CONSTRUCTION METHOD OF APPROXIMATED DEVELOPMENTS OF OPTIONAL LENGTH CIRCULAR ARCS

Mieczysław MAJEWSKI

Technical University of Szczecin , Faculty of Civil Engineering and Architecture
Aleja Piastów 50, 70-311 SZCZECIN, POLAND
e-mail: mmaha@interia.pl

Abstract. This work deals with an analytic-graphical method of construction approximated developments of optional length circular arcs on a straight line. Such problems often appear in the engineering, designing and workshop practices. The method presented by A. Kochański [1] concerns the development of an arc of one-half of a circle, while methods given by S. Polański, A.A. Kowalewski and J. Daniluk [2] as well, as by the author [3] bring approximated results, where the range of measuring errors is 0,1% to 0,6% (for methods included in the work [2]), and 0,0005% to 0,15% (in relation to the work [3]). The solution included in the hitherto elaboration is based on the implementation of a curve - circumference (circular [4]) projection for transformation of final points of given circular arc onto final points of its development on a straight line. The work treats about the geometric idea with it's justification, which is a base of the (called here) analytic-graphical method. The work presents also a practical part of this method. Presented method gives an accuracy up to 0,052% of length of measured arcs.

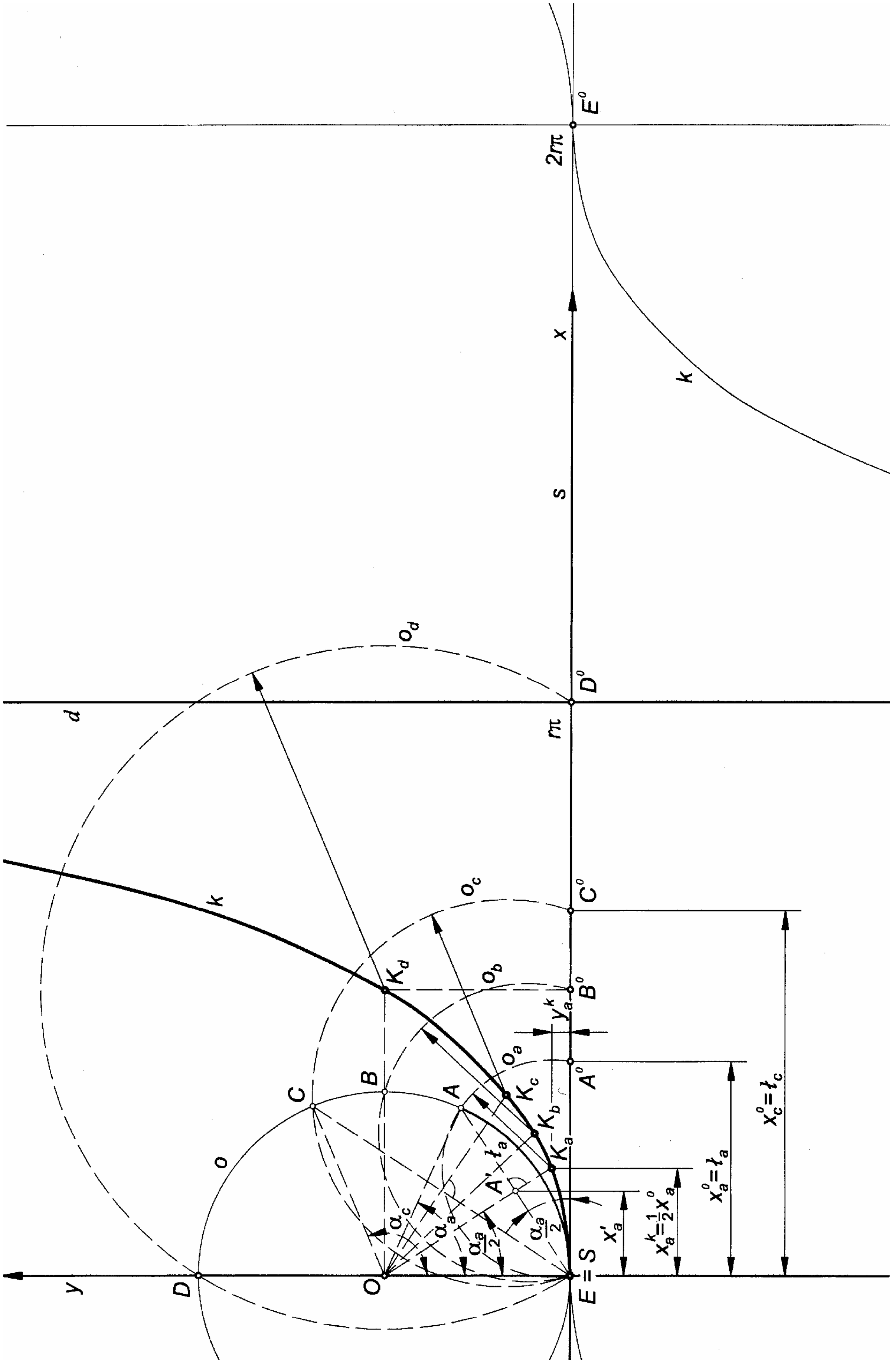
Keywords: circular arc, analytic-graphical method, approximated developments of arc, length of arc

