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ON GEVREY ORDERS OF FORMAL POWER SERIES SOLUTIONS TO THE THIRD AND FIFTH PAINLEVÉ EQUATIONS NEAR INFINITY

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Abstract. The question under consideration is Gevrey summability of formal power series solutions to the third and fifth Painlevé equations near infinity. We consider the fifth Painlevé equation in two cases: when $\alpha\beta\gamma\delta \neq 0$ and when $\alpha\beta\gamma \neq 0$, $\delta = 0$ and the third Painlevé equation when all the parameters of the equation are not equal to zero. In the paper we prove Gevrey summability of the formal solutions to the fifth Painlevé equation and to the third Painlevé equation, respectively.

Keywords: Painlevé equations, Newton polygon, asymptotic expansions, Gevrey orders.

Mathematics Subject Classification: 34M25, 34M55.

1. GENERAL THEORY

Let V be an open sector with a vertex at infinity on the extended complex plane or on the Riemann surface of the logarithm, such that

$$V = V(\varphi_1, \varphi_2) = \{z : |z| > R, \operatorname{Arg} z \in (\varphi_1, \varphi_2)\},$$

$\varphi_1, \varphi_2 \in \mathbb{R}$, $z \in \mathbb{C}$; let w be a function holomorphic on V and $\hat{w} = \sum_{k=0}^{\infty} a_k z^{-k}$ be a formal series belonging to $\mathbb{C}[[1/z]]$.

A function w is said to be *asymptotically approximated by a series* $\hat{w}$ *on* $V(\varphi_1, \varphi_2)$ if for the points z of any closed subsector Y of V and for any $n \in \mathbb{N}$ there exist the constants $M_{Y,n} > 0$:

$$|z^n|\left|w(z) - \sum_{p=0}^{n-1} a_p z^{-p}\right| < M_{Y,n}.$$

If there exists the constants $A_Y > 0$, $C > 0$ such that $M_{Y,n} = C(n!)^{1/k} A_Y^n$, a series $\hat{w}$ is *an asymptotical Gevrey series of order $1/k$ for a function w on $V(\varphi_1, \varphi_2)$*, it is defined like $\hat{w} \in \mathbb{C}[[z]]_{1/k}$.

For $G(z, Y, Y_1, \ldots, Y_n)$ being an analytic function of $n+2$ variables let us consider an equation

$$G(z, w, Dw, \ldots, D^n w) = 0, \tag{1.1}$$

where D is an operator $z\frac{d}{dz}$, $D^2 = z^2\frac{d^2}{dz^2} + z\frac{d}{dz}$, $D^k = D(D^{k-1})$.

Let $\hat{w} \in \mathbb{C}[[1/z]]$ be a formal series, being a formal solution of a differential equation (1.1).

Theorem 1.1 (O. Perron [8], J.-P. Ramis, Y. Sibuya [9, 10], B. Malgrange [6] in different cases)**.** *Let $\hat{w} \in \mathbb{C}[[z]]_{1/k}$ be a solution to the equation* (1.1)*. Then there exists $k' > 0$ such that for every open sector $V(\varphi_1, \varphi_2)$ with the vertex in the infinity, having an angle $|\varphi_2 - \varphi_1|$ less than $\min(\pi/k, \pi/k')$ and for a sufficiently big R there exists a function w, being a solution to the equation* (1.1) *which is asymptotically approximated of Gevrey order $1/k$ by a series $\hat{w}$.*

The next theorem contains conditions on the Newton open polygon. We will describe the process of its construction for a linear differential operator

$$L = \sum_{k=0}^{n} a_k(z) D^k, \quad \text{where} \quad a_k(z) \in \mathbb{C}[z][[1/z]],\, a_k(z) = \sum_{j=j_k}^{\infty} a_{j,k} z^{-j},$$

$$a_{j,k} = \text{const} \in \mathbb{C}.$$

Consider a set of points on the plane: $(k,\, j_k)$, $k = 0, \ldots, n$, called *a support of the operator L*. We define a set

$$N(G) = \bigcup_{k=0}^{n} \{(q_1, q_2) : q_1 \le k,\, q_2 \ge j_k\}$$

and then we construct a convex hull of this set in the half-plane $q_1 \ge 0$. A boundary of this set is called *a Newton open polygon $N(G)$ of the linear differential operator L.*

The Newton open polygon $N(G, \hat{w})$ of the equation (1.1) (meaning, that the equation (1.1) has the form $\mathfrak{G} w = 0$) on a formal solution $\hat{w}$ is defined as a Newton polygon of the linearization on the series $\hat{w}$ of the operator $\mathfrak{G}$.

