into each other. The main property of parabolic balls in our context is the simple scaling property for locally bounded solutions u of the heat equation $(\partial_t-\Delta)u=0$: for all

$$\kappa=(\kappa_1,\ldots,\kappa_d),\kappa_1,\ldots,\kappa_d\ge 0, |\nabla^\kappa u(t_0,x_0)|\lesssim (M^{-j/2})^{|\kappa|}\sup_{\partial_{par}Q^{(j)}(t_0,x_0)}|u| (|\kappa|=\kappa_1+\ldots+\kappa_d),$$

where

$$\partial_{par}Q^{(j)}(t_0,x_0):=\Big(\{t_0-M^j\}\times\bar{B}(x_0,M^{j/2})\Big)\cup\Big([t_0-M^j,t_0)\times\partial B(x_0,M^{j/2})\Big)$$

is the *parabolic boundary* of $Q^{(j)}(t_0,x_0)$. From this, we simply deduce the following: let

$$Q^{(j)}_{(k)}(t_0,x_0):=\{(t,x)\in Q^{(j)}(t_0,x_0)\mid d_{par}((t,x),\partial_{par}Q^{(j)}(t_0,x_0))\ge M^k\}\qquad (k\le j), \tag{4.14}$$

then

$$\sup_{Q^{(j)}_{(k)}(t_0,x_0)}|\nabla^\kappa u|\lesssim (M^{-k/2})^{|\kappa|}\sup_{Q^{(j)}(t_0,x_0)}|u|,$$

which is a quantitative version of the well-known regularizing property of the heat equation: if u is bounded on some j scale parabolic ball $Q^{(j)}$, then $\nabla^\kappa u$ is bounded *away* from the parabolic boundary of $Q^{(j)}$. In particular, since $Q^{(j-1)}(t_0,x_0)\subset Q^{(j)}_{(j-\delta k)}(t_0,x_0)$, one has:

$$\sup_{Q^{(j-1)}(t_0,x_0)}|\nabla^\kappa u|\lesssim (M^{-j/2})^{|\kappa|}\sup_{Q^{(j)}(t_0,x_0)}|u|.$$

Proposition 1 (Schauder estimates) *Let v solve the linear parabolic PDE*

$$(\partial_t-\Delta+a(t,x))u(t,x)=b(t,x)\cdot\nabla u(t,x)+f(t,x) \tag{4.15}$$

on the parabolic ball $Q^{(j)}:=Q^{(j)}(t_0,x_0)$. Assume: u is bounded; $a\ge 0$;

$$\|f\|_\alpha:=\|f\|_{\alpha,Q^{(j)}}:=\sup_{(t,x),(t',x')\in Q^{(j)}}\frac{|f(t,x)-f(t',x')|}{|x-x'|^\alpha+|t-t'|^{\alpha/2}}<\infty \tag{4.16}$$

for some $\alpha\in(0,1)$, and similarly $\|a\|_\alpha,\|b\|_\alpha<\infty$. Then

$$\sup_{Q^{(j-1)}}|\nabla u|\lesssim M^{j/2}R_b^{-1}\left\{M^{j\alpha/2}\|f\|_\alpha+\Big(M^{j\alpha}R_b^{-1}\|b\|_\alpha^2+M^{j\alpha/2}\|a\|_\alpha+M^{-j}\Big)\sup_{Q^{(j)}}|u|\right\}, \tag{4.17}$$

$$||\nabla u||_{\alpha,Q^{(j-1)}} \lesssim M^{-j\alpha/2} R_b^{-(1+\alpha)/2} \ \Big\{ M^{j(1+\alpha)/2} ||f||_\alpha + \Big(M^{j(1+\alpha+\alpha^2)/2\alpha} R_b^{-\frac{1}{2}(1+\alpha)/\alpha} ||b||_\alpha^{(1+\alpha)/\alpha} + M^{j(1+\alpha)/2} ||a||_\alpha + M^{-j/2} \Big) \sup_{Q^{(j)}} |u| \Big\}, \tag{4.18}$$

