ARCHIVES OF MECHANICS
Arch. Mech. **74** (5), 373–389, 2022, DOI: 10.24423/aom.4006

The reflection and transmission of waves at interface between two nonlocal orthotropic thermoelastic halfspaces

D. X. TUNG

Faculty of Civil Engineering, Hanoi Architectural University, Km 10 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam, e-mail: tungdx2783@gmail.com

The present article deals with the propagation of inhomogeneous waves in an orthotropic medium based on Eringen's nonlocal thermoelasticity. For chosen directions of propagation and a real finite inhomogeneity parameter, a complex slowness vector is specified to define the propagation of inhomogeneous incident wave. Then the reflection, transmission of plane waves at a plane interface between two nonlocal orthotropic thermoelastic halfspaces are discussed. In this incidence, horizontal slowness determines the slowness vectors for all reflected, transmitted waves. For each reflected, transmitted wave, the corresponding slowness vector is resolved to define its phase direction, phase velocity and attenuation angle. Appropriate boundary conditions on this wave-field determine the amplitude ratios for reflected, transmitted waves relative to the incident wave. The numerical examples are provided to show the effect of the inhomogeneity of incident wave, nonlocal parameter on the propagation characteristic of the reflected, transmitted waves.

Key words: inhomogeneous, attenuation, orthotropic, nonlocal thermoelastic.

1. Introduction

The generalized thermoelasticity theories have been developed with the aim of removing the paradox of infinite speed of heat propagation inherent in the classical coupled dynamical thermoelasticity theory. Lord and Shulman [1] first modified Fourier's law by introducing the term representing the thermal relaxation time. The heat equation associated with this theory is of hyperbolic type and hence, eliminates the paradox of infinite speed of propagation. Later, Green and Lindsay [2] developed a more general theory of thermoelasticity in which Fourier's law of heat conduction is unchanged, whereas the classical energy equation and Duhamel–Neumann's relations are modified by introducing two constitutive constants having the dimension of time. The study of wave propagation in a generalized thermoelastic media with additional parameters like prestresses, porosity, viscosity, microstructure, temperature and other parameters provide vital information about existence of new or modified waves [3–5]. In general, a mathematical model is derived for the wave propagation in anisotropic generalized thermoelastic medium. The modified Christoffel equations are solved into a quartic equation. The four roots of this equation explain the existence of four quasi-waves in the medium and provide the complex velocities of these waves. Therefore, the four waves propagating in this medium attenuate waves. The attenuating wave motion can be described by inhomogeneous waves with complex slowness vectors. The real and imaginary parts of the complex slowness vector are termed as the propagation vector and attenuation vector, respectively. The inhomogeneity of an attenuating wave, in general, is represented through the difference in the directions of its propagation vector and attenuation vector. CEREVENY and PSENCIK [6, 7] put forward a specification for the complex slowness vector which was calculated from the solution of a complex polynomial equation of degree six. A non-zero value of D implies the propagation of inhomogeneous waves, or else, the attenuating motion propagates as homogeneous waves. The mixed specification offers a simple and general algorithm to study inhomogeneous or homogeneous plane waves propagating in the direction of a known unit vector **N** in unbounded, anisotropic or isotropic, viscoelastic or perfectly elastic media. SHARMA [4] introduced a non-dimensional finite-valued inhomogeneity parameter and formulated a procedure to constitute the complex slowness vector for inhomogeneous propagation in anisotropic dissipative media. The procedure is wave-specific, in which there is no need to solve an algebraic equation of degree six.

The nonlocal continuum theory contains information about long-range forces of atoms or molecules and, thus, an internal length scale parameter can be introduced in the formulation [8, 9]. Nonlocal elasticity theories have been applied to the problems of harmonic plane wave propagation in classical and nonclassical elastic materials. TUNG [10–12] investigated the propagation, the reflection and transmission of waves in nonlocal transversely isotropic liquid-saturated porous solid, in nonlocal orthotropic micropolar elastic solids. The mechanical behaviors of nano-sized devices, especially nanostructured component failure, are inevitably related to temperature: the devices may be subjected to heat from an environment or heat generated by themselves when working. Therefore, the nonlocal thermoelastic theory is derived to analysis at the micro or nanoscale as the characteristic length of the structure become comparable to the internal characteristic length, the wave length, etc. The problems on wave propagation in nonlocal thermoelastic solids are studied by many authors [13–17].

Generally speaking, in most of the above investigations, the propagation of reflected, transmitted waves at an interface between orthotropic thermoelastic half-spaces are affected not only by the inhomogeneity of the incident wave but also by the nonlocality of the medium. However, the research works involving both the inhomogeneity effects and the nonlocal effects have been rare in the published literature so far. Therefore, the aim of the paper is to study the inhomogeneous propagation of waves in the nonlocal orthotropic thermoelastic medium.

2. Plane waves in nonlocal orthotropic thermoelastic medium

2.1. Basic equations

Wave propagation is considered in a nonlocal orthotropic thermoelastic medium, specified by four elastic constants c_{11}, c_{13}, c_{33}, c_{55} and five thermal coefficients β_{11}, β_{33}, C_E, K_{11}, K_{33}. For a two-dimensional problem in which the plane wave is in the plane x_1x_3 of the rectangular Cartesian coordinate system, the resulting wave-field is specified through mechanical displacements (u_1, u_3) and temperature change (T), each being the function of x_1, x_3 and t (time)

$$u_1 = u_1(x_1, x_3, t), \quad u_3 = u_3(x_1, x_3, t), \quad T = T(x_1, x_3, t). \tag{2.1}$$

Within the framework of Eringen's theory of nonlocal elasticity [9, 16], the constitutive relations for a thermoelastic solid are given by

$$\begin{aligned}
(1-\epsilon^2\nabla^2)\sigma_{11} &= \sigma_{11}^L = c_{11}u_{1,1} + c_{13}u_{3,3} - \beta_{11}(T + t_1\dot{T}),\\
(1-\epsilon^2\nabla^2)\sigma_{33} &= \sigma_{33}^L = c_{13}u_{1,1} + c_{33}u_{3,3} - \beta_{33}(T + t_1\dot{T}),\\
(1-\epsilon^2\nabla^2)\sigma_{13} &= \sigma_{13}^L = c_{55}(u_{1,3} + u_{3,1}),\\
(1-\epsilon^2\nabla^2)\rho\eta &= (\rho\eta)^L = \rho C_E T + \beta_{11}u_{1,1} + \beta_{33}u_{3,3}.
\end{aligned} \tag{2.2}$$