Theorem 1.2 (J.-P. Ramis [11])**.** *Let $\hat{w} \in \mathbb{C}[[z]]$ be a formal solution to the equation* (1.1)*, then the series $\hat{w}$ converges or has a Gevrey order equal to s, where $s \in \{0, \frac{1}{k_1}, \ldots, \frac{1}{k_N}\}$ and $0 < k_1 < \ldots < k_N < +\infty$ are all positive slopes of the edges of the Newton open polygon $N(G, \hat{w})$ of the equation* (1.1) *on a formal solution $\hat{w}$ to the X-axis.*

The previous Theorem 1.2 is formulated for a general nonlinear differential equation and its formal solution $\hat{w}$.

In a particular case $\frac{\partial G}{\partial Y_n}(z, \hat{w}, , \ldots, \hat{w}^{(n)}) \neq 0$, a Newton polygon of this equation (1.1) on a formal solution $\hat{w}$ can be constructed (Remark A.2.4.3 of [12]) as a polygon of a linear operator

$$L_0 = \sum_{k=0}^{n} \left(\frac{\partial G}{\partial Y_k}(z, \hat{w}, D\hat{w}, \ldots, D^n \hat{w}) \right) D^k. \tag{1.2}$$

We can show that the operator L_0 (1.2) coincides with the operator $\mathcal{M}$ used to construct exponential expansions of solutions via the methods of Power Geometry [2].

Let $(b_{j,k})$ be a matrix of a transformation from the basis $D, D^2, \ldots, D^n$ to basis $z\frac{d}{dz}, z^2\frac{d^2}{dz^2}, \ldots, z^n\frac{d^n}{dz^n}$ in a vector space of linear differential operators. We can check that the element $b_{j,k}$ is equal to $b_{j,k} = S(j,k)$, where $S(j,k)$ is a Stirling number of the second kind ([13]).This assertion is easily proved by induction using the fact that the elements of the matrix satisfy an equation $b_{j+1,k+1} = b_{j,k} + (k+1)b_{j,k+1}$. The transition matrix is lower triangular, the diagonal elements are equal to one ($S(j,j) = 1$). An inverse matrix $(b^{j,k})$ is also lower triangular, its elements are equal to $b^{j,k} = (-1)^{j-k}s(j,k)$, where $s(j,k)$ is a Stirling number of the first kind ([13]).

Let

$$G(z, w, Dw, \ldots, D^n w) = F\Big(z, w, z\frac{dw}{dz}, \ldots, z^n\frac{d^n w}{dz^n}\Big),$$

i.e.

$$G(z, Y, Y_1, \ldots, Y_n) = F(z, Y, X_1, \ldots, X_n).$$

An operator (1.2) can be rewritten using $z^k\frac{d^k}{dz^k}$ (we use designation $\delta_{j,l}$ for the Kronecker delta):

$$L_0 = \sum_{k=0}^{n}\sum_{l=0}^{n} \frac{\partial F}{\partial X_l}\frac{\partial X_l}{\partial Y_k} D^k = \sum_{k=0}^{n}\sum_{l=0}^{n} \frac{\partial F}{\partial X_l} b^{l,k} \sum_{j=0}^{n} b_{k,j} z^j \frac{d^j}{dz^j} =$$

$$= \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} \sum_{j=0}^{n} z^j \frac{d^j}{dz^j} \sum_{k=0}^{n} b^{l,k} b_{k,j} = \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} \sum_{j=0}^{n} z^j \delta_{j,l} \frac{d^j}{dz^j} = \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} z^l \frac{d^l}{dz^l},$$

and this expression is a first variation of a differential sum F. Being calculated on a series $\hat{w}$, it coincides with an operator $\mathcal{M}$ from [2].

