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# REACTION-DIFFUSION COUPLED INCLUSIONS WITH VARIABLE EXPONENTS AND LARGE DIFFUSION

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**Abstract.** This work concerns the study of asymptotic behavior of coupled systems of $p(x)$-Laplacian differential inclusions. We obtain that the generalized semiflow generated by the coupled system has a global attractor, we prove continuity of the solutions with respect to initial conditions and a triple of parameters and we prove upper semicontinuity of a family of global attractors for reaction-diffusion systems with spatially variable exponents when the exponents go to constants greater than 2 in the topology of $L^\infty(\Omega)$ and the diffusion coefficients go to infinity.

**Keywords:** reaction-diffusion coupled systems, variable exponents, attractors, upper semicontinuity, large diffusion.

**Mathematics Subject Classification:** 35K55, 35K92, 35A16, 35B40, 35B41.

## 1. INTRODUCTION

PDEs for which the flow is essentially determined by an ordinary differential equation have been studied by many researchers, see for example [2, 9, 10, 12, 16–18, 24, 25, 29]. In [27–29] the authors investigated in which way the parameter $p(x)$ affects the dynamic of problems involving the $p(x)$-Laplacian.

In this work we consider the following nonlinear coupled system

$$\begin{cases} \frac{\partial u_s}{\partial t} - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + |u_s|^{p_s(x)-2}u_s \in F(u_s,v_s), & t>0, x\in\Omega,\\ \frac{\partial v_s}{\partial t} - \operatorname{div}(D_s|\nabla v_s|^{q_s(x)-2}\nabla v_s) + |v_s|^{q_s(x)-2}v_s \in G(u_s,v_s), & t>0, x\in\Omega,\\ \dfrac{\partial u_s}{\partial n}(t,x) = \dfrac{\partial v_s}{\partial n}(t,x) = 0, & t\geq 0, x\in\partial\Omega,\\ u_s(0,x)=u_{0s}(x),\ v_s(0,x)=v_{0s}(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $u_{0s}, v_{0s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain, $D_s \in [1,\infty)$, $p_s(\cdot), q_s(\cdot) \in C(\bar{\Omega})$, $p_s^- := \min_{x\in\overline{\Omega}} p_s(x) \geq p$, $q_s^- := \min_{x\in\overline{\Omega}} q_s(x) \geq q$,

 

$p_s^+ := \max_{x\in\overline{\Omega}} p_s(x) \le L$, $q_s^+ := \max_{x\in\overline{\Omega}} q_s(x) \le L$, for all $s \in \mathbb{N}$. We assume that $p_s(\cdot) \to p$, $q_s(\cdot) \to q$ in $L^\infty(\Omega)$ and $D_s \to \infty$ as $s \to \infty$, where $L, p, q > 2$ are positive constants. $F, G : L^2(\Omega) \times L^2(\Omega) \to P(L^2(\Omega))$ are bounded, upper semicontinuous and positively sublinear multivalued maps. We will prove that the dynamics of the coupled reaction-diffusion system of inclusions with large diffusion is governed by an ordinary coupled system of inclusions when the variable exponent become constant. The goal is to prove the convergence of solutions and global attractors to the corresponding ones of a limit ordinary differential coupled inclusion system.

The constant exponent case was considered in [23] and the variable exponent case without perturbation on the main operator and positive finite diffusion was considered in [26]. A partial differential inclusion with multivalued right-hand side of Lipschitz type was considered in [20].

The paper is organized as follows. In Section 2 we remind some definitions, we present properties of the operator and we prove the existence of global solutions and global attractors. In Section 3 we obtain uniform estimates for solutions of (1.1). In Section 4 we prove that the solutions $\{u_s\}$ of (1.1) converge to the solution $u$ of the limit problem (4.1) which is an Ordinary Differential Inclusion (ODI) system, and, after that, we obtain the upper semicontinuity of the global attractors for the problem (1.1).

## 2. EXISTENCE OF GLOBAL SOLUTIONS AND GLOBAL ATTRACTOR

In this work, to study global attractors for the system (1.1) for which we do not have guarantee of uniqueness of solution, we follow the general framework of the works [5, 21, 22]. Let us first remind some definitions. Consider the system

$$\begin{cases} u_t + Au \in F(u,v), & t \in (0,T), \\ v_t + Bv \in G(u,v), & t \in (0,T), \\ u(0) = u_0, v(0) = v_0, \end{cases} \tag{2.1}$$

where $A$ and $B$ are monotone operators of subdifferential type defined in a real Hilbert space $H$.

**Definition 2.1** ([22])**.** A strong solution (weak solution) of (2.1) is a pair $(u, v)$ satisfying: $u, v \in C([0,T]; H)$ for which there exists $f, g \in L^1(0,T;H)$, $f(t) \in F(u(t), v(t))$, $g(t) \in G(u(t), v(t))$ *a.e.* in $(0,T)$, and such that $(u,v)$ is a strong solution (weak solution) (see Definition 3.1 and Theorem 3.4 in [7]) on $(0,T)$ to the system (2.2) below:

$$\begin{cases} u_t + Au = f, \\ v_t + Bv = g, \\ u(0) = u_0, v(0) = v_0. \end{cases} \tag{2.2}$$

**Definition 2.2** ([3, 4, 22])**.** Let $X$ be a real Banach space and $U$ a topological space. A mapping $G : U \to P(X)$ is called upper semicontinuous at $u \in U$ if

(i) $G(u)$ is nonempty, bounded, closed and convex,
(ii) for each open subset $D$ in $X$ satisfying $G(u) \subset D$, there exists a neighborhood $V$ of $u$, such that $G(v) \subset D$, for each $v \in V$.

If $G$ is upper semicontinuous at each $u \in U$, then it is called upper semicontinuous on $U$.

**Definition 2.3** ([3,4,22])**.** Let $X$ be a real Banach space and $M$ a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$. By a *selection* of $E : M \to P(X)$ we mean a function $f : M \to X$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by Sel $E$ the set

$$\text{Sel } E := \{f;\ f : M \to X \text{ is a measurable selection of } E\}.$$

In order to get global solutions we impose suitable conditions on the terms $F$ and $G$.

**Definition 2.4** ([22])**.** The pair $(F, G)$ of maps $F, G : H \times H \to P(H)$, which takes bounded subsets of $H \times H$ into bounded subsets of $H$, is called positively sublinear if there exist $a > 0$, $b > 0$, $c > 0$ and $m_0 > 0$ such that for each $(u, v) \in H \times H$ with $\|u\|_H > m_0$ or $\|v\|_H > m_0$ for which either there exists $f_0 \in F(u, v)$ satisfying $\langle u, f_0\rangle > 0$ or there exists $g_0 \in G(u, v)$ with $\langle v, g_0\rangle > 0$, we have both

$$\|f\|_H \leq a\|u\|_H + b\|v\|_H + c \text{ and } \|g\|_H \leq a\|u\|_H + b\|v\|_H + c$$

for each $f \in F(u, v)$ and each $g \in G(u, v)$.

