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44-100 Gliwice, ul. Krzywoustego 7, POLAND
phone: (+48 32) 237 26 58

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CHARACTERISTIC POINTS OF CONICS IN THE NET-LIKE METHOD OF CONSTRUCTION

Cecylia ŁAPIŃSKA[1], Alicja OGORZAŁEK[2]

[1,2] Warsaw University of Technology, Civil Engineering Faculty,
Plac Politechniki 1, 00-661 Warszawa
[1]email: cecylia.lapinska@is.pw.edu.pl, [2]email: alicja.ogorzalek@gmail.com

Abstract. The aim of this paper is to show how to complete the known net-like method for the case of a parabola or a hyperbola without using advanced methods of projective geometry. Only a construction of proportional segments is applied. Authors present a construction of the vertex of a parabola when its ideal point D^{∞}, a point B, and a point A with the tangent t are given. In the case of a hyperbola defined by its vertices A and B and a point C, the net-like method is completed by a construction of the hyperbola asymptotes. To understand the idea of this construction, a bit more complicated than the previous one, basic skills of elementary geometry, Pythagoras' theorem and Thales' theorem, are sufficient. In the case of a hyperbola defined by its asymptotes and a point, the presented construction of its vertices considering some parallelograms equal in area, follows from the well-known theorem about a line intersecting the hyperbola and its asymptotes.

Keywords: conics, parabola, hyperbola, ellipse, net-like methods, vertices of conics, asymptotes of a hyperbola, Pythagoras' theorem, Thales' theorem, proportional segments

1 Introduction

Former Descriptive Geometry programs included the basics of projective geometry. This was thus reflected in the classic textbooks for this subject (see [5], [6]). In those books, conic curves are defined and analyzed through projection transformations. Projective properties are used to formulate important theorems (Pascal's and Brianchon's) and to construct characteristic points.

Present course programs do not incorporate projective geometry. Therefore, we cannot consider conics as "products" of projection as E. Otto does in [5]. Nevertheless, the net-like method (see [3], p.142) resulting from this approach is presented to students as a way to construct points of an ellipse, parabola and hyperbola ([1], [3]). In order to achieve a satisfactory shape of these curves, it is of course better to know the characteristic points of these conics.

Diligent students using the CAD software are not always satisfied with the shape of the curve achieved by connecting through the "spline" command the consecutive points found by the net-like method. They accurately notice that perhaps the effect would have been better if characteristic points were among the constructed ones.

To use the CAD program to draw an ellipse, its vertices are necessary. This does not pose a problem, since affinity is part of the course, and these missing crucial points can be found by transforming the ellipse into a circle. It is impossible however, to do the same with a parabola or a hyperbola. When searching for the parabola/hyperbola vertices or asymptotes, students cannot transform them into a circle (as it once was standard – see [2] p. 120) because

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central collineation is not part of the course program. There are also no practical exercises in applying the Pascal's theorem, even if there are a few minutes during the lecture to mention it.

The aim of this article is to show how characteristic points may be constructed in the net-like method, relying only on the knowledge gained in high school (the ability to construct proportional segments).

2 The vertex of a parabola when its ideal point D^{∞}, a point A with the tangent t, and a point B are given

In this case students know the net-like method of construction of points of the parabola in the form as presented in Figure 1. We will show that the construction of the parabola vertex can