Let us recall that a formal first variation of a differential sum $g(z,w)$ is a linear differential operator

$$\frac{\delta g}{\delta w} : \mathbb{C}\Big[z, \frac{1}{z}, w, w', \ldots, w^{(n)}\Big] \to \mathbb{C}\Big[z, \frac{1}{z}, w, w', \ldots, w^{(n)}, \frac{d}{dz}, \ldots, \frac{d^n}{dz^n}\Big]$$

defined by the following formulae:

$$\begin{gathered} \frac{\delta C(z)}{\delta w} = 0, \quad \frac{\delta w}{\delta w} = 1, \quad \frac{\delta w^{(k)}}{\delta w} = \frac{d^k}{dz^k}, \\ \frac{\delta(g_1+g_2)}{\delta w} = \frac{\delta g_1}{\delta w} + \frac{\delta g_2}{\delta w}; \quad \frac{\delta(g_1 g_2)}{\delta w} = \frac{\delta g_1}{\delta w} g_2 + g_1 \frac{\delta g_2}{\delta w} \\ \text{for } C(z) \in \mathbb{C}[z, 1/z],\ g_1, g_2 \in \mathbb{C}[z, 1/z, w, w', \ldots, w^{(n)}]. \end{gathered} \tag{1.3}$$

The conclusion is – to construct a Newton polygon of the equation $F(z, w, zw', \dots, z^n w^{(n)}) = 0$ on a solution $\hat{w}$ we need to perform the following steps:

1) calculate the first variation $\frac{\delta F}{\delta w}$ on a solution $\hat{w}$,
2) perform a transformation expressed by a matrix $(-1)^{j-k} s(j,k)$,
3) verify a condition $\frac{\partial G}{\partial Y_n}(z, \hat{w},\, , D\hat{w},\, , \dots,\, D^n\hat{w}) \neq 0$,
4) find the set $(k,\, j_k)$, $k = 0,\, \dots,\, n$,
5) construct a convex hull of this set in the half-plane $q_1 \geq 0$.

Remark 1.3. The steps 2 and 3 can be interchanged, i.e. instead of condition $\frac{\partial G}{\partial Y_n}(z, \hat{w},\, , D\hat{w},\, , \dots,\, D^n\hat{w}) \neq 0$ we can check condition

$$\frac{\partial F}{\partial X_n}(z, \hat{w},\, , \hat{w}',\, , \dots,\, \hat{w}^{(n)}) \neq 0.$$

Remark 1.4. We use a Newton polygon considered in [12] to calculate the Gevrey orders but it can be easily shown that the Gevrey order of the solution can be calculated via constructing a polygon of the eqaution used in Power Geometry ([1]). To calculate the order we should:

1) perform a transrormation $u = \log y$ in the operator $\mathcal{M}(z)$ ([2]), applied to u;
2) reduce an expression by u;
3) construct a polygon for the differential sum obtained.

In the conditions of Theorem 1.2 the tangents are replaced by cotangents.

2. THE FIFTH PAINLEVÉ EQUATION

We consider the fifth Painlevé equation

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (2.1)$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, z is an independent complex variable, w is a dependent one, and we consider its formal power series solutions near infinity. These solutions are obtained in the work [5].

If $\alpha\beta\gamma\delta \neq 0$, there exist the following five formal power series solutions:

$$(-1)^l \sqrt{\frac{\beta}{\delta}}\frac{1}{z} + \left(-\frac{2\beta}{\delta} + (-1)^l \frac{\gamma}{2\delta}\sqrt{\frac{\beta}{\delta}}\right)\frac{1}{z^2} + \sum_{s=3}^{\infty} \frac{c_{-s,l}}{z^s}, \quad l = 1, 2, \quad (2.2)$$

$$-1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_{-s}}{z^s}, \quad (2.3)$$

$$(-1)^l \sqrt{-\frac{\delta}{\alpha}}z + 2 + (-1)^l \frac{\gamma}{2\sqrt{-\alpha\delta}} + \sum_{s=1}^{\infty} \frac{c_{-s,l}}{z^s},\ l = 1, 2. \quad (2.4)$$

If $\alpha\beta\gamma \neq 0$, $\delta = 0$ there exist the following four expansions:

$$(-1)^l \sqrt{-\frac{\beta}{\gamma}} \frac{1}{\sqrt{z}} + \frac{\beta}{\gamma z} + \sum_{s=3}^{\infty} \frac{c_{-s,l}}{z^{s/2}}, \ l = 3, 4, \tag{2.5}$$

$$(-1)^l \sqrt{-\frac{\gamma}{\alpha}} \sqrt{z} + 1 + \sum_{s=1}^{\infty} \frac{c_{-s,l}}{z^{s/2}}, \quad l = 3, 4. \tag{2.6}$$

The coefficients c_s, $c_{s,l} \in \mathbb{C}$, $l = 1, 2, 3, 4$ are uniquely determined constants, i.e. if we fix the values of the parameters $\alpha, \beta, \gamma, \delta$, these coefficients are uniquely determined as solutions of a non-degenerate system of linear equations.