1. The geometric idea with it's justification and practical method of preparation the developments of circular arcs sections.

Let us take optional sector $l_a = \cap SA$ of circular arc determined by a centre O and radius r, which we have to develop onto a straight line s – tangent to the circle o in the point S, in such a way, as for initial point S of the arc SA, being united with the initial point of its development $|SA^0|$ (fig.1). Now, let us discuss such a geometric transformation of the point A into point $A^0 \in s$, as for the sector $X^0_a = |SA^0|$ being equal by length with the given arc $l_a = X^0_a = |SA^0|$. It appears convenient for these conditions the implementation of curve – circumference (circular [4]) projections, where projecting circles o_n posses unchangeable point S, and theirs centres K_n belong to some curve k. Thus, the essence of the problem discussed brings to a determination of a course of curve line k, dependent exclusively on a radius r of a circle o, which following points $K_n, ...,$ are equally distant from points $S, A, ...$ and $A^0, ...$ accordingly.

In order to determine the equation of a curve k within assumed coordinate system S_{xy}, let us discuss geometric dependencies, which band its optional point, for example K_a, with the given arc SA, bearing central angle α_a, and with a sector $x^0_a = l_a = |SA^0|$ of a straight line s, which length is equal with the length of given arc $l_a = |SA^0| = \frac{1}{180}\alpha_a \pi r$. Coordinates $x^k_a = \frac{1}{2} x^0_a$, and y^k_a for the point K_a are determined as a result of equations for: a midperpendicular OA' of the sector SA, and midperpendicular of the sector SA^0. Thus, solving the equation $y = -x \, ctg \frac{\alpha_a}{2} + r$ for the midperpendicular OA' of the sector SA, (which constitutes with the axis $x = s$ an angle equal $\frac{\alpha_a}{2}$), together with the equation $x = x^k_a = \frac{1}{2} x^0_a$ for the

ISSN 1644-9363 / zł 15.00


y
x
s
k
d
o
D
O
C
B
A
E = S
π
2π
E^0
D^0
C^0
B^0
A^0
o_a
o_b
o_c
o_d
K_a
K_b
K_c
K_d
$x_c^0 = l_c$
$x_a^0 = l_a$
$x_a^k = \frac{1}{2} x_a^0$
x'_a
y_a^k
α_a
α_c
$\frac{\alpha_a}{2}$

midperpendicular of the sector SA^0, where $x^0_a = \ell_a = \frac{1}{180}\alpha_a\pi r$, we come to results $x^k_a = \frac{1}{2}x^0_a = \frac{1}{360}\alpha_a \pi r$ and $y^k_a = -\frac{1}{360}\alpha_a \pi r\, ctg\frac{\alpha_a}{2} + r$, which are coordinates of point K_a belonging to the curve k. By making a quantification of coordinate x^k_n of the point K_n dependent on the radius r and from central angle α, i.e.: $x^k_n = \frac{1}{360}\alpha \pi r$ the equation for the curve k is as following: $y = -\frac{1}{360}\alpha \pi r\, ctg\frac{\alpha}{2} + r$, and its asymptote d is expressed by equation $x = r\pi$.