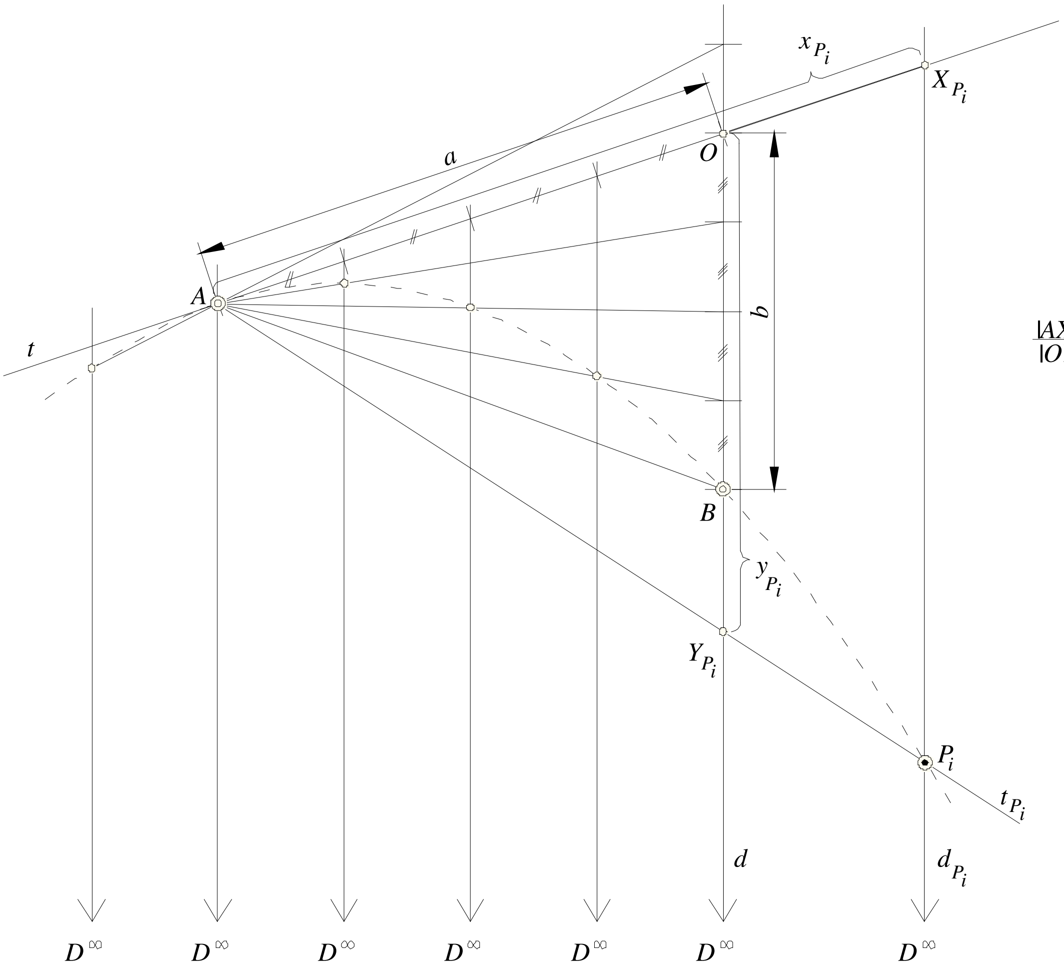

Figure 1 The net-like method of construction of points of the parabola

be based only on the construction of proportional segments, not new to students.

Label d the line passing through D^{∞} and B, and O, the point of the intersection of t and d ; |AO|= a, |BO|= b. A point P_i of the parabola is the point of intersection of two lines, t_i and d_i. The line d_i passes through D^{∞} and X_i, where X_i lies on t, and is distant at $i\frac{a}{n}$ from A. The line t_i passes through A and Y_i, where Y_i lies on d and is distant from O at $i\frac{b}{n}$. By definition $x_i = i\frac{a}{n}$, $y_i = i\frac{b}{n}$, where i is an integer. Therefore, for any "rational" point P_i on the parabola $\frac{AX_i}{OY_i} = \frac{x_i}{y_i} = \frac{a}{b}$, there is no difficulty to generalize this result for any x_P and y_P defining a “real“ point P on the parabola.

Hence, the construction of a missing point Q on the parabola consists in finding two segments AX_Q on t and OY_Q on d such that $|AX_Q| : |OY_Q| = \frac{a}{b}$. The point Q is the point of intersection of t_Q and d_Q, defined similarly as for a “rational” point P_i. If one of them is given, the other can be found.

Using the Thales’ theorem that construction can be made as shown in Figure 2.