If $\gamma = \delta = 0$, the fifth Painlevé equation can be solved directly.

Theorem 2.1. *The series* (2.2), (2.3) *and the regular part of the series* (2.4) *are of Gevrey order* 1*. The series* (2.5), (2.6) *considered as the series in a new variable* $\sqrt{z}$ *are also of Gevrey order* 1*.*

Proof. We apply Theorem 1.2, where (1.1) is equation (2.1) multiplied by $z^2 w(w-1)$ with all the terms of the equation put into the left part:

$$\begin{aligned} f(z,w) \stackrel{def}{=} -z^2 w(w-1)w'' + z^2 \left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' + \\ + (w-1)^3(\alpha w^2 + \beta) + \gamma z w^2 (w-1) + \delta z^2 w^2 (w+1) = 0. \end{aligned} \tag{2.7}$$

We take instead $\hat{w}$ the series (2.2), (2.3), (2.4) in course. If the principal part is not equal to zero, we can easily obtain the case of a zero principal part using a transformation; we speak about the Gevrey order of the regular part of the series.

The first variation of the equation P_5 represented in a form of a differential sum (2.7) is equal to

$$\begin{aligned} \frac{\partial f}{\partial w} &= -z^2 w(w-1)\frac{d^2}{dz^2} + \left(z^2(3w-1)w' - zw(w-1)\right)\frac{d}{dz} - \\ &- z^2(2w-1)w'' + \frac{3z^2(w')^2}{2} - z(2w-1)w' + \\ &+ (w-1)^2(5\alpha w^2 - 2\alpha w + 3\beta) + \gamma z(3w^2 - 2w) + \delta z^2(3w^2 + 2w). \end{aligned} \tag{2.8}$$

We substitute the series (2.2) into the expression (2.8), rewrite it in terms of operators D and D^2, and we select the highest order term in z in each coefficient of D^2, D and D^0. We finally construct the following operator, where only the highest order term appears in each coefficient of D^2, D and D^0:

$$(-1)^l \sqrt{\frac{\beta}{\delta}} \left(\frac{D^2}{z} + \frac{D}{z} + 2\delta z\right), \quad l = 1, 2,$$

the support of such an operator consists of the points $(0, -1)$, $(1, 1)$, $(2, 1)$, the Newton polygon is shown in Figure 1. $\square$

Remark 2.2. Below we will see that in the other cases (considering the series (2.3), (2.4), (2.5), (2.6)) we obtain a shift of Figure 1.

Analogous calculations can be performed for the series (2.3). We obtain an operator

$$-2D^2 + 14\frac{\gamma}{\delta}\frac{D}{z} + \delta z^2,$$

the support of it consists of the points $(0,-2)$, $(1,-1)$, $(2,0)$, its Newton polygon (brought down by a vector $(0,1)$) is shown in Figure 1.

For the series (2.4) we obtain an operator

$$\frac{\delta}{\alpha}z^2D^2 + \Big(-3\ \frac{\delta}{\alpha} - 2\sqrt{-\ \frac{\delta}{\alpha}}\big(2+(-1)^l\frac{\gamma}{2\sqrt{-\alpha\delta}}\big) - \sqrt{-\ \frac{\delta}{\alpha}}\Big)zD - 3\frac{\delta^2}{\alpha}z^4, \quad l = 1,\, 2.$$

Its support consists of the points $(0,-4)$, $(1,-1)$, $(2,-2)$, its Newton polygon (brought down by a vector $(0,3)$) is shown in Figure 1.

As we see in Figure 1, the unique positive tangent of the Newton polygon is equal to 1, using Theorem 1.2 we obtain that series (2.2), series (2.3) and the regular part of series (2.4) are of Gevrey order 1.

The polygon constructed in Power Geometry is obtained from this one by the reflection in the first quadrant bisector, this fact corresponds with Remark 1.4.