The nonlocal generalization of the heat conduction law for thermoelastic materials is postulated as [13, 14]

$$(1-\epsilon^2\nabla^2)(q_i + t_0\dot{q}_i) = (q_i + t_0\dot{q}_i)^L = -K_{ij}T_{,j} \quad (i, j = 1, 3), \tag{2.3}$$

where σ_{mn}, q_i and σ_{mn}^L, q_i^L, $(m, i, j = 1, 3)$ are the nonlocal and local stresses, heat flux vector components, respectively. The constant $\epsilon (= e_0 a)$ is the nonlocal parameter (e_0 is the nonlocal constant and a is the internal characteristic length). The energy equation for the linear theory of a thermoelastic material is (in absence of heat source) [13, 14]

$$-T_0(\rho\dot{\eta})^L = q_{i,i}^L \quad (i = 1, 3). \tag{2.4}$$

Equations of motion (in the absence of body force) have the form

$$\frac{\partial}{\partial x_j}\sigma_{ij}^L = \rho\frac{\partial^2 u_i}{\partial t^2} \quad (i, j = 1, 3). \tag{2.5}$$

Substituting (2.2), (2.3) into (2.4), (2.5), we obtain the field equations in terms of displacement and temperature for the homogeneous and nonlocal orthotropic thermoelastic medium as

$$\begin{aligned} &c_{11}u_{1,11} + (c_{13}+c_{55})u_{3,13} + c_{55}u_{1,33} - \beta_{11}(T + t_1\dot{T})_{,1} = \rho(1-\epsilon^2\nabla^2)\ddot{u}_1, \\ &c_{55}u_{3,11} + (c_{13}+c_{55})u_{1,13} + c_{33}u_{3,33} - \beta_{33}(T + t_1\dot{T})_{,3} = \rho(1-\epsilon^2\nabla^2)\ddot{u}_3, \\ &K_{11}T_{,11} + K_{33}T_{,33} - \rho C_E(\dot{T} + t_0\ddot{T}) \\ &\qquad\qquad = T_0\big(\beta_{11}(\dot{u}_{1,1} + \varepsilon t_0\ddot{u}_{1,1}) + \beta_{33}(\dot{u}_{3,3} + \varepsilon t_0\ddot{u}_{3,3})\big). \end{aligned} \tag{2.6}$$

The relaxation times t_0, t_1 and the parameter ε are chosen to represent generalised thermoelasticity; $\varepsilon = 1$, $t_1 = 0$, $t_0 > 0$ for the Lord–Shulman theory [1], and $\varepsilon = 0$, $t_0 = 0$, $t_1 > 0$ for the Green–Lindsay theory [2].

For harmonic propagation of bulk waves in nonlocal thermoelastic medium, the mechanical and thermal displacements are expressed as

$$(u_1, u_3, T) = (a_1, a_3, A)\exp[i\omega(p_1x_1 + p_3x_3 - t)], \tag{2.7}$$

where $\omega = kC$ is the circular frequency, k is wavenumber, C is velocity, p_1, p_3 are the components of the slowness vector.

Using these expressions, the differential equations (2.6) translate into a homogeneous system of three linear equations. These equations, called Christoffel equations, are written as

$$M_{k1}a_1 + M_{k2}a_3 + M_{k3}A = 0 \quad (k = 1, 2, 3). \tag{2.8}$$

The coefficient matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} c_{11}p_1^2 + c_{55}p_3^2 - \rho^N & (c_{13}+c_{55})p_1p_3 & \beta_{11}\tau_1 p_1 \\ (c_{13}+c_{55})p_1p_3 & c_{55}p_1^2 + c_{33}p_3^2 - \rho^N & \beta_{33}\tau_1 p_3 \\ \beta_{11}T_0\tau_\epsilon p_1 & \beta_{33}T_0\tau_\epsilon p_3 & (K_{11}p_1^2 + K_{33}p_3^2 - \rho C_E\tau_0)/\omega \end{bmatrix}, \tag{2.9}$$

where

$$\rho^N = \rho(1 + \epsilon^2\omega^2(p_1^2 + p_3^2)), \quad \tau_0 = t_0 + \frac{i}{\omega}, \quad \tau_1 = t_1 + \frac{i}{\omega}, \quad \tau_\varepsilon = \varepsilon t_0 + \frac{i}{\omega}.$$

The non-trivial solution (a_1, a_3, A) of the homogeneous system (2.8) is ensured with $\det(\mathbf{M}) = 0$. With (p_1, p_3) given, the characteristic equation $\det(\mathbf{M}) = 0$ is solved into a cubic equation in C^2. Three complex roots of this equation define three complex velocities (C_j, $j = 1, 2, 3$), which imply the anisotropic propagation of three attenuated waves. When sorted in ascending order of $real(1/C_j)$, the waves associated with (C_1, C_2, C_3) are identified as quasi-longitudinal (qL), quasi-transverse (qT) and quasi-thermal (T-mode), respectively [5, 18].

2.2. Statement of the problem

We consider two distinct nonlocal orthotropic elastic half-spaces Ω^+ and Ω^- that represented with four elastic constants $(c_{11}, c_{13}, c_{33}, c_{55})$ and five thermal coefficients $(\beta_{11}, \beta_{33}, C_E, K_{11}, K_{33})$ (see Fig. 1). In this case, the material constants of the two half-spaces are given by

$$c_{ij}, \beta_{ij}, C_E, K_{ij} = \begin{cases} c_{ij}^+, \beta_{ij}^+, C_E^+, K_{ij}^+ & \text{for } \Omega^+, \\ c_{ij}^-, \beta_{ij}^-, C_E^-, K_{ij}^- & \text{for } \Omega^-. \end{cases} \tag{2.10}$$

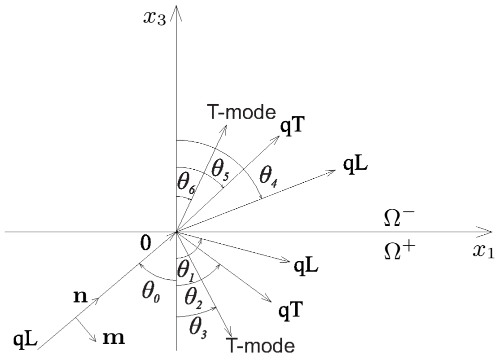


FIG. 1. Geometry of the problem.

Consider the problem shown in Fig. 1. The Ω^- medium occupies the space $x_3 > 0$, which is mechanical bonding with the Ω^+ medium occupied the space $x_3 < 0$. The x_1-axis is taken along the interface and the x_3-axis is directed vertically upwards. For the oblique incidence of the inhomogeneous qL wave from the Ω^+ medium at the interface $x_3 = 0$, all kinds of scattered waves are depicted in Fig. 1. The transmitted wave fields consist of the transmitted qL, qT and T-mode waves; and the reflected wave fields make up of the reflected qL, qT, T-mode waves.

2.3. Slowness vector of inhomogeneous waves

The vector $\mathbf{p} = (p_1, p_3)$ defines the slowness of plane waves propagating in thermoelastic medium. The components of this slowness vector are, in general, complex. Therefore, the slowness vector $\mathbf{p}$ is the complex-valued vector

$$\mathbf{p} = \mathbf{P} + i\mathbf{A}. \tag{2.11}$$

Here $\mathbf{P}$ is the real-valued propagation vector (the direction of propagation), $\mathbf{A}$ is the real-valued attenuation vector (the direction of the maximum decay of amplitude). When $\mathbf{P}$ and $\mathbf{A}$ are parallel, the wave is said to be homogeneous, otherwise it is called inhomogeneous. We introduce the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ in the direction of $\mathbf{P}$ and $\mathbf{A}$, respectively, and the attenuation angle ψ, $0° \leq \psi < 180°$

$$\mathbf{N} = \mathbf{P}/|\mathbf{P}|, \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}|, \quad \cos(\psi) = \mathbf{NM}.$$

We consider the plane waves propagating in the plane of symmetry Σ^S of orthotropic thermoelastic and the coordinate system x_i in such a way that the plane of symmetry Σ^S corresponds to the coordinate x_1x_3.