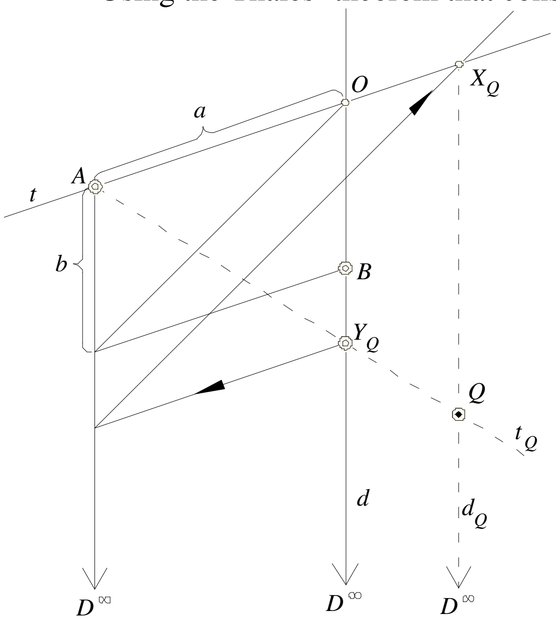

Figure 2 The construction of a missing point Q on the parabola

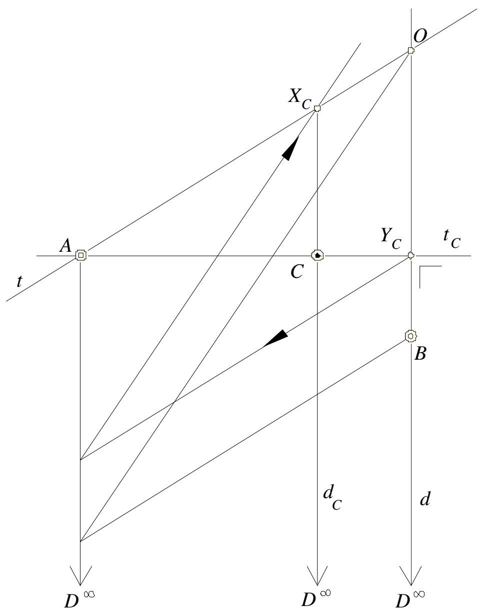

Figure 3 The construction of the point

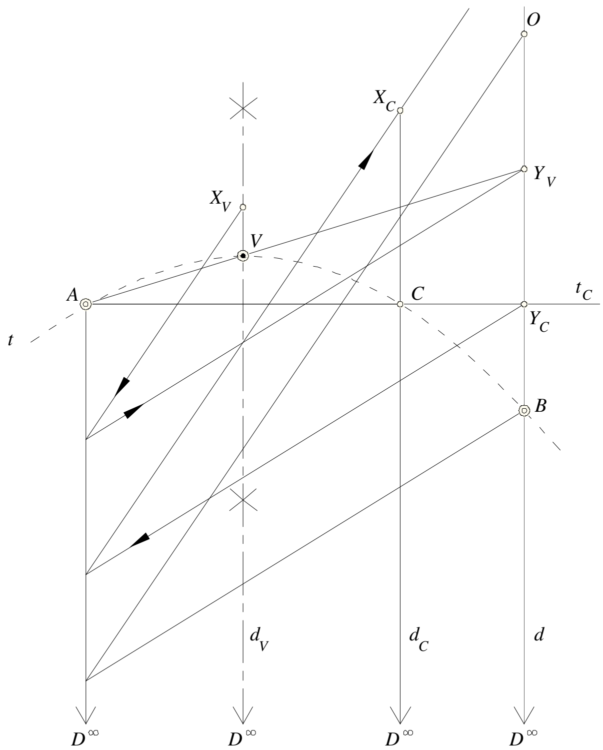

Figure 4 The whole structure from figures 2 and 3

The vertex V of the parabola is characterized by the fact that d_V is the axis of symmetry of the parabola. Thus d_V is the perpendicular bisector of the segment AC, where the point C is symmetric on the parabola to A, lying on t_C perpendicular to d (AY_C ⊚ d). Applying in Figure 3 the construction presented in Figure 2, one finds first X_C, next the line d_C and the required point C.

The perpendicular bisector d_V of AC intersects t at X_V, and Y_V is found applying once more the construction from Figure 2. Figure 4 displays the whole construction.

Remark: the construction can be simplified. Once point X_C is constructed, the line d_V passes also through the midpoint X_V of the segment AX_C.

3 The asymptotes of a hyperbola when its vertices *A* and *B*, and a point *C* are given.

Two perpendicular lines x and y intersecting at the given point C are considered as number lines with zero points, 0_x and 0_y respectively, as it is shown in Figure 5. Let X_i be the point on x with the coordinate x_i, and Y_i the point on y with the coordinate y_i.. Therefore, according to the net-like method, a "rational" point P_i (lying on the hyperbola defined by A, B, and C) is determined as the intersection point of two lines, a_i and b_i, where a_i is passing through A and X_i, and b_i is passing through B and Y_i, and $x_i = i\frac{a}{n}$, $y_i = i\frac{b}{n}$.