Let us calculate the Gevrey order of the series (2.5) and (2.6) obtained if $\delta = 0$. To transform the fifth Painlevé equation to the form (1.1) we substitute $t = \sqrt{z}$ and consider the series (2.5) and (2.6) which are series in decreasing half-integer degrees of z as the series in decreasing interger degrees of t. We calculate the first variation (we define differentiation with respect to t as a dot):

$$\begin{aligned}-\ \frac{t^2w(w-1)}{4}\frac{d^2}{dt^2} + \frac{1}{4}\left(t^2(3w-1)\dot{w} + tw(w-1) - w(w-1)\right)\frac{d}{dt} -& \\ -\frac{(2w-1)(t^2\ddot{w}+t\dot{w})}{4} + \frac{3t^2\dot{w}^2}{8} +& \\ + (w-1)^2(5\alpha w^2 - 2\alpha w + 3\beta) + \gamma t^2(3w^2 - 2w).&\end{aligned} \tag{2.9}$$

We substitute series (2.5) into the expression (2.9), rewrite it in terms of operators D and D^2, and we select the highest order term in z in each coefficient of D^2, D and D^0. We finally construct the following operator, where only the highest order term appears in each coefficient of D^2, D and D^0:

$$\frac{c_{-1/2}}{4t}D_t^2 - \frac{c_{-1/2}}{4t}D_t + 3\gamma c_{-1/2}t,$$

where $D_t = t\frac{d}{dt}$, $c_{-1/2} = (-1)^l\sqrt{\frac{\beta}{\gamma}}$, $l = 1, 2$.

The support of the operator consists of the points $(0,-1)$, $(1,1)$, $(2,1)$, its Newton polygon is shown in Figure 1. We obtain again that series (2.5) is of Gevrey order 1.

The operator corresponding to series (2.6) has the form

$$-\frac{c_{1/2}^2t^2}{4}D_t^2 + \frac{3c_{1/2}^2t^2}{4}D_t - 3\gamma c_{1/2}^2t^4,$$

where $c_{1/2} = (-1)^l\sqrt{-\frac{\gamma}{\alpha}}$, $l = 1, 2$.

The support of the operator consists of the points $(0,-4)$, $(1,-2)$, $(2,-2)$, its Newton polygon (brought down by a vector $(0,3)$) is shown in Figure 1. The regular part of the series (2.6) considered as the Laurent series in a variable t is of Gevrey order 1.

This statement accomplishes the proof of Theorem 2.1.

Corollary 2.3. *For each of the power series* (2.3), (2.4) *and regular part of series* (2.2) *there exist* $k' \geq 1$ *and* $R_0 \in \mathbb{R}_+$ *such that for every open sector*

$$\{z : |z| > R \geq R_0,\ \operatorname{Arg} z \in (\varphi_1, \varphi_2)\}, \quad \varphi_2 - \varphi_1 < \pi/k' \leq \pi,$$

there exists a solution to the fifth Painlevé equation approximated by these Gevrey-1 series.

For each of the power series (2.5), (2.6) *there exist* $k' \geq 1/2$ *and* $R_0 \in \mathbb{R}_+$ *such that for every domain*

$$\{z : |\sqrt{z}| > R \geq R_0,\ \operatorname{Arg} z \in (\varphi_1, \varphi_2)\}, \quad \varphi_2 - \varphi_1 < \pi/k' \leq 2\pi,$$

there exists a solution to the fifth Painlevé equation (*with parameter* $\delta = 0$) *approximated by these Gevrey-1 series.*

This assertion is a consequence of Theorems 1.1 and 2.1.

Other results concerning the fifth Painlevé equation obtained by the author of this article are published in [3, 4, 7].

3. THE THIRD PAINLEVÉ EQUATION

Let us pass on to the consideration of the analogous questions for the formal power series solutions to the third Painlevé equation:

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{3.1}$$

which can also be found in the book [5].

When $\gamma\delta \neq 0$ they form four power expansions:

$$i^l \sqrt[4]{-\frac{\delta}{\gamma}} - \left(\frac{(-1)^l \beta}{4\sqrt{-\gamma\delta}} + \frac{\alpha}{4\gamma}\right)\frac{1}{z} + \sum_{k=2}^{\infty} \frac{c_{l,k}}{z^k}, \quad l = 1,2,3,4, \tag{3.2}$$

where $i^2 = -1$, and we consider the main branch of the root while speaking about the forth root.

We apply Theorem 1.2 taking equation (3.1) multiplied by zw with all the terms of the equation put into the left part as in equation (1.1):

$$-zww'' + z\,(w')^2 - ww' + w(\alpha w^2 + \beta) + \gamma z w^4 + \delta z = 0, \tag{3.3}$$

and taking the series (3.2) as a formal power series solution $\hat{w}$.