$$\sup_{Q^{(j-1)}} |\partial_t u|, \sup_{Q^{(j-1)}} |\nabla^2 u| \lesssim R_b^{-1} \left\{ M^{j\alpha/2} ||f||_\alpha + \Big(M^{j\alpha} R_b^{-1} ||b||_\alpha^2 + M^{j\alpha/2} ||a||_\alpha + M^{-j} \Big) \sup_{Q^{(j)}} |u| \right\}, \tag{4.19}$$

and for every $\alpha' > \alpha$,

$$||\partial_t u||_{\alpha,Q^{(j-1)}}, ||\nabla^2 u||_{\alpha,Q^{(j-1)}} \lesssim M^{-j\alpha/2} R_b^{-(1+\alpha'/2)} \ \Big\{ M^{j\alpha/2} ||f||_\alpha + \Big(M^{j\alpha/2} R_b^{-\frac{1}{2}(2+\alpha')/(1+\alpha)} ||b||_\alpha^{(2+\alpha)/(1+\alpha)} + M^{j\alpha/2} ||a||_\alpha + M^{-j} \Big) \sup_{Q^{(j)}} |u| \Big\}, \tag{4.20}$$

where $R_b := \Big(1 + M^{j/2} |b(t_0, x_0)|\Big)^{-1}$.

Remark: Upon removing the condition $a \geq 0$, we would get the same estimates, multiplied by $e^{M^j \sup_{Q^{(j)}}(-a)}$.

PROOF. Let

$$\tilde{u}(\tilde{t}, \tilde{x}) := u(M^j \tilde{t}, M^{j/2} \tilde{x}),$$
$$\tilde{b}(\tilde{t}, \tilde{x}) := M^{j/2} b(M^j \tilde{t}, M^{j/2} \tilde{x}),$$
$$\tilde{f}(\tilde{t}, \tilde{x}) := M^j f(M^j \tilde{t}, M^{j/2} \tilde{x}),$$
$$\tilde{a}(\tilde{t}, \tilde{x}) := M^j a(M^j \tilde{t}, M^{j/2} \tilde{x}).$$

Then, the PDE $(\partial_t - \Delta + a)u = b \cdot \nabla u + f$ on $Q^{(j)}$ reduces to an equivalent PDE, $(\partial_{\tilde{t}} - \tilde{\Delta} + \tilde{a})\tilde{u} = \tilde{b} \cdot \tilde{\nabla} \tilde{u} + \tilde{f}$ on a parabolic ball $\tilde{Q}$ of size unity. *Assume* (leaving out for sake of conciseness the powers of $R_b = (1 + |\tilde{b}(t_0, x_0)|)^{-1}$) that we have proved an inequality of the type

$$\sup_{\tilde{Q}^{(-1)}} |\tilde{\nabla}^\kappa \tilde{u}| \lesssim \left(||\tilde{f}||_\alpha + (||\tilde{b}||_\alpha^\beta + ||\tilde{a}||_\alpha + 1) \sup_{\tilde{Q}} |\tilde{u}| \right), \tag{4.21}$$

respectively

$$||\tilde{\nabla}^\kappa \tilde{u}||_{\alpha,\tilde{Q}^{(-1)}} \lesssim \left(||\tilde{f}||_\alpha + (||\tilde{b}||_\alpha^\beta + ||\tilde{a}||_\alpha + 1) \sup_{\tilde{Q}} |\tilde{u}| \right). \tag{4.22}$$

By rescaling, we get

$$\sup_{Q^{(j-1)}} |\nabla^\kappa u| \lesssim (M^{-j/2})^\kappa \left(M^{j(1+\alpha/2)} ||f||_\alpha + \left(((M^{j/2})^{1+\alpha} ||b||_\alpha)^\beta + (M^j)^{1+\alpha/2} ||a||_\alpha + 1 \right) \sup_{Q^{(j)}} |u| \right), \tag{4.23}$$

$$||\tilde{\nabla}^\kappa u||_{\alpha, Q^{(j-1)}} \lesssim$$
$$(M^{-j/2})^{\kappa+\alpha} \left(M^{j(1+\alpha/2)} ||f||_\alpha + \left(((M^{j/2})^{1+\alpha} ||b||_\alpha)^\beta + (M^j)^{1+\alpha/2} ||a||_\alpha + 1 \right) \sup_{Q^{(j)}} |u| \right). \tag{4.24}$$

This gives the correct scaling factors in (4.17, 4.18, 4.19, 4.20). Thus, we may assume that $j = 0$. In the sequel we write, for short, $|| \cdot ||_\alpha$ instead of $|| \cdot ||_{\alpha, Q^{(0)}}$, and $|| \cdot ||_\infty$ instead of $\sup_{Q^{(0)}} | \cdot |$.