Taking into consideration selected specific practical aims, the construction of developments of circular arcs sections will be realized with the use of prepared before curve k. A complex analytic-graphical method will be used here. This method is less comfortable (because of preparation of curve k for a selected circular arc), but gives accurate results when the curve k is suitably good contracted. In this case, the sequence of steps in solving the development of optional sector of circular arc, e,g.: $\ell_a = \cap SC$ of a circle o – determined by its final points S and C or by central angle $\alpha_c = \sphericalangle SOC$, is following: after assuming axes $x = s$, tangent to a circle o in the point S, and $y = SO$ – of perpendicular co-ordinate system S_{xy}, on a base of an equation: $y = -\frac{1}{360}\alpha\pi r\, ctg\frac{\alpha}{2} + r$ we draw a curve k (a sector SK_c is enough, assuming quantities of an angle α from $0°$ to α_c). Than we draw a midperpendicular of a sector SC, which in an intersection with a curve k is determining a point K_c – a centre of projecting circle $o_c = SCC^0$. The circle o_c defined by its centre $K_c \in k$ and a radius $r_c = |K_cS| = |K_cC| = |K_cC^0|$ intersecting with the tangent $s = x$, determines a point C^0, and at the same time the sector $\ell_c = |SC^0|$, which is a development of the circular arc $\ell_c = \cap SC$ onto the straight line s.

Let us notice, that for developments of circular arcs of a length from 0 to $2r\pi$ (α – from $0°$ to $360°$), centres K_n of projecting circles o_n belong only to positive branch of the curve k, which coordinates y^k_n take values from 0 to ∞.

2. Construction procedure of approximated development of circular arcs sections, which have a length up to $\frac{1}{4}$ of their circumference

On the base of analysis of a course of curve k expressed by an equation $y = -\frac{1}{360}\alpha \pi r\, ctg\frac{\alpha}{2} + r$, particularly of its initial sector determined by coordinates x^k_n, from $x^k_n = 0$ to $x^k_n = x^k_d$, it has been stated, that this sector may be approximated with a high precision (sufficient for practical use) by accordingly defined circular arc o_k (fig. 2). By such replacement of some initial sector of curve k, for example delimited by point K_d, which is applied for determining the development of an arc $\ell_d = \cap SD$, equal $\frac{1}{4}$ circumference of a circle o – by adequate circular arc o_k, presented construction of development of circular arc up to $\frac{1}{4}$ circumference of a circle appears simple and applicable for practical use.

Thus, let us determine firstly a circular arc o_k, which is approximating a sector SK_d of the curve $k = S\,K_a\,K_b\,K_c\,K_d$, in the following way. We assume some points, for example A, B, and C, which belong to a quarter SD of circular arc o, in such a way, that they determine accordingly central angels: point A – an angle $\alpha_a = 30°$, point B – an angle $\alpha_b = 45°$ and point C – an angle $\alpha_c = 60°$. Than, for assumed values of angles α_n we determine quantities of coordinates $x^k_n = \frac{1}{360}\alpha_a \pi r$ and $y^k_n = -\frac{1}{360}\alpha_n \pi r\, ctg\frac{\alpha}{2} + r$ for points K_a, K_b, K_c and K_d of the curve k. For three selected ternary points, for example S, K_a, K_d; S, K_b, K_d and S, K_c, K_d of the curve k, we write three triple systems of equations for circles determined by according three ternary points, it is $x^2 + y^2 = r_1^2$; $(x^k_a - a_1)^2 + (y^k_a - b_1)^2 = r_1^2$ and $(x^k_d - a_1)^2 + (y^k_d - b_1)^2 = r_1^2, \ldots$. After solution of each system of equations we come to results, which are coordinates $a_1, \ldots, b_1, \ldots$ of centres O^1_k of circles $o^1_k, \ldots$ and radiuses $r_1, \ldots$ of three circles, that successively approximate to the searched circle o_k. Quantities of coordinates a_k and b_k of the centre O_k and the radius r_k of the searched circle o_k, approximating a curve k on the