The first variation of the equation (3.3) is equal to

$$-zw\frac{d^2}{dz^2}+(2zw'-w)\,\frac{d}{dz}-zw''-w'+3\alpha w^2+\beta+4\gamma zw^3. \tag{3.4}$$

We substitute the series (3.2) into the expression (3.4), rewrite it in terms of operators D and D^2, and we select the highest order term in z in each coefficient of D_t^2, D_t and D_t^0. We finally construct the following operator, where only the highest order term appears in each coefficient of D_t^2, D_t and D_t^0:

$$i^k\sqrt[4]{-\frac{\delta}{\gamma}}\frac{D^2}{z}+3\left(\frac{(-1)^l\beta}{4\sqrt{-\gamma\delta}}+\frac{\alpha}{4\gamma}\right)\frac{D}{z^2}+4i^{3k}\sqrt[4]{-\delta^3\gamma}z, \quad l=1,2,3,4,$$

the support of such an operator consists of the points $(0,-1)$, $(1,2)$, $(2,1)$, the Newton polygon is shown in Figure 1.

As we see in Figure 1, the unique positive tangent of the Newton polygon is equal to 1, using Theorem 1.2 we obtain that the series (3.2) are of Gevrey order 1.

So, we obtain the following theorem.

Theorem 3.1. *The series* (3.2) *are of Gevrey order equal to one. There exists* $k'\geq 1$, $R_0\in\mathbb{R}_+$ *such that for any open sector*

$$\{z:|z|>R\geq R_0,\,\mathrm{Arg}\,z\in(\varphi_1,\varphi_2)\},\quad \varphi_2-\varphi_1<\pi/k'\leq\pi,$$

there exists a solution to the third Painlevé equation approximated by this Gevrey-1 series.

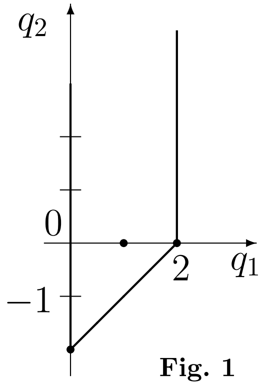

Fig. 1

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REFERENCES

[1] A.D. Bruno, *Asymptotic behavior and expansions of solutions to an ordinary differential equation*, Uspekhi Mat. Nauk **59** (2004) 3, 31–80; English transl.: Russian Math. Surveys **59** (2004), 429–480.

[2] A.D. Bruno, *Exponential expansions of solutions to an ordinary differential equation*, Dokl. Akad. Nauk **443** (2012) 5, 539–544; English transl.: Doklady Mathematics **85** (2012) 2, 259–264.

[3] A.D. Bruno, A.V. Parusnikova, *Expansions of solutions to the fifth Painlevé equation in a neighbourhood of its nonsingular point*, Dokl. Akad. Nauk **442** (2012) 5, 582–589; English transl.: Doklady Mathematics **85** (2012) 1, 87–92.

[4] A.D. Bruno, A.V. Parusnikova, *Local expansions of solutions to the fifth Painlevé equation*, Dokl. Akad. Nauk **438**, no. 4, 439–443; English transl.: Doklady Mathematics **83** (2011), 348–352.

[5] V.I. Gromak, I. Laine, S. Shimomoura, *Painleve Differential Equations in the Complex Plane*, Walter de Gruter, Berlin, New York, 2002, 303 p.

[6] B. Malgrange, *Sur le théorème de Maillet*, Asymptot. Anal. **2** (1989), 1–4.

[7] A.V. Parusnikova, *Asymptotic expansions of solutions to the fifth Painlevé equation in neighbourhoods of singular and nonsingular points of the equation*, [in:] Banach Center Publications. Vol. 97: Formal and Analytic Solutions of Differential and Difference Equations, Polish Academy of Sciences, Warszawa, 2012, 113–124.

[8] O. Perron, *Uber lineare und Differentialgleichungen mit rationalen Koeffizienten*, Acta Math. **34** (1910), 139–163.

[9] J.-P. Ramis, *Dévissage Gevrey, Journées singuliéres de Dijon*, Astérisque **59**–**60** (1978), 173–204.

[10] J.P. Ramis, Y. Sibuya, *Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type*, Asymptotic Analysis **2** (1989), 39–84

[11] J.P. Ramis, *Divergent series and asymptotic theory*, Institute of Computer Research, Moscow-Ijevsk, 2002 [in Russian].

[12] Y. Sibuya, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Providence: AMS, 1985.

[13] N.Ya. Vilenkin, A.N. Vilenkin, P.A. Vilenkin, Combinatorika, FIMA, MCCME, 2006. 400 p. [in Russian].

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