The general principle underlying the proof of the Schauder estimates in Wang (2006) is the following. Let $(t_1, x_1) \in Q^{(0)}_{(-k_1)}$. One rewrites $u(t_1, x_1)$ as the sum of the series $u(t_1, x_1) = u_{k_1+1}(t_1, x_1) + \sum_{k=k_1+1}^{+\infty} (u_{k+1}(t_1, x_1) - u_k(t_1, x_1))$, where u_k, $k \geq k_1 + 1$ is the solution on $Q_1^{(-k)} := Q^{(-k)}(t_1, x_1)$ of the 'frozen' PDE

$$(\partial_t - \Delta + a(t_1, x_1)) u_k(t, x) = b(t_1, x_1) \cdot \nabla u_k(t, x) + f(t_1, x_1) \tag{4.25}$$

with the initial-boundary condition $u_k\big|_{\partial_{par} Q_1^{(-k)}} = u\big|_{\partial_{par} Q_1^{(-k)}}$. We split the proof into several steps.

(i) (estimates for $|u_{k+1} - u_k|$) One first remarks that $u_k - u$, $k \geq k_1 + 1$ solves on $Q_1^{(-k)}$ the heat equation

$$(\partial_t - \Delta + a(t_1, x_1) - b(t_1, x_1) \cdot \nabla)(u_k - u) = (b(t_1, x_1) - b) \cdot \nabla u + (f(t_1, x_1) - f) - (a(t_1, x_1) - a) u \tag{4.26}$$

with zero initial-boundary condition $(u_k - u)\big|_{\partial_{par} Q_1^{(-k)}} = 0$, implying by the maximum principle

$$\sup_{Q_1^{(-k-1)}} |u_{k+1} - u_k| \leq \sup_{Q_1^{(-k-1)}} |u_{k+1} - u| + \sup_{Q_1^{(-k)}} |u_k - u|$$
$$\lesssim M^{-k(1+\alpha/2)} \left(||f||_\alpha + ||a||_\alpha ||u||_\infty + ||b||_\alpha \sup_{Q_1^{(-k)}} |\nabla u| \right). \tag{4.27}$$

(ii) (estimates for higher-order derivatives of u_{k_1+1}) Recall that u_{k_1+1} is a solution of the heat equation $(\partial_t - \Delta - b(t_1, x_1) \cdot \nabla) u_{k_1+1} = f(t_1, x_1)$ with initial-boundary condition $u_{k_1+1}\big|_{\partial_{par} Q_1^{-(k_1+1)}} = u\big|_{\partial_{par} Q_1^{-(k_1+1)}}$.
Assume first $|b(t_1, x_1)| \lesssim 1$. As follows from the standard estimates, recalled before Proposition 1.

$$||\nabla u_{k_1+1}||_{\alpha, Q_1^{-(k_1+2)}} \lesssim (M^{k_1/2})^{1+\alpha} ||u||_\infty, \quad \sup_{Q_1^{-(k_1+2)}} |\partial_t u_{k_1+1}|, \sup_{Q_1^{-(k_1+2)}} |\nabla^2 u_{k_1+1}| \lesssim M^{k_1} ||u||_\infty,$$

$$(4.28)$$

$$||\nabla^2 u_{k_1+1}||_{\alpha, Q_1^{-(k_1+2)}} \lesssim (M^{k_1})^{1+\alpha/2} ||u||_\infty. \tag{4.29}$$