2.4. Incident wave

Let a longitudinal displacement plane wave, the incident angle θ_0 and inhomogeneity parameter D propagates in the half-space Ω^+. For $D = 0$, the plane wave is homogeneous, and for $D \neq 0$ it is inhomogeneous. In order to determine the incident wave belongs to θ_0 and D, we use the mixed specification of slowness vector $\mathbf{p} = \sigma\mathbf{n} + iD\mathbf{m}$ (see [6, 19]) with $\mathbf{n}(n_1, 0, n_3)$ and $\mathbf{m}(m_1, 0, m_3)$ are given:

$$n_1 = \sin\theta_0, \quad n_3 = \cos\theta_0, \quad m_1 = \cos\theta_0, \quad m_3 = -\sin\theta_0. \tag{2.12}$$

Due to this choice of the unit vector $\mathbf{m}$, the propagation-attenuation plane coincides with the symmetry plane. For the incident wave, it is note that medium is orthotropic, when the equation $\det(\mathbf{M}) = 0$ leads to,

$$t_4\sigma^4 + t_3\sigma^3 + t_2\sigma^2 + t_1\sigma + t_0 = 0, \tag{2.13}$$

where the coefficients t_0, t_1, t_2, t_3, t_4 are functions of θ_0, D. These coefficients can be determined through some symbolic routines in a mathematical software (MATLAB). Note that, when the incident wave is homogeneous ($D = 0$), Eq. (2.13) reduces to a quadratic equation for σ^2. We can determine relevant terms. They are given by relations [6, 19]:

$$\begin{aligned} &\mathbf{P} = \mathbf{n}\,\mathrm{Re}(\sigma), \quad |\mathbf{P}| = |\mathrm{Re}(\sigma)|, \\ &\mathbf{A} = \mathbf{n}Im(\sigma) + D\mathbf{m}, \quad |\mathbf{A}| = [\mathrm{Im}(\sigma)^2 + D^2]^{1/2}, \\ &c_0 = 1/|\mathbf{P}|, \quad \cos\psi_0 = \frac{\mathrm{Re}(\sigma)}{|\mathrm{Re}(\sigma)|}\,\mathrm{Im}(\sigma)/[(\mathrm{Im}(\sigma)^2 + D^2)]^{1/2}, \end{aligned} \tag{2.14}$$

where c_0, ψ_0 is the phase velocity, the attenuation angle of the incident wave.

2.5. Reflected and transmitted waves

In order to study the reflection and transmission of waves, the slowness vector $\mathbf{p}$ is expressed in terms of a known real-valued unit vector $\mathbf{n}$ (the unit vector of x_3 axis) and a complex-valued vector $\mathbf{p}_0$ as follows [6, 7, 19]. Note that plane Σ^S coincides to $x_3 = 0$ one,

$$\mathbf{p} = \delta\mathbf{n} + \mathbf{p}_0, \quad \text{with} \quad \mathbf{p}_0\mathbf{n} = 0. \tag{2.15}$$

In fact, $\mathbf{p}_0$ represents a known vectorial component of the slowness vector $\mathbf{p}$ in the plane $x_3 = 0$ perpendicular to $\mathbf{n}$. For the reflection-transmission problems, from the Snell law $\mathbf{p}_0$ a known vectorial component of slowness vector of an incident wave. The unknown complex-valued quantity δ must be determined by inserting (2.15) into (2.9). That leads to a cubic equation in δ^2

$$h_6\delta^6 + h_4\delta^4 + h_2\delta^2 + h_0 = 0, \tag{2.16}$$

where the coefficients h_6, h_4, h_2, h_0 are given in Appendix. Once a value of δ has been found from (2.16) for qL, qT, T-mode waves, we can determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, their unit vectors $\mathbf{N}$, $\mathbf{M}$ respectively, phase velocity c and the attenuation angle ψ. They are given by relations [6, 19]:

$$\begin{aligned}
&\mathbf{P} = \mathbf{n}\,\mathrm{Re}(\delta) + \mathrm{Re}(\mathbf{p}_0), \quad |\mathbf{P}| = [\mathrm{Re}(\delta)^2 + \mathrm{Re}(\mathbf{p}_0)\,\mathrm{Re}(\mathbf{p}_0)]^{1/2},\\
&\mathbf{A} = \mathbf{n}\,\mathrm{Im}(\delta) + \mathrm{Im}(\mathbf{p}_0), \quad |\mathbf{A}| = [\mathrm{Im}(\delta)^2 + \mathrm{Im}(\mathbf{p}_0)\,\mathrm{Im}(\mathbf{p}_0)]^{1/2},\\
&\mathbf{N} = \mathbf{P}/|\mathbf{P}|, \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}|, \quad c = 1/|\mathbf{P}|,\\
&\cos\psi = [\mathrm{Re}(\mathbf{p}_0)\,\mathrm{Im}(\mathbf{p}_0) + \mathrm{Re}(\delta)\,\mathrm{Im}(\delta)]/|\mathbf{P}|\,|\mathbf{A}|.
\end{aligned} \tag{2.17}$$

The corresponding slowness vectors $\mathbf{p}^j = (p_1^j, p_3^j)$ for reflected, transmitted waves are obtained. The index $(j = 1, 2, 3)$ is associated with three reflected waves T-mode, qT, qL; the one $(j = 4, 5, 6)$ are associated with three transmitted waves T-mode, qT, qL and $p_1^j = p_1^0$ of the incident wave qL.

Remark. The terms σ in (2.13) and δ (2.16) must be selected, which propagate away from the interface for reflected, transmitted waves and propagate towards interface for an incident wave. As you know, for isotropic perfectly elastic media and real-valued of slowness vector, the selection criterion is simple. For anisotropic elastic media and of the slowness vector, analogous selection criteria are based on the directions of the time averaged energy flux, not on the directions of the slowness vector. Actually the criterion based on the direction of the time-averaged energy flux is universal, valid also for isotropic media, as the time-averaged energy flux is parallel to the slowness vector in isotropic media. For thermoelastic as well as viscoelastic media, however, the selection criteria are still a subject of research, see Krebes *et al.* [20, 21], Richards [22], Ruud [23], etc. The problem of the selection criteria, however, is not treated here. In this paper, we investigate a simple problem where the transmitted waves are upgoing

in the upper half-space, and the reflected waves are downgoing in the lower half-space and these waves ensure the decay of wave-field with depth; the incident wave in the lower half-space is towards interface $x_3 = 0$ (see [6, 7, 18, 19]). This means that:
(i) The incident wave is selected if $\mathrm{Re}(\sigma) > 0$; $\mathrm{Im}(\sigma) < 0$.
(ii) The reflected waves are selected if $\mathrm{Im}(p_3^j) < 0$ $(j = 1, 2, 3)$.
(iii) The transmitted waves are selected if $\mathrm{Im}(p_3^j) > 0$ $(j = 4, 5, 6)$.