Now, let us remind the definitions of Lebesgue and Sobolev spaces with variable exponents. Considering $p \in L^\infty_+(\Omega) := \{q \in L^\infty(\Omega) :\ \text{ess inf}\, q \geq 1\}$, then

$$L^{p(\cdot)}(\Omega) := \left\{ u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$$

is a Banach space with the norm

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0; \rho\Big(\frac{u}{\lambda}\Big) \leq 1\right\},$$

where $\rho(u) := \int_\Omega |u(x)|^{p(x)} dx$. The following inequality will be used later

$$\min\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\} \leq \int_\Omega |u(x)|^{p(x)} dx \leq \max\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\}. \tag{2.3}$$

Furthermore,

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

which is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|\nabla u\|_{p(x)} + \|u\|_{p(x)}.$$

We refer the reader to [14, 15] for more details on Lebesgue and Sobolev spaces with variable exponents.

The authors in [30] (see also [31]) proved that the operator

$$A^s u := -\mathrm{div}(D_s|\nabla u|^{p_s(x)-2}\nabla u) + |u|^{p_s(x)-2}u$$

is the realization in $H$ of the operator $A_1^s : X_s \to X_s^*$, $X_s := W^{1,p_s(\cdot)}(\Omega)$,

$$A_1^s u(v) := \int_\Omega D_s|\nabla u(x)|^{p_s(x)-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |u(x)|^{p_s(x)-2}u(x)v(x)dx,$$

i.e., $A^s(u) = A_1^s u$, if $u \in \mathcal{D}(A^s) := \{u \in X_s;\ A_1^s u \in H\}$ and it is a maximal monotone operator in $H$. Besides, $A^s$ generates a compact semigroup and is the subdifferential of the proper, convex and lower semicontinuous function $\varphi_{p_s(x)} : H \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\varphi_{p_s(x)}(u) := \begin{cases} \Big[\int_\Omega \frac{D_s}{p_s(x)}|\nabla u|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u|^{p_s(x)}dx\Big], & \text{if } u \in X_s, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.4}$$

Moreover, they proved in [30] that the system (1.1) has a strong global solution $(u_s, v_s)$.

Using the following elementary assertion we can obtain estimates on the operator only by considering two cases.

**Proposition 2.5** ([1]). *Let $\lambda, \mu$ be arbitrary nonnegative numbers. For every positive $\alpha, \beta$, $\alpha \geq \beta$,*

$$\lambda^\alpha + \mu^\beta \geq \frac{1}{2^\alpha}\begin{cases} (\lambda+\mu)^\alpha & \text{if } \lambda+\mu < 1, \\ (\lambda+\mu)^\beta & \text{if } \lambda+\mu \geq 1. \end{cases}$$

Then it is easy to show that for every $u \in X_s$

$$\langle A^s u, u\rangle_{X_s^*, X_s} \geq \frac{1}{2^{p_s^+}}\begin{cases} \|u\|_{X_s}^{p_s^+} & \text{if } \|u\|_{X_s} < 1, \\ \|u\|_{X_s}^{p_s^-} & \text{if } \|u\|_{X_s} \geq 1. \end{cases} \tag{2.5}$$

From now on, we will denote $X_s := W^{1,p_s(\cdot)}(\Omega)$, $Y_s := W^{1,q_s(\cdot)}(\Omega)$, $X := W^{1,p}(\Omega)$ and $Y := W^{1,q}(\Omega)$.

It is a known result that $X_s, Y_s \subset H$ with continuous and dense embeddings (see [14, 25]). Moreover, it is easy to see that

$$\|u_s\|_H \leq 4(|\Omega|+1)^2\|u_s\|_{X_s}, \tag{2.6}$$

for all $u_s \in X_s$ and for all $s \in \mathbb{N}$.

By [22], we obtain that if $\mathcal{R}_s(u_0, v_0)$ is the set of all solutions of (1.1) with initial data $(u_0, v_0)$, then

$$\mathbb{G}_s := \bigcup_{(u_0,v_0)\in H\times H} \mathcal{R}_s(u_0, v_0)$$

is a generalized semiflow in $H \times H$ (which is called the generalized semiflow associated with (1.1)), i.e., $\mathbb{G}_s$ is a family of maps $\varphi : [0,\infty) \to H \times H$ satisfying the conditions:

(H1) For each $z \in H \times H$ there exists at least one $\varphi \in \mathbb{G}_s$ with $\varphi(0) = z$.
(H2) If $\varphi \in \mathbb{G}_s$ and $\tau \geq 0$, then $\varphi^\tau \in \mathbb{G}_s$, where $\varphi^\tau(t) := \varphi(t+\tau)$ for all $t \in [0, \infty)$.
(H3) If $\varphi, \psi \in \mathbb{G}_s$, and $\psi(0) = \varphi(t)$ for some $t \geq 0$, then $\theta \in \mathbb{G}_s$, where

$$\theta(\tau) \doteq \begin{cases} \varphi(\tau) & \text{for } \ \tau \in [0, t], \\ \psi(\tau - t) & \text{for } \ \tau \in (t, \infty). \end{cases}$$

(H4) If $\{\varphi_j\}_{j=1}^{\infty} \subset \mathbb{G}_s$ and $\varphi_j(0) \to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathbb{G}_s$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Let us review some concepts from [21]:

**Definition 2.6.** Let $\mathbb{G}$ be a generalized semiflow in a complete metric space $M$.

(a) $\mathbb{G}$ is bounded dissipative or B-dissipative if there is a bounded global B-attractor for $\mathbb{G}$.
(b) $\mathbb{G}$ is point dissipative if there is a bounded global point attractor for $\mathbb{G}$.
(c) We say that $\mathbb{G}$ is $\varphi$-dissipative if there is a bounded set $B_0$ such that, for any $\varphi \in \mathbb{G}$, $\varphi(t) \in B_0$ for all sufficiently large t.
(d) $\mathbb{G}$ is eventually bounded if for any bounded set $B \subset M$ there exists $\tau = \tau(B) \geq 0$ such that $\gamma_\tau^+(B) := \bigcup_{t \geq \tau} \{\varphi(t);\ \varphi \in \mathbb{G} \text{ with } \varphi(0) \in B\}$ is a bounded set in $M$.

**Remark 2.7.** The following implications hold for a generalized semiflow $\mathbb{G}$ in a complete metric space:

$$\mathbb{G} \text{ is bounded dissipative} \Rightarrow \mathbb{G} \text{ is point dissipative} \Rightarrow \mathbb{G} \text{ is } \varphi\text{-dissipative.}$$

Moreover,

$$\mathbb{G} \text{ is bounded dissipative} \Rightarrow \mathbb{G} \text{ is eventually bounded.}$$

**Definition 2.8.** A generalized semiflow $\mathbb{G}$ in a complete metric space $M$ is asymptotically compact if for any sequence $\{\varphi_j\} \subset \mathbb{G}$ with $\{\varphi_j(0)\}$ being a bounded set in $M$, and for any sequence $\{t_j\}$, $t_j \to \infty$, the sequence $\{\varphi_j(t_j)\}$ has a convergent subsequence.