If $|b(t_0,x_0)| \gg 1$, then one makes the Galilean transformation $x \mapsto x - b(t_0,x_0)t$ to get rid of the drift, after which the boundary of $Q_1^{-(k_1+1)}$ lies at distance $R = O(M^{-k_1/2}/|b(t_0,x_0)|)$ instead of $O(M^{-k_1/2})$ of (t_1,x_1); thus, in general,

$$\begin{aligned} &||\nabla u_{k_1+1}||_{\alpha, Q_1^{-(k_1+2)}} \lesssim R_b^{-(1+\alpha)/2} (M^{k_1/2})^{1+\alpha} ||u||_\infty, \\ &\sup_{Q_1^{-(k_1+2)}} |\partial_t u_{k_1+1}|, \ \sup_{Q_1^{-(k_1+2)}} |\nabla^2 u_{k_1+1}| \lesssim R_b^{-1} M^{k_1} ||u||_\infty, \end{aligned} \tag{4.30}$$

$$||\nabla^2 u_{k_1+1}||_{\alpha, Q_1^{-(k_1+2)}} \lesssim R_b^{-(1+\alpha/2)} (M^{k_1})^{1+\alpha/2} ||u||_\infty. \tag{4.31}$$

(iii) (estimates for higher-order derivatives of $u_{k+1} - u_k$) Similarly to (ii), we note that $u_{k+1} - u_k$ is a solution on $Q_1^{(-k-1)}$ of the heat equation

$$(\partial_t - \Delta + a(t_1,x_1) - b(t_1,x_1) \cdot \nabla)(u_{k+1} - u_k) = 0.$$

Thus

$$\sup_{Q_1^{(-k-2)}} |\partial_t (u_{k+1} - u_k)|, \ \sup_{Q_1^{(-k-2)}} |\nabla^2 (u_{k+1} - u_k)| \lesssim M^k R_b^{-1} \sup_{Q_1^{(-k-1)}} |u_{k+1} - u_k|, \tag{4.32}$$

$$||\nabla^2 (u_{k+1} - u_k)||_{\alpha', Q_1^{(-k-2)}} \lesssim (M^k)^{1+\alpha'/2} R_b^{-(1+\alpha'/2)} \sup_{Q_1^{(-k-1)}} |u_{k+1} - u_k| \tag{4.33}$$

is bounded using (i) in terms of R_b, $||b||_\alpha$, $||f||_\alpha$, and $\sup_{Q_1^{(-k)}} |\nabla u|$.

(iv) (Schauder estimates for higher-order derivatives of u) Summing up the estimates in (i), (ii), (iii), and noting that $\cdots \subset Q_1^{(-k_1-2)} \subset Q_1^{(-k_1-1)} \subset Q^{(0)}_{(-k_1-\delta k)}$ for $\delta k = \delta k(M)$ large enough, one obtains

$$\begin{aligned} &M^{-k_1} \sup_{Q^{(0)}_{(-k_1)}} |\partial_t u|, M^{-k_1} \sup_{Q^{(0)}_{(-k_1)}} |\nabla^2 u| \lesssim \\ &\qquad R_b^{-1} \left\{ (M^{-k_1})^{1+\alpha/2} \left(||f||_\alpha + ||a||_\alpha ||u||_\infty + ||b||_\alpha \sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| \right) + ||u||_\infty \right\}. \end{aligned} \tag{4.34}$$

By interpolation (see immediately thereafter), $\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u|$ is bounded in terms of $||u||_\infty$ and $\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla^2 u|$. Thus, in principle, (4.34) gives a bound for $\nabla^2 u$. *However*, since $Q^{(0)}_{(-k_1-\delta k)} \supsetneq Q^{(0)}_{(-k_1)}$, one *cannot* fix k_1. Instead we shall bound $\sup_{k_1} M^{-k_1} \sup_{Q^{(0)}_{(-k_1)}} |\nabla^2 u|$, and similarly for the different gradient/Hölder norms considered in the Proposition. This explains *why* ultimately we must consider the values of ∇u, $\nabla^2 u$ on the whole parabolic ball $Q^{(0)}$, not only on the subset $Q^{(-1)}$, where our results are stated.