3. The reflection and transmission coefficients

3.1. Solutions for the upper and lower half-spaces

(i) Incident qL wave

The qL wave incident at the boundary in the lower nonlocal thermoelastic half-space Ω^+ whose displacement, temperature change can be given as

$$\begin{bmatrix} u_1^0 \\ u_3^0 \\ T^0 \end{bmatrix} = A_0 \begin{bmatrix} \alpha_0 \\ \gamma_0 \\ 1 \end{bmatrix} e^{i\omega(p_1^0 x_1 + p_3^0 x_3 - t)}. \tag{3.1}$$

(ii) Reflected, transmitted waves

After reflecting, transmitting at the interface $x_3 = 0$, the reflected waves T-mode, qT and qL (for $j = 1, 2, 3$) in originating medium Ω^+ and the transmitted waves T-mode, qT and qL (for $j = 4, 5, 6$) in the continuing medium Ω^- may be written as

$$\begin{bmatrix} u_1^j \\ u_3^j \\ T^j \end{bmatrix} = A_j \begin{bmatrix} \alpha_j \\ \gamma_j \\ 1 \end{bmatrix} e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)}, \tag{3.2}$$

where A_j $(j = 0, \ldots, 6)$ are the amplitudes of the incident, reflected, transmitted waves, respectively and:

$$\begin{aligned} \alpha_j &= \frac{(c_{13} + c_{55}) p_1^0 (p_3^j)^2 \beta_{33} \tau_1 - \big(c_{55} (p_1^0)^2 + c_{33} (p_3^j)^2 - \rho^N\big) \beta_{11} \tau_1 p_1^0}{\Delta_j}, \\ \gamma_j &= \frac{(c_{13} + c_{55}) (p_1^0)^2 p_3^j \beta_{11} \tau_1 - \big(c_{11} (p_1^0)^2 + c_{55} (p_3^j)^2 - \rho^N\big) \beta_{33} \tau_1 p_3^j}{\Delta_j}, \\ \Delta_j &= \big(c_{11} (p_1^0)^2 + c_{55} (p_3^j)^2 - \rho^N\big)\big(c_{55} (p_1^0)^2 + c_{33} (p_3^j)^2 - \rho^N\big) \\ &\quad - (c_{13} + c_{55})^2 (p_1^0 p_3^j)^2 \quad (j = 0, \ldots, 6). \end{aligned} \tag{3.3}$$

The generalized Snell law has been taken into account in (3.2) and (3.3).

3.2. Boundary conditions

At the common interface between two halfspaces, the welded contact is maintained through the continuity of stresses and displacements. The continuity of temperature as well as heat flux ensures the isothermal conditions at the interface [3, 15]:

$$\begin{aligned} &u_1^+ = u_1^-, \quad u_3^+ = u_3^-, \quad T^+ = T^-,\\ &\sigma_{13}^+ = \sigma_{13}^-, \quad \sigma_{33}^+ = \sigma_{33}^-, \quad K_{33}^+ \frac{\partial T^+}{\partial x_3} = K_{33}^- \frac{\partial T^-}{\partial x_3}. \end{aligned} \tag{3.4}$$

To facilitate later calculations, we provide the displacement, stress field of two halfspaces Ω^+ and Ω^-, respectively:

$$\begin{aligned} u_1^+ &= \sum_{j=0}^{3} u_1^j = \sum_{j=0}^{3} \alpha_j A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ u_3^+ &= \sum_{j=0}^{3} u_3^j = \sum_{j=0}^{3} \gamma_j A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ T^+ &= \sum_{j=0}^{3} T^j = \sum_{j=0}^{3} A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ u_1^- &= \sum_{j=4}^{6} u_1^j = \sum_{j=4}^{6} \alpha_j A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ u_3^- &= \sum_{j=4}^{6} u_3^j = \sum_{j=4}^{6} \gamma_j A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ T^- &= \sum_{j=4}^{6} T^j = \sum_{j=4}^{6} A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)}; \end{aligned} \tag{3.5}$$

$$\begin{aligned} \sigma_{13}^+ &= \sum_{j=0}^{3} c_{55}^+ i\omega\big(\alpha_j p_3^j + \gamma_j p_1^0\big) A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ \sigma_{13}^- &= \sum_{j=4}^{6} c_{55}^- i\omega\big(\alpha_j p_3^j + \gamma_j p_1^0\big) A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ \sigma_{33}^+ &= \sum_{j=0}^{3} i\omega\big(c_{13}^+ \alpha_j p_1^0 + c_{33}^+ \gamma_j p_3^j + \beta_{33}^+ \tau_1\big) A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)},\\ \sigma_{33}^- &= \sum_{j=4}^{6} i\omega\big(c_{13}^- \alpha_j p_1^0 + c_{33}^- \gamma_j p_3^j + \beta_{33}^- \tau_1\big) A_j e^{i\omega(p_1^0 x_1 + p_3^j x_3 - t)}, \end{aligned} \tag{3.6}$$

$$K_{33}^{+}\frac{\partial T}{\partial x_3}=\sum_{j=0}^{3}K_{33}^{+}i\omega p_3^jA_je^{i\omega(p_1^0x_1+p_3^jx_3-t)},$$
(3.6)[cont.]
$$K_{33}^{-}\frac{\partial T}{\partial x_3}=\sum_{j=4}^{6}K_{33}^{-}i\omega p_3^jA_je^{i\omega(p_1^0x_1+p_3^jx_3-t)}.$$

For the displacements, temperature in (3.5) and the stresses, heat flux in (3.6), these conditions (3.4) yield six linear equations, which are expressed as follows:

$$\begin{cases}-\alpha_1A_1-\alpha_2A_2-\alpha_3A_3+\alpha_4A_4+\alpha_5A_5+\alpha_6A_6=\alpha_0A_0,\\ -\gamma_1A_1-\gamma_2A_2-\gamma_3A_3+\gamma_4A_4+\gamma_5A_5+\gamma_6A_6=\gamma_0A_0,\\ -A_1-A_2-A_3+A_4+A_5+A_6=A_0,\\ -c_{55}^{+}(\alpha_1p_3^1+\gamma_1p_1^0)A_1-c_{55}^{+}(\alpha_2p_3^2+\gamma_2p_1^0)A_2-c_{55}^{+}(\alpha_3p_3^3+\gamma_3p_1^0)A_3\\ \quad+c_{55}^{-}(\alpha_4p_3^4+\gamma_4p_1^0)A_4+c_{55}^{-}(\alpha_5p_3^5+\gamma_5p_1^0)A_5+c_{55}^{-}(\alpha_6p_3^6+\gamma_6p_1^0)A_6\\ \qquad=c_{55}^{+}(\alpha_0p_3^0+\gamma_0p_1^0)A_0,\\ -(c_{13}^{+}\alpha_1p_1^0+c_{33}^{+}\gamma_1p_3^1+\beta_{33}^{+}\tau_1^{+})A_1-(c_{13}^{+}\alpha_2p_1^0+c_{33}^{+}\gamma_2p_3^2+\beta_{33}^{+}\tau_1^{+})A_2\\ \qquad-(c_{13}^{+}\alpha_3p_1^0+c_{33}^{+}\gamma_3p_3^3+\beta_{33}^{+}\tau_1^{+})A_3+(c_{13}^{-}\alpha_4p_1^0+c_{33}^{-}\gamma_4p_3^4+\beta_{33}^{-}\tau_1^{-})A_4\\ \qquad+(c_{13}^{-}\alpha_5p_1^0+c_{33}^{-}\gamma_5p_3^5+\beta_{33}^{-}\tau_1^{-})A_5+(c_{13}^{-}\alpha_6p_1^0+c_{33}^{-}\gamma_6p_3^6+\beta_{33}^{-}\tau_1^{-})A_6\\ \qquad=(c_{13}^{+}\alpha_0p_1^0+c_{33}^{+}\gamma_0p_3^0+\beta_{33}^{+}\tau_1^{+})A_0,\\ -K_{33}^{+}p_3^1A_1-K_{33}^{+}p_3^2A_2-K_{33}^{+}p_3^3A_3+K_{33}^{-}p_3^4A_4+K_{33}^{-}p_3^5A_5\\ \qquad+K_{33}^{-}p_3^6A_6=K_{33}^{+}p_3^0A_0.\end{cases}\tag{3.7}$$

After solving the system of equations (3.7) for the ratios A_i/A_0 $(i=1,\dots,6)$, the reflection, transmission coefficients (RTCs) are defined by the ratios of the reflected/transmitted amplitudes to the incident amplitude [12]:

$$R_1=\frac{\sqrt{1+\alpha_1^2+\gamma_1^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_1}{A_0},\ R_2=\frac{\sqrt{1+\alpha_2^2+\gamma_2^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_2}{A_0},\ R_3=\frac{\sqrt{1+\alpha_3^2+\gamma_3^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_3}{A_0},$$
(3.8)
$$T_1=\frac{\sqrt{1+\alpha_4^2+\gamma_4^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_4}{A_0},\ T_2=\frac{\sqrt{1+\alpha_5^2+\gamma_5^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_5}{A_0},\ T_3=\frac{\sqrt{1+\alpha_6^2+\gamma_6^2}}{\sqrt{1+\alpha_0^2+\gamma_0^2}}\frac{A_6}{A_0}.$$

4. Numerical results and discussions

In order to illustrate theoretical results obtained in the preceding sections, we have computed them numerically for a particular model for which the values of relevant elastic parameters are taken from Kumar *et al.* [3] as:

For halfspace Ω^+-Magnessium:

$$\begin{aligned}
c_{11}^+ &= 5.974\times 10^{10}\,\mathrm{N\cdot m^{-2}},\ c_{13}^+ = 6.17\times 10^{10}\,\mathrm{N\cdot m^{-2}},\\
c_{33}^+ &= 6.17\times 10^{10}\,\mathrm{N\cdot m^{-2}},\ c_{55}^+ = 3.278\times 10^{10}\,\mathrm{N\cdot m^{-2}},\\
\beta_{11}^+ &= 2.68\times 10^{6}\,\mathrm{N\cdot m^{-2}\cdot deg^{-1}},\ \beta_{33}^+ = 2.68\times 10^{6}\,\mathrm{N\cdot m^{-2}\cdot deg^{-1}},\\
K_{11}^+ &= 1.7\times 10^{2}\,\mathrm{W\cdot m^{-1}\cdot deg^{-1}},\ K_{33}^+ = 1.7\times 10^{2}\,\mathrm{W\cdot m^{-1}\cdot deg^{-1}},\\
\rho^+ &= 1.74\times 10^{3}\,\mathrm{kg\cdot m^{-3}},\ c_E^+ = 1.04\times 10^{3}\,\mathrm{J\cdot kg^{-1}\cdot deg^{-1}},\ T_0^+ = 298\,\mathrm{K}.
\end{aligned}$$

For halfspace Ω^--Cobalt:

$$\begin{aligned}
c_{11}^- &= 3.071\times 10^{11}\,\mathrm{N\cdot m^{-2}},\ c_{13}^- = 1.027\times 10^{11}\,\mathrm{N\cdot m^{-2}},\\
c_{33}^- &= 3.581\times 10^{11}\,\mathrm{N\cdot m^{-2}},\ c_{55}^- = 1.510\times 10^{10}\,\mathrm{N\cdot m^{-2}},\\
\beta_{11}^- &= 7.04\times 10^{6}\,\mathrm{N\cdot m^{-2}\cdot deg^{-1}},\ \beta_{33}^- = 6.90\times 10^{6}\,\mathrm{N\cdot m^{-2}\cdot deg^{-1}},\\
K_{11}^- &= 0.690\times 10^{2}\,\mathrm{W\cdot m^{-1}\cdot deg^{-1}},\ K_{33}^- = 0.690\times 10^{2}\,\mathrm{W\cdot m^{-1}\cdot deg^{-1}},\\
\rho^- &= 8.836\times 10^{3}\,\mathrm{kg\cdot m^{-3}},\ c_E^- = 4.27\times 10^{2}\,\mathrm{J\cdot kg^{-1}\cdot deg^{-1}},\ T_0^- = 298\,\mathrm{K}.
\end{aligned}$$

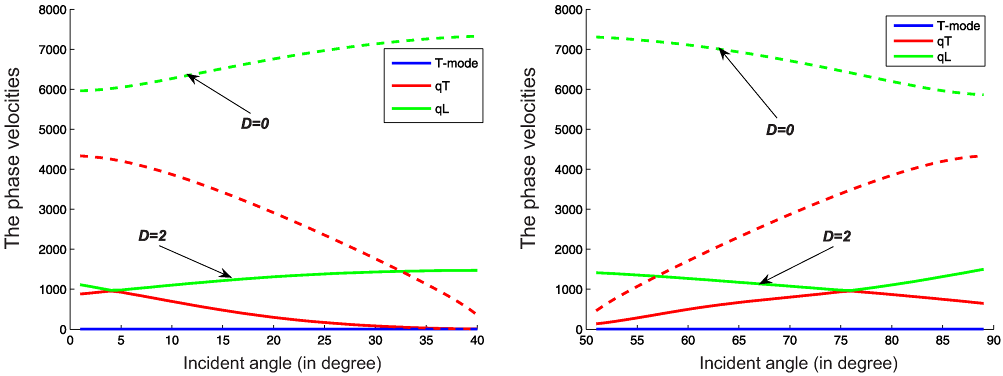
The thermal relaxation times t_0, t_1 and the parameter ε for two halfspaces are chosen to apply the generalised thermoelasticity theory of Lord and Shulman. The frequency for harmonic propagation ω is fixed at 100 Hz.

$$\omega = 100, \quad \varepsilon^+ = \varepsilon^- = 1, \quad t_0^+ = 0.15, \quad t_1^+ = 0, \quad t_0^- = 0.05, \quad t_1^- = 0.$$

The propagation characteristics of the reflected, transmitted waves are computed for the incidence of qL wave, in turn. These characteristics include the phase direction (θ_k), the attenuation direction (ψ_k), amplitude ratios (R_i, T_i) ($i = 1, \ldots, 3$, $k = 1, \ldots, 6$). Variations of these characteristics are analysed with the direction (θ_0) and the inhomogeneity parameter (D) of incident qL wave. Moreover, the effects of elastic nonlocality parameter on the phase velocity of all the propagating waves have been studied numerically.