According to Theorem 9 in [21], in order to assure the existence of a compact invariant global attractor for (1.1), it is enough to guarantee that the generalized semiflow $\mathbb{G}_s$ defined by (1.1) is asymptotically compact and $\varphi$-dissipative.

In this work we denote

$$A^s(w) := -\mathrm{div}(D_s |\nabla w|^{p_s(x)-2} \nabla w) + |w|^{p_s(x)-2} w,$$

and analogously

$$B^s(w) := -\mathrm{div}(D_s |\nabla w|^{q_s(x)-2} \nabla w) + |w|^{q_s(x)-2} w,$$

$S^s$ the semigroup generated by $A^s$ and $T_s$ the multivalued semigroup defined by $\mathbb{G}_s$.

In the next result we will prove that the generalized semiflow $\mathbb{G}_s$ defined by (1.1) is bounded dissipative and so, eventually bounded and $\varphi$-dissipative (see Remark 2.7).

**Theorem 2.9.** *Let $F, G : H \times H \to P(H)$ be bounded, upper semicontinuous and positively sublinear maps. Then, there exist a bounded set $B_s$ in $H \times H$ and $t_0 > 0$ such that for any $\varphi_s \in \mathbb{G}_s$, $\varphi_s(t) \in B_s$ for all $t \geq t_0$. Thus, in particular, the generalized semiflow $\mathbb{G}_s$ defined by* (1.1) *is bounded dissipative.*

*Proof.* Let $\varphi_s = (u_s, v_s) \in \mathbb{G}_s$ a solution of (1.1). Then there exists a pair $(f_s, g_s) \in \text{Sel } F(u_s, v_s) \times \text{Sel } G(u_s, v_s)$, $f_s, g_s \in L^1(0,T;H)$ for each $T > 0$ such that $u_s, v_s$ satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s & \text{in } (0,T) \times \Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s & \text{in } (0,T) \times \Omega, \\ u_s(0,x) = u_{0s}(x),\ v_s(0,x) = v_{0s}(x) & \text{in } \Omega. \end{cases} \tag{2.7}$$

Multiplying the first equation by $u_s$ we obtain

$$\Big\langle \frac{\partial u_s(t)}{\partial t}, u_s(t) \Big\rangle_H + \Big\langle A^s(u_s(t)), u_s(t) \Big\rangle_H = \langle f_s(t), u_s(t) \rangle_H.$$

Let $I := (0,T)$, $I_{1s} := \{t \in I : \|u_s(t)\|_{X_s} < 1\}$ and $I_{2s} := \{t \in I : \|u_s(t)\|_{X_s} \geq 1\}$. Then by (2.5)

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \frac{1}{2^{p_s^+}}\|u_s(t)\|_{X_s}^{p_s^+} \leq \langle f_s(t), u_s(t) \rangle_H \quad \text{if } t \in I_{1s},$$

and

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \frac{1}{2^{p_s^+}}\|u_s(t)\|_{X_s}^{p_s^-} \leq \langle f_s(t), u_s(t) \rangle_H \quad \text{if } t \in I_{2s}.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq \begin{cases} -\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^+} + \langle f_s(t), u_s(t) \rangle_H & \text{if } t \in I_{1s}, \\ -\frac{\sigma}{\alpha^{p_s^-}}\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t) \rangle_H & \text{if } t \in I_{2s}, \end{cases} \tag{2.8}$$

where $\alpha := 4(|\Omega| + 1)^2$ and $\sigma := \frac{1}{2^L}$.

In an analogous way, multiplying the second equation in (2.7) by $v_s$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq \begin{cases} -\frac{\sigma}{\alpha^{q_s^+}}\|v_s(t)\|_H^{q_s^+} + \langle g_s(t), v_s(t) \rangle_H & \text{if } t \in \tilde{I_{1s}} \\ -\frac{\sigma}{\alpha^{q_s^-}}\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t) \rangle_H & \text{if } t \in \tilde{I_{2s}}, \end{cases}$$

where $\tilde{I_{1s}} := \{t \in I : \|v_s(t)\|_{Y_s} < 1\}$, $\tilde{I_{2s}} := \{t \in I : \|v_s(t)\|_{Y_s} \geq 1\}$.

Now, let $r_s := \frac{p_s^+}{p_s^-} > 1$ and $r_s'$ such that $\frac{1}{r_s} + \frac{1}{r_s'} = 1$ then by Young's inequality

$$\|u_s(t)\|_H^{p_s^-} \leq \frac{1}{r_s'} + \frac{1}{r_s}\|u_s(t)\|_H^{p_s^+}$$

and so

$$-\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^+} \leq r_s\Big( -\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^-} + \frac{\sigma}{\alpha^{p_s^+} r_s'}\Big). \tag{2.9}$$

Using (2.9) in (2.8) we get

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1 \quad \text{for all } t \in I = (0,T),$$

where $C_2 := \frac{1}{(2\alpha)^L}$ and $C_1 := \frac{L\sigma}{p\alpha^p}$.

In an analogous way, taking $\tilde{r}_s := \frac{q_s^+}{q_s^-} > 1$ and $\tilde{r}_s'$ such that $\frac{1}{\tilde{r}_s} + \frac{1}{\tilde{r}_s'} = 1$ we get

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \quad \text{for all } t \in I = (0,T),$$

where $\tilde{C}_2 = C_2 = \frac{1}{(2\alpha)^L}$ and $\tilde{C}_1 := \frac{L\sigma}{q\alpha^q}$.

Thus, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1, \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1, \end{cases} \tag{2.10}$$

where $C_2, \tilde{C}_2, C_1, \tilde{C}_1$ are positive real numbers depending on $|\Omega|, L, p, q$.

We can suppose, without loss of generality that $p_s^- \geq q_s^-$. If $p_s^- = q_s^-$ we obtain a similar expression as (2.10) with $q_s^-$ in place of $p_s^-$. If $p_s^- > q_s^-$, taking $\theta_s := \frac{p_s^-}{q_s^-} > 1$, $\theta_s'$ such that $\frac{1}{\theta_s} + \frac{1}{\theta_s'} = 1$ and $\epsilon > 0$ we have

$$\|u_s(t)\|_H^{q_s^-} = \frac{\epsilon}{\epsilon}\|u_s(t)\|_H^{q_s^-} \leq \frac{1}{\theta_s'\epsilon^{\theta_s'}} + \frac{1}{\theta_s}\epsilon^{\theta_s}\|u_s(t)\|_H^{p_s^-}$$

and then

$$-C_2\|u_s(t)\|_H^{p_s^-} \leq \frac{\theta_s}{\epsilon^{\theta_s}}\Big[\frac{C_2}{\theta_s'\epsilon^{\theta_s'}} - C_2\|u_s(t)\|_H^{q_s^-}\Big].$$