Now

$$\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| \lesssim \left(\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla^2 u| \right)^{1/2} (||u||_\infty)^{1/2} \lesssim \varepsilon^2 \sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla^2 u| + \varepsilon^{-2} ||u||_\infty \quad (4.35)$$

for every $\varepsilon > 0$. Hence (using (4.34)), choosing $\varepsilon^2 \approx R_b/||b||_\alpha$, one gets

$$\sup_{k_1 \geq 0} M^{-k_1} \sup_{Q^{(0)}_{(-k_1)}} |\nabla^2 u| \lesssim R_b^{-1} \left\{ (M^{-k_1})^{1+\alpha/2} \left(||f||_\alpha + (||a||_\alpha + R_b^{-1} ||b||^2_\alpha) ||u||_\infty \right) + ||u||_\infty \right\}, \quad (4.36)$$

implying, in particular, the bound (4.19) for $\nabla^2 u$, from which (4.35, 4.34) yields the bound (4.19) for $\partial_t u$.
Using the estimates (4.19) and (4.35) with $\varepsilon = 1$ yields also the gradient bound (4.17).

(v) (Schauder estimates for Hölder norms)
Let us now bound

$$||\nabla^2 u||_{\alpha, Q^{(0)}_{-(k_1-1)}} \approx \sup_{(t_1,x_1),(t_2,x_2) \in Q^{(0)}_{-(k_1-1)}} \frac{|\nabla^2 u(t_2,x_1) - \nabla^2 u(t_2,x_2)|}{d_{par}((t_1,x_1),(t_2,x_2))^\alpha}$$

or, equivalently, $||\partial_t u||_{\alpha, Q^{(0)}_{(-k_1-1)}}$. Assume, e.g., $t_1 \geq t_2$, and $(t_2, x_2) \in Q^{(-k_2)}(t_1, x_1)$, $k_2 \geq k_1 + 1$, with $d_{par}((t_1, x_1), (t_2, x_2)) \approx M^{-k_2/2}$. The hypothesis $k_2 \geq k_1 + 1$ excludes the case where $d_{par}((t_1, x_1), (t_2, x_2))$ is comparable to $M^{-k_1/2}$, a case which is not needed, since it is already covered by the estimates proved in (iv). Then,

$$|\nabla^2 u(t,x) - \nabla^2 u(t',x')| \leq I_1 + I_2 + I_3 + I_4,$$

with (using (4.33) for I_1, I_2 and (4.32) for I_3, I_4)

$$I_1 = |\nabla^2 u_{k_1}(t_1,x_1) - \nabla^2 u_{k_1}(t_2,x_2)| \lesssim (M^{k_1})^{1+\alpha/2} R_b^{-(1+\alpha/2)} ||u||_\infty d_{par}(t_1,x_1;t_2,x_2)^\alpha; \quad (4.37)$$

$$\begin{aligned} I_2 &= \sum_{k=k_1}^{k_2-1} |\nabla^2(u_{k+1}-u_k)(t_1,x_1) - \nabla^2(u_{k+1}-u_k)(t_2,x_2)| \\ &\lesssim R_b^{-(1+\alpha'/2)} d_{par}(t_1,x_1;t_2,x_2)^{\alpha'} \left(\sum_{k=k_1}^{k_2-1} (M^{k/2})^{\alpha'-\alpha} \right) \left(||f||_\alpha + ||a||_\alpha ||u||_\infty + ||b||_\alpha \sup_{Q_1^{(-k)}} |\nabla u| \right) \\ &\lesssim d_{par}(t_1,x_1;t_2,x_2)^\alpha R_b^{-(1+\alpha'/2)} \left(||f||_\alpha + ||a||_\alpha ||u||_\infty + ||b||_\alpha \sup_{Q_1^{(-k_1)}} |\nabla u| \right); \end{aligned} \quad (4.38)$$

and

$$I_3 := \sum_{k\ge k_2} |\nabla^2(u_{k+1}-u_k)(t_1,x_1)|,\ I_4 := \sum_{k\ge k_0} |\nabla^2(u_{k+1}-u_k)(t_2,x_2)| \tag{4.39}$$

are

$$\lesssim d_{par}(t_1,x_1;t_2,x_2)^{\alpha} R_b^{-1}\left(||f||_{\alpha} + ||a||_{\alpha}||u||_{\infty} + ||b||_{\alpha} \sup_{Q_1^{(-k_2)}} |\nabla u| \right). \tag{4.40}$$