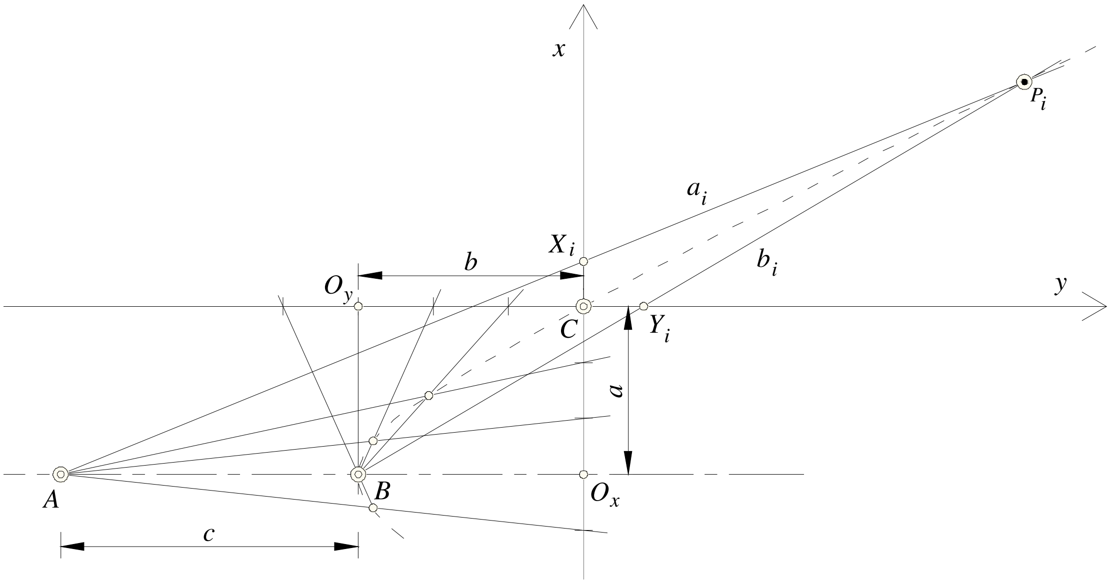

Figure 5 Two perpendicular lines x and y intersecting at the given point C are considered as numberlines with zero points, 0x and 0y respectively

Considering $n \to \infty$, one can describe a point P on the hyperbola as common to two lines a_P and b_P such that a_P is defined by A and X_P (with the coordinate x_P), b_P is defined by B and Y_P on y (with coordinate y_P), and $x_P : y_P = \frac{a}{b}$.

An asymptote q of the hyperbola is passing through an ideal point Q^∞. For Q^∞ lying on the hyperbola, $a_Q \parallel b_Q$, with $\frac{x_Q}{y_Q} = \frac{a}{b}$ (Fig. 6).

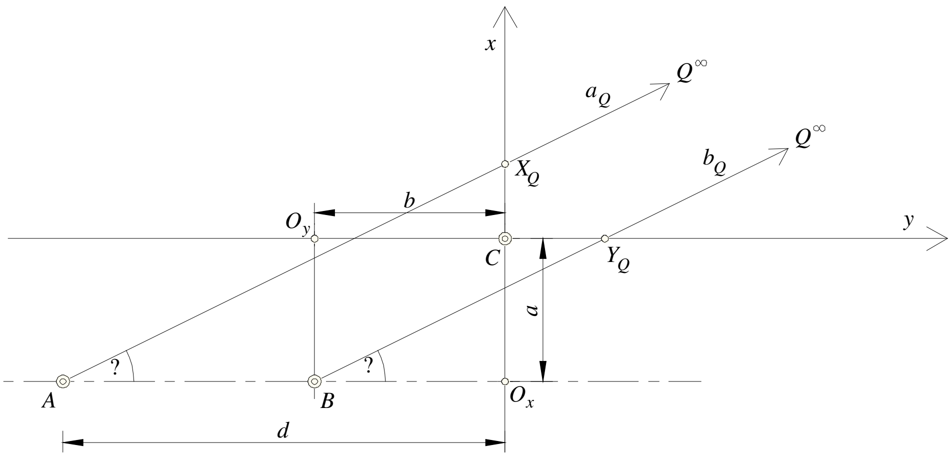

Figure 6 First step of the determination of the asymptote q

Therefore, in order to determine the required asymptotes, one must find the coordinates x_Q and y_Q such that:

$$\frac{x_Q}{y_Q}=\frac{a}{b}\,;\ \tan\varphi = \frac{x_Q}{d}=\frac{a}{y_Q}\,,\ \text{where } d = b+c. \tag{1}$$