So, we have that

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -\frac{C_2\theta_s}{\epsilon^{\theta_s}}\|u_s(t)\|_H^{q_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1 + \frac{\theta_s C_2}{\theta_s'\epsilon^{\theta_s}\epsilon^{\theta_s'}}, \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1. \end{cases} \tag{2.11}$$

Now, we use that $(F,G)$ is positively sublinear (see Definition 2.4) to estimate $\langle f_s(t), u_s(t)\rangle_H$ and $\langle g_s(t), v_s(t)\rangle_H$. To do this, we have to consider the following three cases:

*Case 1.* If $\|u_s(t)\|_H \leq m_0$ and $\|v_s(t)\|_H \leq m_0$ then as $F$ and $G$ map bounded subsets of $H \times H$ into bounded subsets of $H$ there exists $C > 0$ such that

$$\langle f_s(t), u_s(t)\rangle_H \leq \|f_s(t)\|_H\|u_s(t)\|_H \leq Cm_0$$

and

$$\langle g_s(t), v_s(t)\rangle_H \leq \|g_s(t)\|\|v_s(t)\| \leq Cm_0.$$

*Case 2.* If $\|u_s(t)\|_H > m_0$ or $\|v_s(t)\|_H > m_0$ and $\langle f_0, u_s(t)\rangle \le 0$ and $\langle g_0, v_s(t)\rangle \le 0$ for all $f_0 \in F(u_s(t), v_s(t))$ and for all $g_0 \in G(u_s(t), v_s(t))$ then $\langle f_s(t), u_s(t)\rangle_H \le 0$ and $\langle g_s(t), v_s(t)\rangle_H \le 0$.

*Case 3.* If $\|u_s(t)\|_H > m_0$ or $\|v_s(t)\|_H > m_0$ and $\langle f_0, u_s(t)\rangle > 0$ or $\langle g_0, v_s(t)\rangle > 0$ for some $f_0 \in F(u_s(t), v_s(t))$ or for some $g_0 \in G(u_s(t), v_s(t))$ then, for $\epsilon > 0$, $\kappa_s := \frac{q_s^-}{2} > 1$ and $\nu_s := \frac{q_s^-}{(q_s^-)'} > 1$, we get

$$\begin{aligned}
\langle f_s(t), u_s(t)\rangle &\le \|f_s(t)\|_H \|u_s(t)\|_H \\
&\le \frac{\epsilon}{\epsilon} a \|u_s(t)\|_H^2 + \frac{\epsilon}{\epsilon} b \|u_s(t)\|_H \|v_s(t)\|_H + \frac{\epsilon}{\epsilon} c \|u_s(t)\|_H \\
&\le \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{\kappa_s}\epsilon^{\kappa_s}\|u_s(t)\|_H^{q_s^-} + \frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\|v_s(t)\|_H^{(q_s^-)'} \\
&\quad + \frac{1}{q_s^-}\epsilon^{q_s^-}\|u_s(t)\|_H^{q_s^-} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'} + \frac{1}{q_s^-}\epsilon^{q_s^-}\|u_s(t)\|_H^{q_s^-} \\
&= \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\|v_s(t)\|_H^{(q_s^-)'} \\
&\quad + \Big(\frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big) \\
&\le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} \\
&\quad + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big]
\end{aligned}$$

and in an analogous way

$$\begin{aligned}
\langle g_s(t), v_s(t)\rangle &\le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-} \\
&\quad + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big].
\end{aligned}$$

Therefore, in all cases we get

$$\begin{aligned}
\langle f_s(t), u_s(t)\rangle &\le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} + m_0 C \\
&\quad + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big]
\end{aligned}$$

and

$$\begin{aligned}
\langle g_s(t), v_s(t)\rangle &\le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-} + m_0 C \\
&\quad + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big].
\end{aligned}$$

Using the last two inequalities in (2.11) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq \Big(-\frac{C_2\theta_s}{\epsilon^{\theta_s}} + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} + C_3(\epsilon,s)$$

and

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq \Big(-\tilde{C}_2 + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-} + C_4(\epsilon,s),$$

where

$$C_3(\epsilon,s) = m_0C + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big] + C_1 + \frac{\theta_s C_2}{\theta_s'\epsilon^{\theta_s}\epsilon^{\theta_s'}}$$

and

$$C_4(\epsilon,s) = m_0C + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big] + \tilde{C}_1.$$

Thus, adding the last two inequalities we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big)\\
&\leq \Big(-\frac{C_2\theta_s}{\epsilon^{\theta_s}} + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\Big)\|u_s(t)\|_H^{q_s^-}\\
&\quad + \Big(-\tilde{C}_2 + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\Big)\|v_s(t)\|_H^{q_s^-} + C_3(\epsilon,s) + C_4(\epsilon,s).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we can take $\epsilon_0$ sufficiently small such that

$$\frac{2}{q_s^-}\epsilon_0^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon_0^{q_s^-} + \frac{\epsilon_0^{\nu_s}}{\nu_s} < \frac{\tilde{C}_2}{2} \quad \text{and} \quad \frac{C_2\theta_s}{\epsilon_0^{\theta_s}} \geq \tilde{C}_2.$$

Then

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big) \leq -C_5\Big(\|u(t)\|_H^{q_s^-} + \|v_s(t)\|_H^{q_s^-}\Big) + C_6(s),$$

where $C_5 := \frac{\tilde{C}_2}{2} > 0$ and $C_6(s) = C_3(\epsilon_0,s) + C_4(\epsilon_0,s) > 0$.

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2+\|v_s(t)\|_H^2\Big) &\le -C_5\Big(\|u_s(t)\|_H^{2\frac{q_s^-}{2}}+\|v_s(t)\|_H^{2\frac{q_s^-}{2}}\Big)+C_6(s)\\ &\le -\frac{C_5}{2^{\frac{q_s^-}{2}}}\Big(\|u_s(t)\|_H^2+\|v_s(t)\|_H^2\Big)^{\frac{q_s^-}{2}}+C_6(s).\end{aligned}$$

Therefore, the function $y_s(t):=\|u_s(t)\|_H^2+\|v_s(t)\|_H^2$ satisfies the inequality

$$y_s'(t)\le -\frac{2C_5}{2^{\frac{q_s^-}{2}}}y_s(t)^{\frac{q_s^-}{2}}+2C_6(s),\quad t>0.$$

From Lemma 5.1 in [33] we obtain

$$y_s(t)\le\left(\frac{2C_6(s)}{\frac{2C_5}{2^{q_s^-/2}}}\right)^{2/q_s^-}+\left[\frac{2C_5}{2^{q_s^-/2}}\left(\frac{q_s^-}{2}-1\right)t\right]^{\frac{-1}{\left(\frac{q_s^-}{2}-1\right)}}.$$