In Fig. 2, it exhibits the variations in phase velocities of incident waves with the phase direction θ_0. The phase velocities of qL, qT and T-mode waves are computed from (2.14) for aforementioned physical constants for the range $0^\circ < \theta_0 < 90^\circ$ except the range $40^\circ < \theta_0 < 50^\circ$. The values of each phase velocity for $D = 0$ in the range $40^\circ < \theta_0 < 50^\circ$ become quite large due to the quite small values of each $\mathrm{Re}(\sigma)$. Therefore, Fig. 2 is divided into two subfigures (a) and (b) for easy comparison. The comparison of solid curves $D = 2$ and dashed curves $D = 0$ in these figures shows the inhomogeneity of incident wave effects on these phase velocities of incident waves.

As shown in Remark, the selection of the incident wave must satisfy the condition $\mathrm{Re}(\sigma) > 0$ and $\mathrm{Im}(\sigma) < 0$. Calculating on this particular model,



(a) (b)

FIG. 2. Variations in phase velocities of incident waves with phase direction θ_0 for two cases: $D = 2$ and $D = 0$.

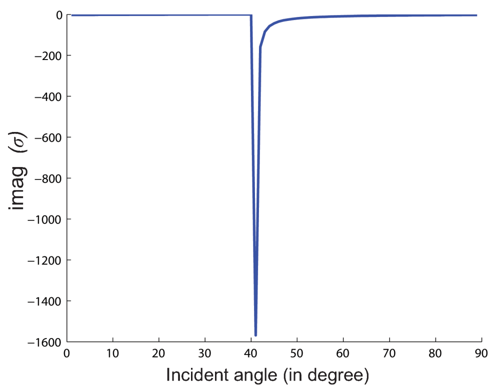


FIG. 3. Condition for the existence of incident wave qL: $-\infty < D < 3.2$.

we find that there can exist one, two or three incident waves for the range $-\infty < D < 3.2$. On the contrary, there will be no incident waves satisfying the above condition. Figure 3 represents the existence of the incident qL wave in the range $-\infty < D < 3.2$.

The phase directions (θ_k), attenuation angles (ψ_k) and amplitude ratios (R_i, T_i) of reflected, transmitted (qL, qT, T-mode) waves vary with phase direction (θ_0) of the incident wave are presented in Figs. 4–9. The solid and dashed curves in various plots identify the reflected, transmitted waves from inhomogeneous and homogeneous incidence, respectively. Again, in any plot, the different curves (i.e., solid, dashed) can be compared, in turn, to check the influence of presence of incident wave inhomogeneity on these characteristics.

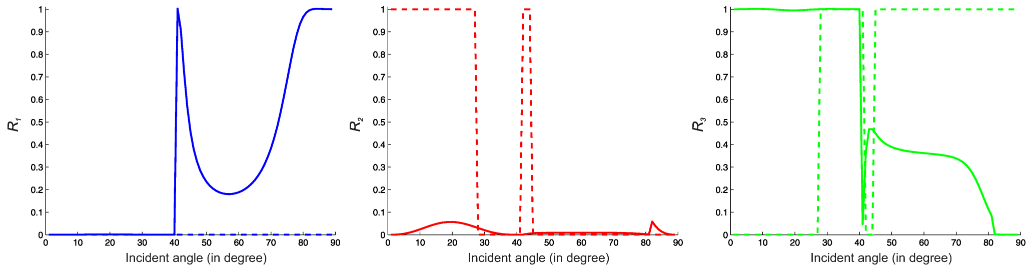


FIG. 4. The reflection coefficients R_1, R_2, R_3 for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

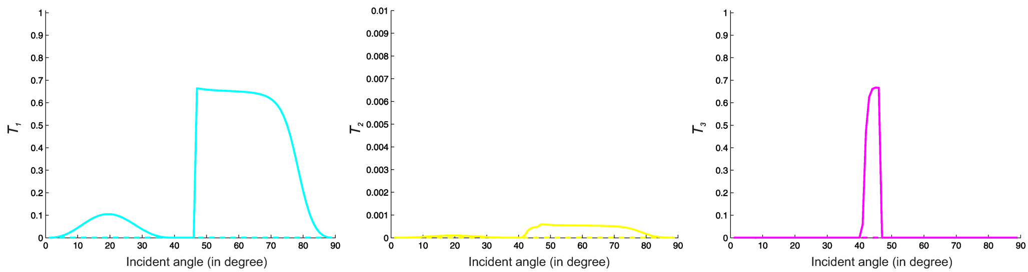


FIG. 5. The transmission coefficients T_1, T_2, T_3 for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

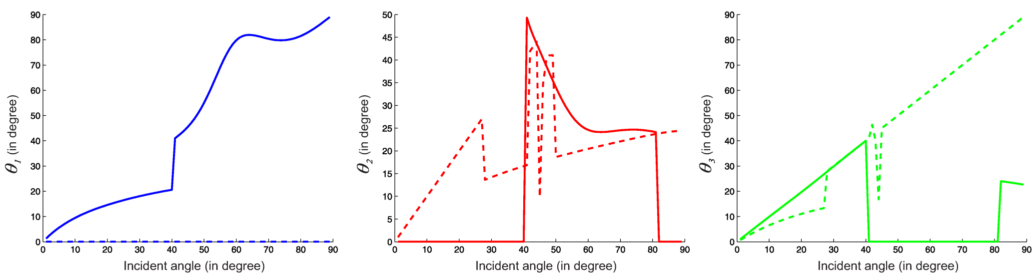


FIG. 6. The phase direction of reflected waves for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

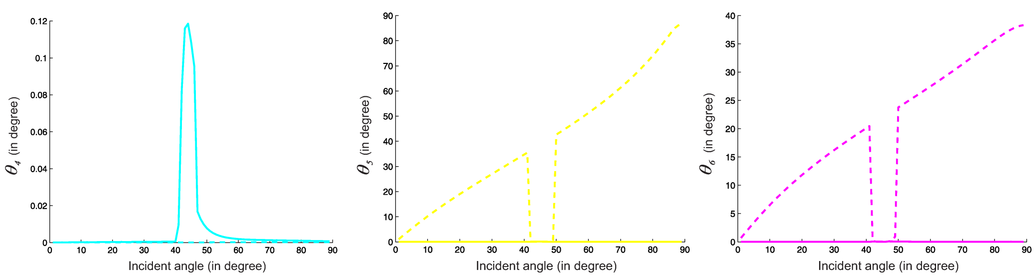


FIG. 7. The phase direction of transmitted waves for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