From these relations we have:

$$x_Q^2\, b = a^2 d. \tag{2}$$

After calculations we obtain a ratio:

$$\frac{x_Q}{a}=\frac{\sqrt{d}}{\sqrt{b}}\,. \tag{3}$$

The last proportion is not easy to construct directly. Because segments of the form $\sqrt{xy}$ can be constructed for any given segments x and y (see [4], p.18), we change the obtained equality into the following:

$$\frac{x_Q}{a}=\frac{\sqrt{ad}}{\sqrt{ab}}\,. \tag{4}$$

Now the construction of the required asymptote can be realized using the Thales' theorem, as it is shown in Fig. 7.

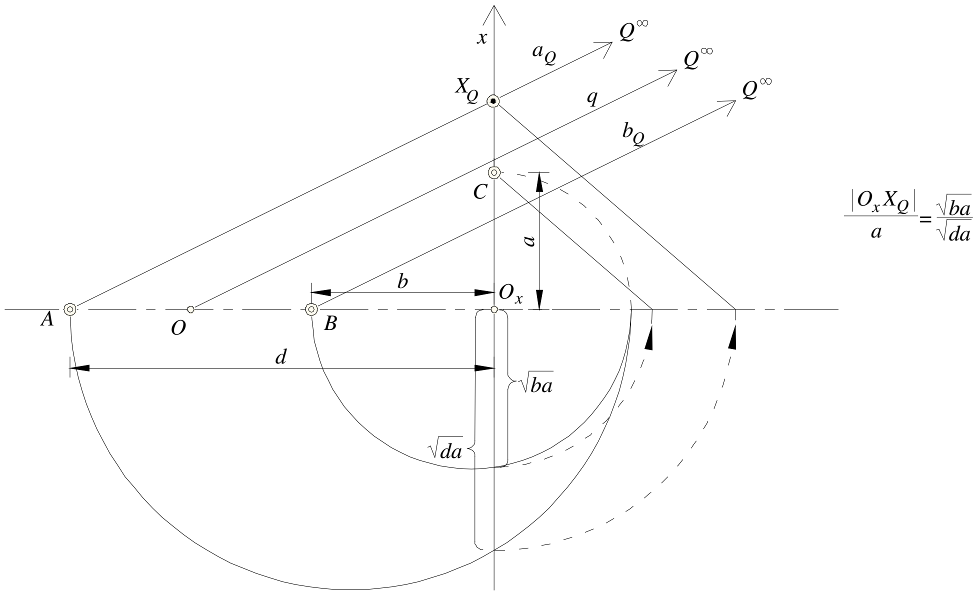

Figure 7 The construction of the asymptote q by using the Thales' theorem

The asymptote q (tangent to the hyperbola at the ideal point Q^∞) is passing through Q^∞ and the midpoint O of the segment AB. The other asymptote is symmetric to q with respect to the hyperbola axis (the line AB).

4 Vertices of a hyperbola when its asymptotes *s* and *t*, and a point *C* are given

In the case of a hyperbola defined by its asymptotes and a point, in order to construct the missing points of the hyperbola one generally uses the method based on the following well known property (see for example [1], [3]):

I. *Segments of any line intersecting a hyperbola, included between the hyperbola and its asymptotes, are equal in length.*

This fact will be used to show another useful property of a hyperbola.