So, considering

$$r_s:=\left(\frac{2C_6(s)2^{q_s^-/2}}{2C_5}\right)^{2/q_s^-}+\left[\frac{2C_5}{2^{q_s^-/2}}\left(\frac{q_s^-}{2}-1\right)\right]^{\frac{-1}{\left(\frac{q_s^-}{2}-1\right)}}$$

and $t_0=1$ we have $\|u_s(t)\|_H^2+\|v_s(t)\|_H^2\le r_s$, for all $t\ge t_0$. □

**Corollary 2.10.** *$\mathbb{G}_s$ is asymptotically compact.*

*Proof.* Let $\{\varphi_j\}\subset\mathbb{G}_s$ with $\{\varphi_j(0)\}$ bounded in $H\times H$, and $\{\varphi_j(t_j)\}$ a sequence in $H\times H$ with $t_j\to\infty$. We want to show that $\{\varphi_j(t_j)\}$ has a convergent subsequence. By definition $\varphi_j=(u_j,v_j)$, $\varphi_j(0)=(u_j(0),v_j(0))\in H\times H$. As $t_j\to\infty$ we can suppose $t_j\ge 1$ for all $j\in\mathbb{N}$ and as $\mathbb{G}_s$ is a generalized semiflow $\widetilde{\varphi_j}=\varphi_j^{t_j-1}=(u_j^{t_j-1},v_j^{t_j-1})\in\mathbb{G}_s$. Then for each $j\in\mathbb{N}$ there exist $f_j,\ g_j\in L^1(0,1;H)$, $f_j\in\mathrm{Sel}\ F(u_j^{t_j-1},v_j^{t_j-1})$, $g_j\in\mathrm{Sel}\ G(u_j^{t_j-1},v_j^{t_j-1})$, where $(u_j^{t_j-1},v_j^{t_j-1})$ is the solution of (2.7) on $(0,1)\times\Omega$.

Let $K_1=\{f_j,j\in\mathbb{N}\}$, $K_2=\{g_j,j\in\mathbb{N}\}$, $M(K_1)(t)=\{u_j^{t_j-1}(t),j\in\mathbb{N}\}$ and $M(K_2)(t)=\{v_j^{t_j-1}(t),j\in\mathbb{N}\}$, $t\in[0,1]$, and $\{S^s(t),\ t\ge 0\}$ the compact semigroup generated by $A^s$ on $H$.

Now, let $h>0$ be such that $1-h\in[0,1]$. We define $T_h: M(K_1)(1)\to H$ by setting $u_j^{t_j-1}(1)\mapsto S^s(h)u_j^{t_j-1}(1-h)$.

It is easy to show that $K_1$ is a bounded set in $L^1(0,1;H)$ and is uniformly integrable in $L^1(0,1;H)$. As $\mathbb{G}_s$ is eventually bounded, $\{\widetilde{\varphi_j}(0)\} = \{\varphi_j(t_j-1)\}$ is a bounded subset of $H \times H$ if $j$ is big enough. Consequently, $M(K_1)(1-h)$ is a bounded subset of $H$. So, $T_h$ is a compact operator. Moreover, we have

$$\parallel S^s(h)u_j^{t_j-1}(1-h) - u_j^{t_j-1}(1) \parallel \le \int_{1-h}^{1} \parallel f_j(s) \parallel ds, \; \forall\, j \in \mathbb{N}.$$

So we have that $\lim_{h\to 0} T_h = I$, uniformly in $M(K_1)(1)$. Therefore the map $I : M(K_1)(1) \to M(K_1)(1)$ is a compact operator and then, $M(K_1)(1)$ is relatively compact in $H$. The same arguments show that $M(K_2)(1)$ is relatively compact in $H$, therefore $\{\varphi_j(t_j)\}$ has a convergent subsequence in $H \times H$. $\square$

We conclude that $\mathbb{G}_s$ has a compact invariant global attractor $\mathcal{A}_s$. The global attractor $\mathcal{A}_s$ is unique and given by $\mathcal{A}_s = \bigcup_{B\in B(H\times H)} \omega_s(B)$, where $B(H\times H)$ means the bounded subsets of $H \times H$. Furthermore $\mathcal{A}_s$ is the maximal compact invariant subset of $H \times H$, and is minimal among all closed global attractors of bounded sets. We also have that $\mathcal{A}_s$ is the union of all complete bounded orbits in $H \times H$ (see Theorem 15 in [21]).

## 3. UNIFORM ESTIMATES

In this section we prove uniform estimates in $H \times H$ and $X_s \times Y_s$ for the solutions of (1.1).

**Lemma 3.1.** *If $(u_s, v_s)$ is a solution of (1.1), then there exist positive numbers $r_0$ and a constant $t_0 > 0$ such that $\|(u_s(t), v_s(t))\|_{H\times H} \le r_0$, for each $t \ge t_0$ and $s \in \mathbb{N}$.*

*Proof.* The same arguments employed in the proof of Theorem 2.9 can also be applied here, but now, in order to obtain uniform estimates, we use from the beginning the hypothesis $p_s^- \ge p$, $q_s^- \ge q$, $p_s^+ \le L$, $q_s^+ \le L$ for all $s \in \mathbb{N}$, and we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \le -C_2\|u_s(t)\|_H^p + \langle f_s(t), u_s(t)\rangle_H + C_1 \quad \text{for all } t \in I = (0,T)$$

and

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \le -\tilde{C}_2\|v_s(t)\|_H^q + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \quad \text{for all } t \in I = (0,T).$$

Thus, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \le -C_2\|u_s(t)\|_H^p + \langle f_s(t), u_s(t)\rangle_H + C_1, \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \le -\tilde{C}_2\|v_s(t)\|_H^q + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1, \end{cases}$$

where $C_2, \tilde{C}_2, C_1, \tilde{C}_1$ are positive real numbers depending on $|\Omega|, L, p, q$. Now, repeating the procedure with $\theta := p/q,\ \kappa := q/2,\ \nu := q/q'$ we obtain the result. $\square$

**Remark 3.2.** The constants $r_0$ and $t_0$ in Lemma 3.1 are independent of the initial values and can be chosen uniformly in $s \in \mathbb{N}$.