Hence

$$(M^{-k_1})^{1+\alpha/2}||\partial_t u||_{\alpha,Q^{(0)}_{-(k_1-1)}}, (M^{-k_1})^{1+\alpha/2}||\nabla^2 u||_{\alpha,Q^{(0)}_{-(k_1-1)}} \lesssim R_b^{-(1+\alpha'/2)}\cdot$$
$$\cdot \left\{ (M^{-k_1})^{1+\alpha/2}\left(||f||_{\alpha} + ||b||_{\alpha} \sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| + ||a||_{\alpha}||u||_{\infty}\right) + ||u||_{\infty}\right\}, \tag{4.41}$$

compare with (4.34).

By standard Hölder interpolation inequalities (see Lieberman, 1996),

$$\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| \lesssim ||\nabla^2 u||^{1/(2+\alpha)}_{\alpha,Q^{(0)}_{(-k_1-\delta k)}} \big(\sup_{Q^{(0)}_{(-k_1-\delta k)}} |u|\big)^{(1+\alpha)/(2+\alpha)} \lesssim$$
$$\varepsilon^{2+\alpha}||\nabla^2 u||_{\alpha,Q^{(0)}_{(-k_1-\delta k)}} + \varepsilon^{-(2+\alpha)/(1+\alpha)}||u||_{\infty} \tag{4.42}$$

for every $\varepsilon > 0$. Choosing $\varepsilon^{2+\alpha} \approx R_b^{1+\alpha'/2}/||b||_{\alpha}$ yields as in (iv) a bound for $\sup_{k_1\ge 0}(M^{-k_1})^{1+\alpha/2}||\nabla^2 u||_{\alpha,Q^{(0)}_{-(k_1-1)}}$, from which one deduces, in particular, (4.20).

In order to obtain the bound (4.18) for $||\nabla u||_{\alpha,Q^{(-1)}}$, we proceed initially in the same way, with the only difference being that one may take $\alpha' = \alpha$ in (4.38), since one gets a series $\sum_{k=k_1}^{k_2-1} M^{-k/2}$ of order $O(1)$. Thus, (4.41) becomes

$$(M^{-k_1/2})^{1+\alpha}||\nabla u||_{\alpha,Q^{(0)}_{-(k_1-1)}} \lesssim$$
$$R_b^{-(1+\alpha)/2}\left\{ (M^{-k_1/2})^{1+\alpha}\left(||f||_{\alpha} + ||b||_{\alpha} \sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| + ||a||_{\alpha}||u||_{\infty}\right) + ||u||_{\infty}\right\}. \tag{4.43}$$

One now uses Hölder interpolation inequalities to bound ∇u in terms of $||u||_{\infty}$ and $\nabla^2 u$. Instead of (4.44), one has here

$$\sup_{Q^{(0)}_{(-k_1-\delta k)}} |\nabla u| \lesssim ||\nabla u||^{1/(1+\alpha)}_{\alpha,Q^{(0)}_{(-k_1-\delta k)}} \big(\sup_{Q^{(0)}_{(-k_1-\delta k)}} |u|\big)^{\alpha/(1+\alpha)} \lesssim \varepsilon^{1+\alpha}||\nabla u||_{\alpha,Q^{(0)}_{(-k_1-\delta k)}} + \varepsilon^{-(1+\alpha)/\alpha}||u||_{\infty} \tag{4.44}$$

for every $\varepsilon > 0$. Choosing $\varepsilon^{1+\alpha} \approx R_b^{(1+\alpha)/2}/||b||_{\alpha}$ yields as in (iv) a bound for $\sup_{k_1\ge 0}(M^{-k_1})^{(1+\alpha)/2}||\nabla u||_{\alpha,Q^{(0)}_{-(k_1-1)}}$, from which one deduces, in particular, (4.18).

■

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