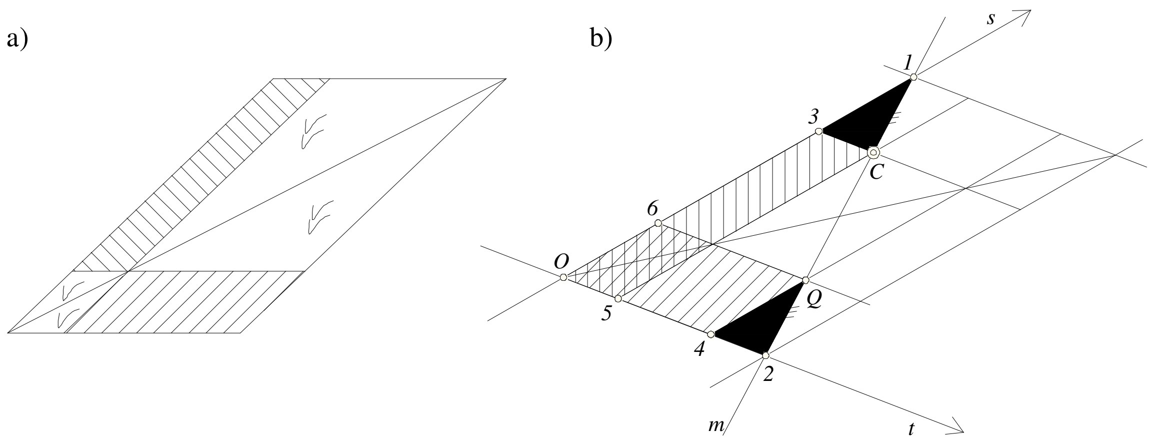

Figure 8 Illustration of: a) the property: two lines passing through a point on the parallelogram's diagonal, parallel to the parallelogram's sides, determine two parallelograms equal in area (crosshatched), b) the property II

Notice that any parallelogram is divided by its diagonal onto two triangles equal in area. Thus, two lines passing through a point on the parallelogram's diagonal, parallel to the parallelogram's sides, determine two parallelograms equal in area (crosshatched in Figure 8a). Consider now two points *C* and *Q* on a hyperbola, with its asymptotes *s* and *t* and lines passing through these points parallel to the asymptotes (see Figure 8b). As $|C1| = |Q2|$ by the Property I, the triangles *C13* and *Q24* are congruent according to the criterion ASA (angle, side, angle). Consequently, the parallelograms *O5C3* and *O4Q6* are equal in area. Therefore, the following property is true as well.

II. *Given a hyperbola with asymptotes intersecting at O, parallelograms with sides parallel to the asymptotes, with one vertex at O and the other on the hyperbola, are equal in area.*

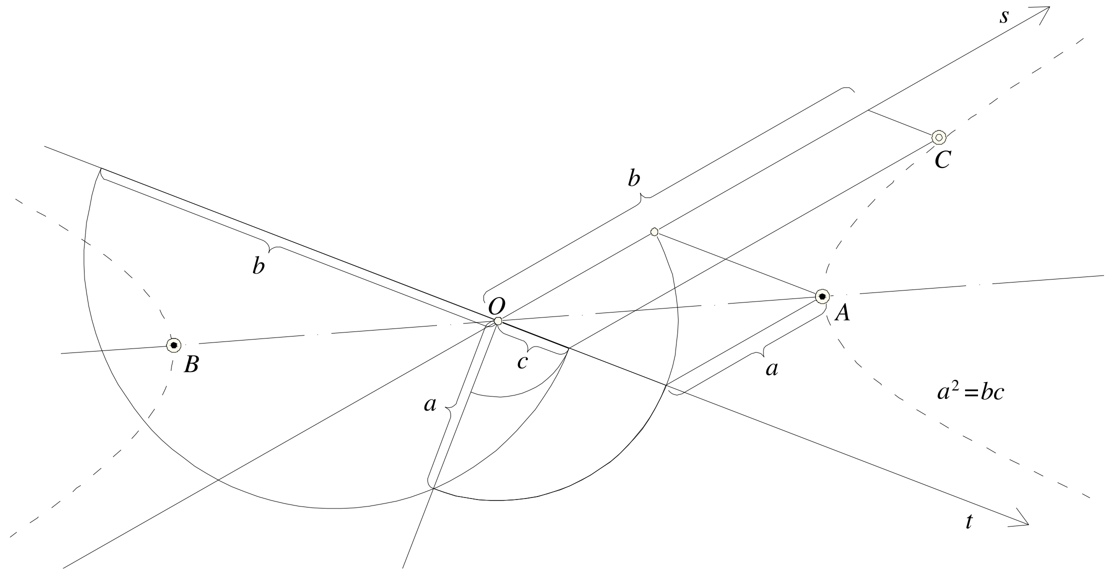

Figure 9 Illustration of the construction of a segment *a*

Accordingly, consider now a hyperbola when its asymptotes *s* and *t* together with a point C are given. The parallelogram determined by *C* has the area equal to *b*sin*α*. As a vertex *A* of a hyperbola defines a parallelogram with equal sides, therefore in order to construct it, one may find a segment *a* such that $a^2\sin\alpha = bc\sin\alpha$, i.e. $a^2 = bc$. The construction is shown in Figure 9.