**Corollary 3.3.** *There exists a bounded set $B_0$ in $H \times H$ such that $\mathcal{A}_s \subset B_0$ for all $s \in \mathbb{N}$.*

**Lemma 3.4.** *If $(u_s, v_s)$ is a solution of (1.1), then there exists a positive number $K = K(u_{0s}, v_{0s}, t_0)$ such that $\|(u_s(t), v_s(t))\|_{H\times H} \leq K$, for all $t \in [0, t_0]$. If the initial values are all in a bounded set of $H \times H$, then $K$ is uniform in $s$ and we have that $\|(u_s(t), v_s(t))\|_{H\times H} \leq K$, for each $s$ and for each $t \in [0, t_0]$. In this case we can consider $t_0 = 0$ in Lemma 3.1.*

*Proof.* As $(u_s, v_s)$ is a solution of (1.1), there exists a pair $(f_s, g_s) \in \operatorname{Sel} F(u_s, v_s) \times \operatorname{Sel} G(u_s, v_s)$, $f_s, g_s \in L^1(0, T; H)$ such that $u_s, v_s$ satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s & \text{in } (0,T)\times\Omega, \\ u_s(0,x) = u_{0s}(x),\ v_s(0,x) = v_{0s}(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Then, multiplying the first equation on (3.1) by $u_s(t)$ and the second one by $v_s(t)$, summing up and using that $\langle A^s(u_s(t)), u_s(t)\rangle \geq 0$ and $\langle B^s(v_s(t)), v_s(t)\rangle \geq 0$ it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) \leq \langle f_s(t)), u_s(t)\rangle + \langle g_s(t)), v_s(t)\rangle.$$

Now, we use that $(F, G)$ is positively sublinear to estimate $\langle f_s(t)), u_s(t)\rangle$ and $\langle g_s(t)), v_s(t)\rangle$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) \leq C_1\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) + C_2, \tag{3.2}$$

where $C_1$ is a positive real number depending on $a, b, c$ and $C_2$ is a positive real number depending on $m_0$.

Integrating (3.2) from 0 to $t \leq t_0$ we obtain

$$\begin{aligned} &\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2 \\ &\leq \left(\|u_{0s}\|_H^2 + \|v_{0s}\|_H^2\right) + \int_0^t 2C_1\left(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\right)d\tau + 2C_2 t_0. \end{aligned}$$

By the Gronwall–Bellman inequality

$$\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2 \leq \left(\|u_{0s}\|_H^2 + \|v_{0s}\|_H^2 + 2C_2 t_0\right)e^{2C_1 t_0} \quad \text{for all } t \in [0, t_0],$$

and the assertion of the lemma follows. □

**Lemma 3.5.** *If $\varphi_s := (u_s, v_s) \in \mathbb{G}_s$, then there exist positive constants $K > 0$ and $t_1 > t_0$, independent of $s$, such that*

$$\|\varphi_s(t)\|_{X_s\times Y_s} = \|u_s(t)\|_{X_s} + \|v_s(t)\|_{Y_s} < K$$

*for every $t \geq t_1$ and $s \in \mathbb{N}$, where $t_0$ is the positive constant in Lemma 3.1.*

*Proof.* Take $t_1 > t_0$. As $(u_s, v_s)$ is a solution of (1.1) there exists a pair $(f_s, g_s) \in \text{Sel}\, F(u_s, v_s) \times \text{Sel}\, G(u_s, v_s)$, $f_s, g_s \in L^1(0,T;H)$ such that $u_s, v_s$ satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s & \text{in } (0,T)\times\Omega. \end{cases}$$

Considering $\varphi_{p_s(x)}$ as in (2.4) we obtain

$$\begin{aligned} \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) &= \left\langle \partial\varphi_{p_s(x)}(u_s(t)), \frac{\partial u_s}{\partial t}(t) \right\rangle \\ &= \left\langle f_s(t) - \frac{\partial u_s}{\partial t}(t), \frac{\partial u_s}{\partial t}(t) - f_s(t) + f_s(t) \right\rangle \\ &= -\left\| f_s(t) - \frac{\partial u_s}{\partial t}(t) \right\|_H^2 + \left\langle f_s(t) - \frac{\partial u_s}{\partial t}(t), f_s(t) \right\rangle \end{aligned}$$

for a.e. $t$ in $(0,T)$. Therefore,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) + \frac{1}{2}\left\| f_s(t) - \frac{\partial u_s}{\partial t}(t) \right\|_H^2 \le \frac{1}{2}\|f_s(t)\|_H^2.$$

Now by using Lemma 3.1 and the fact that $F$ and $G$ are bounded, there exists a positive constant $C_0$ such that $\|f_s(t)\|_H \le C_0$ for all $t \ge t_0$ and $s \in \mathbb{N}$. In particular,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \le \frac{1}{2}\|f_s(t)\|_H^2 \le \frac{1}{2}C_0^2 \quad \text{for all } t \ge t_0,\ s \in \mathbb{N}. \tag{3.3}$$

By definition of the subdifferential we have the following inequality

$$\varphi_{p_s(x)}(u_s(t)) \le \langle \partial\varphi_{p_s(x)}(u_s(t)), u_s(t)\rangle.$$

Thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \varphi_{p_s(x)}(u_s(t)) &\le \left\langle \frac{\partial u_s}{\partial t}(t), u_s(t) \right\rangle + \langle \partial\varphi_{p_s(x)}(u_s(t)), u_s(t)\rangle \\ &= \langle f_s(t), u_s(t)\rangle \\ &\le \|f_s(t)\|_H \|u_s(t)\|_H \le C_0 r_0 \end{aligned} \tag{3.4}$$

for all $t \ge t_0$ and $s \in \mathbb{N}$. Let $t \ge t_0$ and $r := t_1 - t_0 > 0$. Integrating (3.4) from $t$ to $t+r$ we obtain

$$\int_t^{t+r} \varphi_{p_s(x)}(u_s(\tau))d\tau \le \frac{1}{2}\|u_s(t)\|_H^2 + C_0 r_0 r \le \frac{1}{2}r_0^2 + C_0 r_0 r =: A \tag{3.5}$$

for all $s \in \mathbb{N}$. From (3.3), (3.5) and the Uniform Gronwall Lemma (see [33]), we obtain

$$\varphi_{p_s(x)}(u_s(t)) \le \frac{A}{r} + \frac{1}{2}C_0^2 r =: \tilde{r_1},$$

for all $t \geq t_1$ and $s \in \mathbb{N}$. Using (2.4) we obtain $\|u_s(t)\|_{X_s} \leq K_1$ for all $t \geq t_1$ and $s \in \mathbb{N}$ for a positive constant $K_1$. In a similar way, we conclude $\|v_s(t)\|_{Y_s} \leq K_2$ for all $t \geq t_1$ and $s \in \mathbb{N}$ for a positive constant $K_2$ and the assertion of the lemma follows. □

**Corollary 3.6.**

(a) *There exists a bounded set $B_1^s$ in $X_s \times Y_s$ such that $\mathcal{A}_s \subset B_1^s$.*
(b) *Let $(u_s, v_s)$ be a solution of problem* (1.1)*. Given $t_1 > 0$ there exists a positive constant $r_2$, independent of $s$, such that*

$$\|u_s(t)\|_X + \|v_s(t)\|_Y < r_2,$$

*for all $t \geq t_1$ and $s \in \mathbb{N}$, where $X = W^{1,p}(\Omega)$ and $Y = W^{1,q}(\Omega)$.*
(c) $\mathcal{A} := \overline{\bigcup_{s\in\mathbb{N}} \mathcal{A}_s}$ *is a compact subset of $H \times H$.*

**Lemma 3.7.** *If $(u_s, v_s) \in \mathbb{G}_s$ and there exists $C > 0$ such that $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$, then we have that there exists a positive constant $\tilde{K}$ such that*

$$\|(u_s(t), v_s(t)\|_{X_s \times Y_s} \leq \tilde{K} \quad \text{for all } s \in \mathbb{N}, t \in [0, t_1].$$