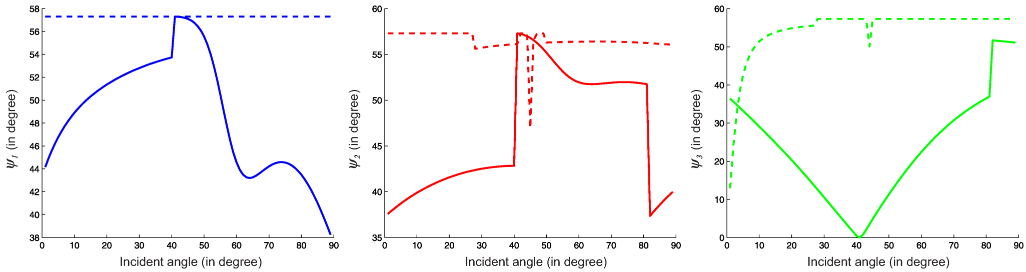


FIG. 8. The attenuation angle of reflected waves for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

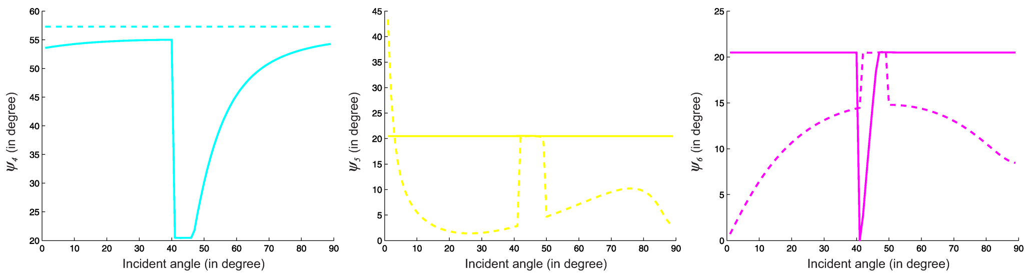


FIG. 9. The attenuation angle of transmitted waves for inhomogeneous and homogeneous incident wave; solid line represents $D = 2$, dashed line for $D = 0$.

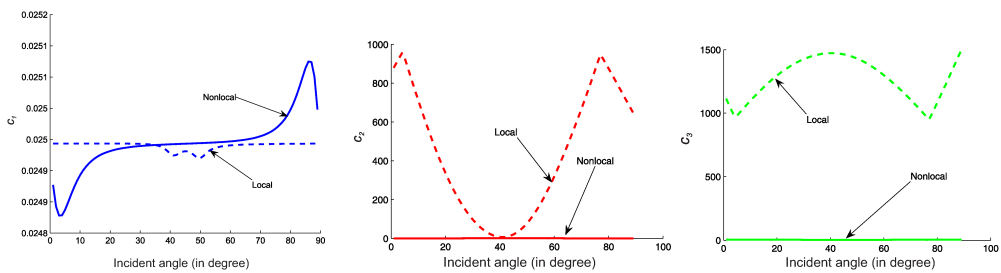


FIG. 10. Comparison of phase velocities with respect to the incident angle θ_0 for local ($\epsilon = 0$) and nonlocal ($\epsilon > 0$) medium.

Since the nonlocal parameters of Magnessium and Cobalt are not available, we choose the internal characteristic lengths of the two halfspaces which are $e_0^+ = 0.39$, $a^+ = 0.121 \times 10^{-9}$ and $e_0^- = 0.32$, $a^- = 0.421 \times 10^{-9}$, respectively. When $\omega = 100\,\text{Hz}$ the relative difference between velocities of local and nonlocal theories is very small. We also see that the effect of the nonlocality is substantial when the considered wave-length is comparable with the characteristic length of the medium, namely $\omega = 10^{14}\,\text{Hz}$ (see Fig. 10). This suggests the necessity of the nonlocal property when working with the wave of very short wave-length or very high wave frequency.

5. Conclusions

In conclusion, a mathematical study of reflection and transmission of waves at an interface separating two nonlocal orthotropic thermoelastic half-spaces is made when a qL wave is incident. It is observed that:

(i) for incident wave: a specification of the slowness vector, based on the use of unit vectors **n** and **m**, and the inhomogeneity parameter D, can be used to study homogeneous or inhomogeneous waves propagating in thermoelastic media;

(ii) for reflected, transmitted waves: another of the slowness vector, based on the Snell's law, horizontal slowness determines the slowness vectors for all reflected, transmitted waves. The propagation characteristics of the reflected, transmitted waves include the phase velocity (c_k), the phase direction (θ_k), the attenuation direction (ψ_k) are computed from the slowness vector of each wave;

(iii) based on the continuity of stresses and displacements at interface between two halfspaces and the continuity of temperature as well as heat flux ensures the isothermal conditions at the interface, the amplitude ratios (R_i, T_i) $(i = 1, 2, 3)$ for reflected, transmitted waves relative to the incident wave are given.

From numerical analysis, it is shown that the phase velocities, phase directions, attenuation directions and reflection, transmission coefficients of various plane waves are affected significantly due to the presence of the inhomogeneity of incident wave as well as nonlocality parameter.

Appendix

The coefficients of the equation for δ of the reflected, transmitted waves:

$$
\begin{aligned}
h_6 = {}& K_{33}(C_{33} - \epsilon^2\omega^2\rho)(C_{55} - \epsilon^2\omega^2\rho),\\
h_4 = {}& -C_{13}^2K_{33}p_0^2 - 2C_{13}C_{55}K_{33}p_0^2 + C_{33}(C_{11}K_{33}p_0^2 + C_{55}(K_{11}p_0^2 - c_E\rho\tau_0)\\
& - \rho(K_{33} + \epsilon^2K_{33}\omega^2p_0^2 + \epsilon^2\omega^2(K_{11}p_0^2 - c_E\rho\tau_0)))\\
& - \rho(C_{55}(K_{33} + 2\epsilon^2K_{33}\omega^2p_0^2 + \epsilon^2\omega^2(K_{11}p_0^2 - c_E\rho\tau_0))\\
& + \epsilon^2\omega^2(C_{11}K_{33}p_0^2 + \rho(-2K_{33}(1 + \epsilon^2\omega^2p_0^2) + \epsilon^2\omega^2(-K_{11}p_0^2 + c_E\rho\tau 0))))\\
& + \beta_{33}^2\omega(-C_{55} + \epsilon^2\omega^2\rho)T_0\tau_1\tau_\epsilon,\\
h_2 = {}& -C_{33}K_{11}p_0^2\rho - C_{55}K_{11}p_0^2\rho - C_{55}K_{33}p_0^2\rho - C_{33}\epsilon^2K_{11}\omega^2p_0^4\rho\\
& - 2C_{55}\epsilon^2K_{11}\omega^2p_0^4\rho - C_{55}\epsilon^2K_{33}\omega^2p_0^4\rho + K_{33}\rho^2 + 2\epsilon^2K_{11}\omega^2p_0^2\rho^2\\
& + 2\epsilon^2K_{33}\omega^2p_0^2\rho^2 + 2\epsilon^4K_{11}\omega^4p_0^4\rho^2 + \epsilon^4K_{33}\omega^4p_0^4\rho^2 + C_{33}c_E\rho^2\tau_0\\
& + C_{55}c_E\rho^2\tau_0 + C_{33}c_E\epsilon^2\omega^2p_0^2\rho^2\tau_0 + 2C_{55}c_E\epsilon^2\omega^2p_0^2\rho^2\tau_0 - 2c_E\epsilon^2\omega^2\rho^3\tau_0\\
& - 2c_E\epsilon^4\omega^4p_0^2\rho^3\tau_0 + C_{13}^2(-K_{11}p_0^4 + c_Ep_0^2\rho\tau_0)
\end{aligned}
$$