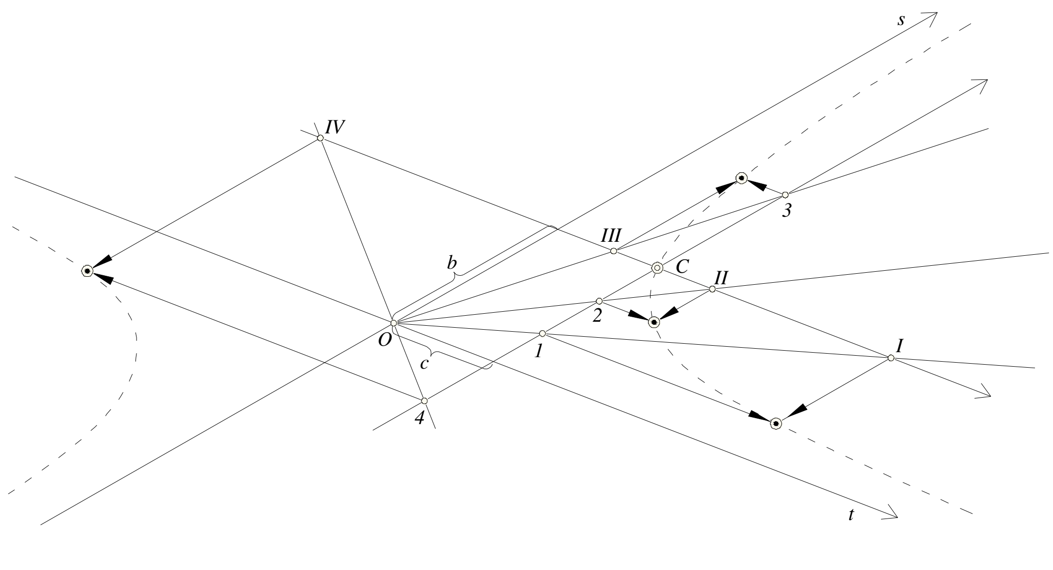

Figure 10 The construction of points of a hyperbola defined by a point and asymptotes

The Property II allows us to determine a method (see Figure 10) of points of a hyperbola defined by a point and asymptotes, similar to that for equilateral hyperbolas (see [3], p.140)

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PUNKTY CHARAKTERYSTYCZNE W SIATKOWYCH KONSTRUKCJACH UZUPEŁNIANIA PUNKTÓW STOŻKOWYCH

Celem tej pracy jest pokazanie jak uzupełnić metody siatkowe wyznaczania punktów hiperboli lub paraboli przez podanie konstrukcji punktów charakterystycznych tych krzywych, bez odwoływania się do zaawansowanych treści geometrii rzutowej. Autorki pokazują konstrukcję wierzchołka paraboli określonej przez dany kierunek D^{∞}, punkt *C,* punkt *A* ze styczną *t.* Wykorzystywana jest tylko konstrukcja odcinków proporcjonalnych. W przypadku hiperboli określonej przez dane wierzchołki *A* i *B* oraz punkt C konstrukcja siatkowa jest uzupełniona o sposób wyznaczania asymptot tej hiperboli. Metoda jest nieco bardziej złożona niż w poprzednim przypadku, ale do jej zrozumienia także wystarcza znajomość geometrii elementarnej, twierdzeń Pitagorasa i Talesa. W przypadku hiperboli określonej przez dany jej punkt *C* oraz asymptoty *s* i *t*, podana konstrukcja jej wierzchołka, wykorzystująca tylko równość pól odpowiednich równoległoboków, opiera się na znanym twierdzeniu o odcinkach prostej przecinającej hiperbolę i jej asymptoty.