*In this case we can consider $t_1 = 0$ in Lemma 3.5.*

*Proof.* Given $t_1 > 0$, if $(u_s, v_s)$ is a solution of (1.1) then multiplying the first equation by $\frac{\partial u_s}{\partial t}(t)$ we have that

$$\left\|\frac{\partial u_s}{\partial t}(t)\right\|_H^2 + \left\langle A^s(u_s(t)), \frac{\partial u_s}{\partial t}(t)\right\rangle = \left\langle f_s(t), \frac{\partial u_s}{\partial t}(t)\right\rangle.$$

As $\langle A^s(u_s(t)), \frac{\partial u_s}{\partial t}(t)\rangle = \frac{d}{dt}\varphi_{p_s(x)}(u_s(t))$, we obtain

$$\frac{1}{2}\left\|\frac{\partial u_s}{\partial t}(t)\right\|_H^2 + \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2$$

and then

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2.$$

Using Lemma 3.4 and the fact that $F$ is bounded we conclude

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq C_1 \quad \text{for all } t \in [0, t_1], s \in \mathbb{N},$$

where $C_1 > 0$ is a constant. Therefore, integrating the equation above from 0 to $\tau$, for $\tau \leq t_1$, we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + C_1 t_1 \quad \text{for all } \tau \in [0, t_1], s \in \mathbb{N}.$$

In a similar way, we obtain

$$\varphi_{q_s(x)}(v_s(\tau)) \leq \varphi_{q_s(x)}(v_{0s}) + C_2 t_1 \quad \text{for all } \tau \in [0, t_1], s \in \mathbb{N},$$

where $C_2 > 0$ is a constant and the result follows. □

**Corollary 3.8.** *Let* $(u_s, v_s)$ *be a solution of* (1.1) *with initial value* $u_{0s}, v_{0s}$. *If there is* $C > 0$ *such that* $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ *for all* $s \in \mathbb{N}$, *then given* $t_1 > 0$ *there exists a positive constant* $\widetilde{R_1}$ *such that*

$$\|u_s(t)\|_X + \|v_s(t)\|_Y \leq \widetilde{R_1},$$

*for all* $t \in [0, t_1]$ *and* $s \in \mathbb{N}$, *where* $X = W^{1,p}(\Omega)$ *and* $Y = W^{1,q}(\Omega)$.

## 4. THE LIMIT PROBLEM AND CONVERGENCE PROPERTIES

Our objective in this section is to prove that the limit problem of problem (1.1) as $D_s$ increases to infinity and $p_s(\cdot) \to p > 2$, $q_s(\cdot) \to q > 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by an ordinary differential system. Firstly we observe that the gradients of the solutions of problem (1.1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem

$$\begin{cases} \dot{u} + \phi_p(u) \in \widetilde{F}(u, v), \\ \dot{v} + \phi_q(v) \in \widetilde{G}(u, v), \\ u(0) = u_0, v(0) = v_0, \end{cases} \tag{4.1}$$

where $\phi_p(w) := |w|^{p-2}w$, $\widetilde{F} := F_{|\mathbb{R}\times\mathbb{R}}, \widetilde{G} := G_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ if we identify $\mathbb{R}$ with the constant functions which are in $H$, since $\Omega$ is a bounded set.

The proof of the next result follows the ideas of [24], but some adjustments are necessary for this variable exponent case. To obtain the limit system we first prove the following theorem.

**Theorem 4.1.** *If* $(u_s, v_s)$ *is a solution of* (1.1), *then for each* $t > t_1$, *the sequences of real numbers* $\{\|\nabla u_s(t)\|_H\}_{s\in\mathbb{N}}$ *and* $\{\|\nabla v_s(t)\|_H\}_{s\in\mathbb{N}}$ *both possess subsequences* $\{\|\nabla u_{s_j}(t)\|_H\}$ *and* $\{\|\nabla v_{s_j}(t)\|_H\}$ *that converge to zero as* $j \to \infty$, *where* $t_1$ *is the positive constant in Lemma 3.5.*

*Proof.* Let $T > 0$ and $t \in (t_1, T)$. Let $(u_s, v_s)$ be a solution of the problem (1.1). Therefore, there are $f_s, g_s \in L^1(0, T; H)$, with $f_s(t) \in F(u_s(t), v_s(t))$, $g_s(t) \in G(u_s(t), v_s(t))$ *a.e.* in $(0, T)$, such that $(u_s, v_s)$ is a solution of the system

$$\begin{cases} \frac{du_s}{dt} + A^s u_s = f_s & \text{in } (0, T), \\ \frac{dv_s}{dt} + B^s = g_s & \text{in } (0, T), \\ u_s(0) = u_{0s}, v_s(0) = v_{0s}. \end{cases} \tag{4.2}$$

Taking the inner product of the first equation of (4.2) with $u_s(\tau)$, yields

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 + D_s \int_\Omega |\nabla u_s(\tau)|^{p_s(x)} dx + \int_\Omega |u_s(\tau)|^{p_s(x)} dx = \langle f_s(\tau), u_s(\tau) \rangle. \tag{4.3}$$

Analogously, we have that

$$\frac{1}{2}\frac{d}{dt}\|v_s(\tau)\|_H^2 + D_s\int\limits_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx + \int\limits_\Omega |v_s(\tau)|^{q_s(x)}dx = \langle g_s(\tau), v_s(\tau)\rangle. \tag{4.4}$$

Now by using Lemma 3.1 and the fact that $F$ and $G$ are bounded, there exists a positive constant $C_0$ such that $\|f_s(\tau)\|_H \leq C_0$ and $\|g_s(\tau)\|_H \leq C_0$ for all $\tau \geq t_0$ and $s \in \mathbb{N}$. Thus

$$\langle f_s(\tau), u_s(\tau)\rangle \leq \|f_s(t)\|_H\|u_s(\tau)\|_H \leq C_0 r_0$$

and

$$\langle g_s(\tau), v_s(\tau)\rangle \leq \|g_s(t)\|_H\|v_s(\tau)\|_H \leq C_0 r_0.$$

Then, adding the equations (4.3) and (4.4), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\Big)\\&+ D_s\int\limits_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx + D_s\int\limits_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx\\&+ \int\limits_\Omega |u_s(\tau)|^{p_s(x)}dx + \int\limits_\Omega |v_s(\tau)|^{q_s(x)}dx \leq C_3,\ a.e. \text{ in } (t_1, T).\end{aligned}$$

As

$$\int\limits_\Omega |u_s(\tau)|^{p_s(x)}dx + \int\limits_\Omega |v_s(\tau)|^{q_s(x)}dx \geq 0,$$

we have in particular that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\Big)\\&+ D_s\int\limits_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx + D_s\int\limits_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx \leq C_3,\end{aligned} \tag{4.5}$$

a.e. in $(t_1, T)$.