$$
\begin{aligned}
&+\omega(\beta_{11}(-\beta_{11}C_{33}+2\beta_{33}C_{55})p_0^2+(\beta_{33}^2+(\beta_{11}^2+\beta_{33}^2)\epsilon^2\omega^2p_0^2)\rho)T_0\tau_1\tau_\epsilon \\
&+2C_{13}p_0^2(-C_{55}K_{11}p_0^2+C_{55}c_E\rho\tau_0 \\
&+\beta_{11}\beta_{33}\omega T_0\tau_1\tau_\epsilon+C_{11}p_0^2(C_{55}K_{33}p_0^2+C_{33}(K_{11}p_0^2-c_E\rho\tau_0) \\
&-\rho(K_{33}+\epsilon^2K_{33}\omega^2p_0^2+\epsilon^2\omega^2(K_{11}p_0^2-c_E\rho\tau_0))-\beta_{33}^2\omega T_0\tau_1\tau_\epsilon)), \\
h_0 =\ &-(-\rho+p_0^2(C_{55}-\epsilon^2\omega^2\rho))(-(\rho+p_0^2(-C_{11}+\epsilon^2\omega^2\rho)) \\
&\times(-K_{11}p_0^2+c_E\rho\tau_0)+\beta_{11}^2\omega p_0^2T_0\tau_1\tau_\epsilon).
\end{aligned}
$$

References

1. H.W. Lord, Y. Shulman, *A generalized dynamical theory of thermoelasticity*, Journal of the Mechanics and Physic of Solids, **15**, 299–309, 1967.
2. A.E. Green, K.A. Lindsay, *Thermoelasticity*, Journal of Elasticity, **2**, 1–7, 1972.
3. R. Kumar, *Reflection/transmission of plane waves at an imperfectly bonded interface of two orthotropic generalized thermoelastic half-spaces*, Materials Science and Engineering, A **472**, 83–96, 2008.
4. M.D. Sharma, *Propagation of inhomogeneous plane waves in anisotropic viscoelastic media*, Acta Mechanica, **200**, 145–154, 2008.
5. B. Singh, *Wave propagation in an initially stressed transversely isotropic thermoelastic solid half-space*, Applied Mathematics and Computation, **217**, 705–715, 2010.
6. V. Cerveny, I. Psencik, *Plane waves in viscoelastic anisotropic media – I, Theory*, Geophysical Journal International, **161**, 197–212, 2005.
7. V. Cerveny, I. Psencik, *Plane waves in viscoelastic anisotropic media – II, Numerical examples*, Geophysical Journal International, **161**, 213–229, 2005.
8. A.C. Eringen, *Theory of nonlocal thermoelasticity*, International Journal of Engineering Science, **12**, 1063–1077, 1974.
9. A.C. Eringen, *Nonlocal Continuum Theories*, Springer, New York, 2002.
10. D.X. Tung, *Surface waves in nonlocal transversely isotropic liquid-saturated porous solid*, Archive of Applied Mechanics, **91**, 2881–2892, 2021.
11. D.X. Tung, *Wave propagation in nonlocal orthotropic micropolar elastic solids*, Archives of Mechanics, **73**, 3, 237–251, 2021.
12. D.X. Tung, *The reflection and transmission of waves at an imperfect interface between two nonlocal transversely isotropic liquid-saturated porous halfspaces*, Waves in Random and Complex Media, 2021, doi: 10.1080/17455030.2021.1954265.
13. S. Biswas, *Surface waves in porous nonlocal thermoelastic orthotropic medium*, Acta Mechanica, **231**, 2741–2760, 2020, doi: 10.1007/s00707-020-02670-2.
14. S. Biswas, *Fundamental solution of steady oscillations equations in nonlocal thermoelastic medium with voids*, Journal of Thermal Stresses, **43**, 3, 2020, doi: 10.1080/01495739.2019.1699482.

15. S. Deswal, R. Kumar, S.K. Sheokand, K.K. Kalkal, *Reflection and transmission between an orthotropic thermoelastic half-space and an initially stressed orthotropic rotating thermoelastic diffusive half-space*, Journal of Thermal Stresses, **43**, 6, 739–761, 2020.

16. S. Modal, N. Sarkar, N. Sarkar, *Waves in dual-phase-lag thermoelastic materials with voids based on Eringen's nonlocal elasticity*, Journal of Thermal Stresses, **42**, 8, 2019, doi: 10.1080/01495739.2019.1591249.

17. Y.J. Yu, X.G. Tian, Q.L. Xiong, *Nonlocal thermoelasticity based on nonlocal heat conduction and nonlocal elasticity*, European Journal of Mechanics/A Solids, **60**, 238–253, 2016.

18. M.D. Sharma, *Correct procedure to study reflection in orthotropic thermoelastic medium: inhomogeneous propagation of waves*, Applied Mathematics and Computation, **391**, 125692, 2021, doi: 10.1016/j.amc.2020.125692.

19. V. Cerveny, *Reflection/transmission laws for slowness vectors in viscoelastic anisotropic media*, Studia Geophysica et Geodaetica, **51**, 391–410, 2007.

20. E.S. Krebes, P.F. Daley, *Difficulties with computing anelastic plane-wave reflection and transmission coefficients*, Geophysical Journal International, **170**, 205–216, 2007.

21. E.S. Krebes, S. Moradi, *Conditions for the occurence of unphysical negative values of the anelastic SH plane wave energy-based transmission coefficient*, Studia Geophysica et Geodaetica, **56**, 323–347, 2012, doi: 10.1007/s11200-011-9040-6.

22. P.G. Richards, *On wave fronts and interfaces in anelastic media*, Bulletin of the Seismological Society of America, **74**, 6, 2157–2165, 1984.

23. B.O. Ruud, *Ambiguous reflection coefficients for anelastic media*, Studia Geophysica et Geodaetica, **50**, 479–498, 2006.

Received February 28, 2022; revised version July 14, 2022.
Published online October 20, 2022.