base of its points $\boldsymbol{S, K_a, K_b, K_c}$ and $\boldsymbol{K_d}$, we determine as arithmetic means of analogical coordinates $\boldsymbol{a_1, a_2, a_3}$ and $\boldsymbol{b_1, b_2, b_3}$, and of radiuses $\boldsymbol{r_1, r_2, r_3}$ – determined for three circles $\boldsymbol{o^1{}_k}$, $\boldsymbol{o^2{}_k}$, $\boldsymbol{o^3{}_k}$, defined by three ternary points $\boldsymbol{S, K_a, K_d}$; $\boldsymbol{S, K_b, K_d}$ and $\boldsymbol{S, K_c, K_d}$.
The three presented values $\boldsymbol{a_k, b_k}$ and $\boldsymbol{r_k}$ – for the determined circle $\boldsymbol{o_k}$, expressed by quantity of a radius $\boldsymbol{r}$ of the circle $\boldsymbol{o}$ (to which the given arc belongs) are quantified according to results of calculations: $\boldsymbol{a_k = -0{,}008972\ r}$, $\boldsymbol{b_k = 1{,}576414\ r}$ and $\boldsymbol{r_k = 1{,}576438\ r}$.

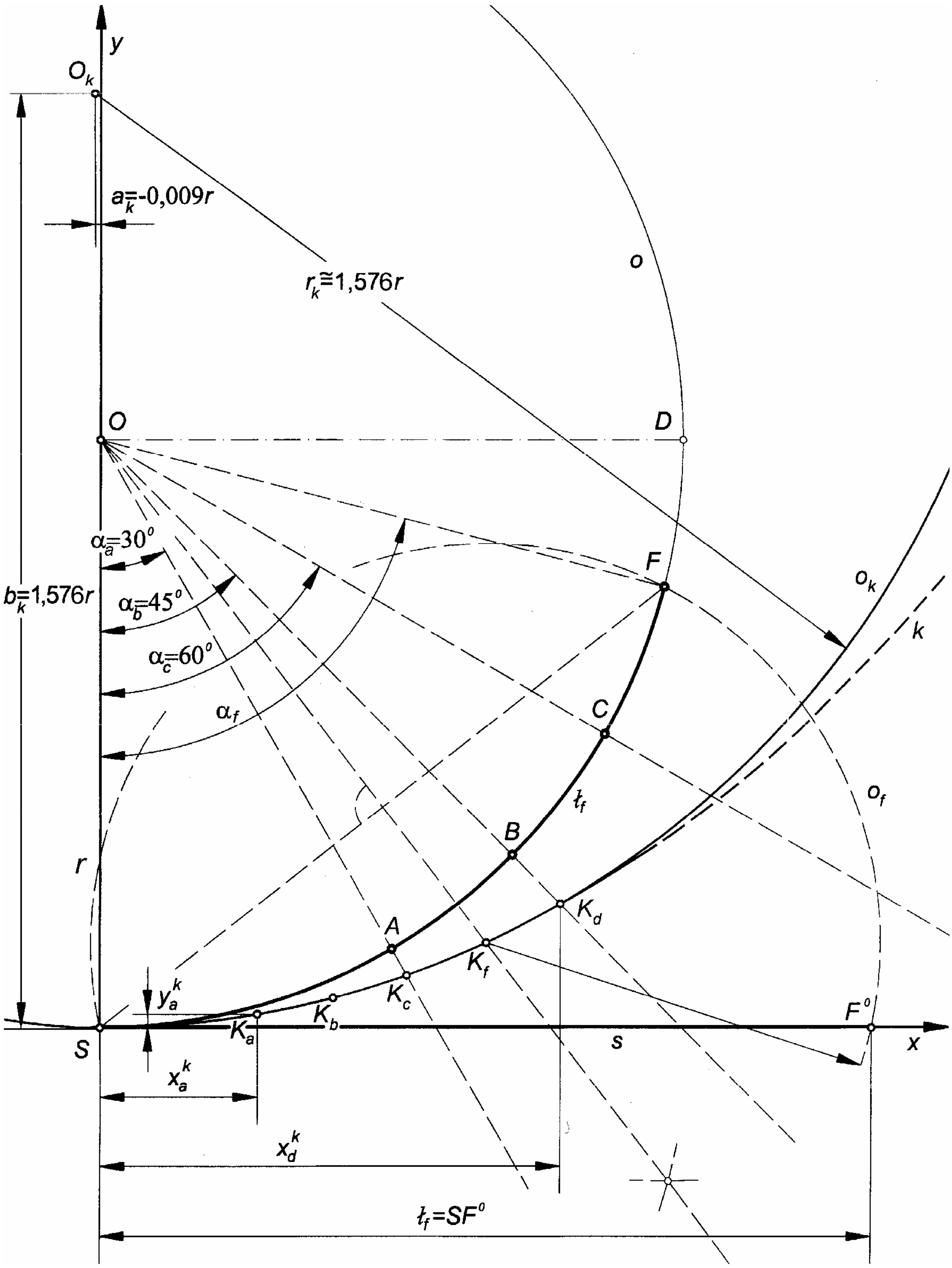


Fig. 2

In practical circumstances, which usually don't require such high precision, as resulting from above calculations it is suggested to accept accuracy to one per thousand in cases where higher precision is required, it is $\boldsymbol{a_k = -0{,}009\, r}$, $\boldsymbol{b_k = 1{,}576\, r}$ and $\boldsymbol{r_k \cong 1{,}576\, r}$, while for other cases, where such high precision is not required it is proposed to limit accuracy up to one per cent, it means $\boldsymbol{a_k = -0{,}01\, r}$, $\boldsymbol{b_k = 0{,}58\, r}$ and $\boldsymbol{r_k \cong 0{,}58\, r}$.
The determination of development of given circular arc $\boldsymbol{l_f = \cap SF}$ (for angles $\boldsymbol{\alpha_f < 90°}$) onto straight line $\boldsymbol{s = x}$, tangent in initial point $\boldsymbol{S}$ of the arc by presented approximated method we proceed as following (fig. 2). After drawing an axes $\boldsymbol{y = SO}$, we determine location of center $\boldsymbol{O_k}$ of the circle $\boldsymbol{o_k}$, which coordinates are $\boldsymbol{x^0{}_k = a_k = -0{,}009\, r}$ and $\boldsymbol{y^0{}_k = b_k = 1{,}576\, r}$, than draw a circular arc $\boldsymbol{o_k}$, determined by the centre $\boldsymbol{O_k}$ and radius $\boldsymbol{r_k \cong 1{,}576\, r}$ through a point $\boldsymbol{S}$. Than we draw a midperpendicular of a chord $\boldsymbol{SF}$, determining a point $\boldsymbol{K_f}$ of its intersection with an arc $\boldsymbol{o_k}$, and after that from the point $\boldsymbol{K_f \in o_k}$ as a centre, we draw an arc of projecting circle $\boldsymbol{o_f}$, which intersecting with a straight line $\boldsymbol{s = x}$, determines point $\boldsymbol{F^0}$. This point together with the point $\boldsymbol{S}$ constitute a sector $\boldsymbol{l_f = |SF^0|}$, being a development of an arc $\boldsymbol{l_f = \cap SF}$ onto straight line $\boldsymbol{s = x}$.
Presented analytic-graphical method gives a measurements error up to 0,048%, when the accuracy of centres' $\boldsymbol{O_k}$ coordinates of circles $\boldsymbol{o_k}$ is up to 0,001, and – a measurements error up to 0,052%, when the accuracy of centres' $\boldsymbol{O_k}$ coordinates of circles $\boldsymbol{o_k}$ is up to 0,01. But the maximal measure occurs when the circular arcs sections are close to 1/4 of their circumference.

Bibliography:

[1] WOCJAN S., LEWANDOWSKI Z., SZYMAŃSKI E.: *Rysunek techniczny*, PWN 1968.

[2] POLAŃSKI S., KOWALEWSKI A., DANILUK J.: *Geometria dla konstruktorów*, PWN 1965.

[3] MAJEWSKI M.: *Konstrukcje geometryczne służące do wyznaczania przybliżonych długości łuków krzywych stożkowych*, Międzyuczelniane Czasopismo Naukowe Geometria Wykreślna i Grafika Inżynierska, t. I, Łódź-Wrocław 1995.

[4] BIEDA K.: *Wiązka stożkowych jako aparat rzutujący*, Zeszyty Naukowe Geometria Wykreślna, t. VIII, Poznań 1973.

Recenzent: dr inż. Tadeusz DYDUCH

Wpłynęło do Redakcji w kwietniu 2002 roku