Integrating the inequality (4.5) from $t_1$ to $T$, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big(\|u_s(T)\|_H^2 + \|v_s(T)\|_H^2\Big) \\
&+ D_s \int_{t_1}^{T}\int_{\Omega} |\nabla u_s(\tau)|^{p_s(x)} dx d\tau + D_s \int_{t_1}^{T}\int_{\Omega} |\nabla v_s(\tau)|^{q_s(x)} dx d\tau \\
&\leq \int_{t_1}^{T} C_3 d\tau + \frac{1}{2}\Big(\|u_s(t_1)\|_H^2 + \|v_s(t_1)\|_H^2\Big) \\
&\leq C_3 T + r_0^2 := k(T).
\end{aligned}$$

In particular

$$D_s \int_{t_1}^{T}\int_{\Omega} |\nabla u_s(\tau)|^{p_s(x)} dx d\tau \leq k(T)$$

and

$$D_s \int_{t_1}^{T}\int_{\Omega} |\nabla v_s(\tau)|^{q_s(x)} dx d\tau \leq k(T),$$

which implies

$$\int_{t_1}^{T}\int_{\Omega} |\nabla u_s(\tau)|^{p_s(x)} dx d\tau \leq \frac{1}{D_s} k(T) \to 0 \quad \text{as } s \to \infty.$$

Therefore there exists a subsequence $s_j$ such that

$$\int_{\Omega} |\nabla u_{s_j}(\tau)|^{p_{s_j}(x)} dx \to 0 \quad \text{as } j \to \infty, \tau\text{-a.e. in } (t_1, T),$$

and so there exists a subset $J \subset (t_1, T)$ with Lebesgue measure $m((t_1, T)/J) = 0$ such that

$$\int_{\Omega} |\nabla u_{s_j}(\tau)|^{p_{s_j}(x)} dx \to 0 \quad \text{as } \; j \to \infty, \text{ for all } \tau \in J.$$

Given $t \in (t_1, T)$, we pick one $\nu \in J$ with $t_1 < \nu < t$ and let $h = t - \nu$. Let $\varepsilon > 0$ and $j_0 = j_0(\varepsilon) > 0$ be such that if $j > j_0$ then

$$\int_{\Omega} |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)} dx < \frac{\varepsilon}{L}.$$

We have that

$$\frac{d}{d\tau}\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)) = \left\langle \partial\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle \quad \text{a.e. in } (0, T).$$

Therefore

$$\begin{aligned}
&\int_\Omega \frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx + \int_\Omega \frac{1}{p_{s_j}(x)}|u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx\\
&-\int_\Omega \frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx - \int_\Omega \frac{1}{p_{s_j}(x)}|u_{s_j}(\nu)|^{p_{s_j}(x)}dx\\
&=\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+h)) - \varphi_{p_{s_j}(x)}(u_{s_j}(\nu))\\
&=\int_0^h \frac{d}{d\tau}\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau))\,d\tau\\
&=\int_0^h \left\langle \partial\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)), \frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\rangle d\tau\\
&=\int_0^h \left\langle f_{s_j}(\nu+\tau) - \frac{d}{d\tau}u_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\rangle d\tau\\
&=\int_0^h \left\langle f_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\rangle d\tau\\
&\quad -\int_0^h \left\langle \frac{d}{d\tau}u_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\rangle d\tau\\
&\le \frac{1}{2}\int_0^h \|f_{s_j}(\nu+\tau)\|_H^2\,d\tau - \frac{1}{2}\int_0^h \left\|\frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\|_H^2 d\tau\\
&\le \frac{1}{2}\int_0^h \|f_{s_j}(\nu+\tau)\|_H^2\,d\tau \le \frac{1}{2}\int_0^h C_0^2 d\tau = \frac{1}{2}C_0^2 h.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega \frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx\\
&\le \int_\Omega \frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \int_\Omega \frac{1}{p_{s_j}(x)}|u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \frac{1}{2}C_0^2 h.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Omega |\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx \\
&\leq \frac{L}{2}\int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \frac{L}{2D_{s_j}}\int_\Omega |u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \frac{LC_0^2 h}{2D_{s_j}}.
\end{aligned}$$

So, using (2.3) and Lemma 3.5

$$\begin{aligned}
&\int_\Omega |\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx \\
&\leq \frac{L}{2}\int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \frac{L}{2D_{s_j}}K^L + \frac{LC_0^2|T-t_1|}{2D_{s_j}},
\end{aligned}$$

where $K$ is the positive constant which appears in the Lemma 3.5.

Thus, choose $j_1 = j_1(\varepsilon)$ sufficiently large such that

$$\frac{L}{2D_{s_j}}K^L + \frac{LC_0^2|T-t_1|}{2D_{s_j}} < \varepsilon/2,$$

whenever $j > j_1$ and, moreover, we consider $j_2 = j_2(\varepsilon) = \max\{j_0, j_1\}$. For $j > j_2$ we have

$$\begin{aligned}
\int_\Omega |\nabla u_{s_j}(t)|^{p_{s_j}(x)}dx &= \int_\Omega |\nabla u_{s_j}(\nu+t-\nu)|^{p_{s_j}(x)}dx \\
&\leq \frac{L}{2}\int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx + \frac{L}{2D_{s_j}}K^L + \frac{LC_0^2|T-t_1|}{2D_{s_j}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, for $j > j_2$

$$\min\{\|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}^{p_{s_j}^-}, \|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}^{p_{s_j}^+}\} \leq \int_\Omega |\nabla u_{s_j}(t)|^{p_{s_j}(x)}dx < \varepsilon.$$

As $p_{s_j}(x) > 2$, $\|\nabla u_{s_j}(t)\|_H \leq 2(|\Omega|+1)\|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}$ we obtain

$$\| \nabla u_{s_j}(t) \|_H \to 0 \quad \text{as } j \to \infty.$$

Analogously we conclude that $\| \nabla v_{s_j}(t) \|_H \to 0$ as $j \to \infty$. □

**Proposition 4.2.** *If $(u_s, v_s)$ is a solution of problem (1.1) in $(0, t_1)$, then for each $t \in [0, t_1]$, the sequences $\{\|\nabla u_s(t)\|_p\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_q\}_{s\in\mathbb{N}}$ remain bounded as $s \to \infty$ whenever the initial values will be such that $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$. If the initial data are equal to a same constant, i.e., if $(u_s(0), v_s(0)) = (u_0, v_0) \in \mathbb{R}\times\mathbb{R}$ for each $s \in \mathbb{N}$, then for each $t \in [0, t_1]$, the sequences of real numbers $\{\|\nabla u_s(t)\|_p\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_q\}_{s\in\mathbb{N}}$ converges to zero as $s \to \infty$, respectively.*

*Proof.* In fact, let $(u_s, v_s)$ be a solution of problem (1.1) in $(0, t_1)$. Therefore, there are $f_s, g_s \in L^1(0, t_1; H)$, with $f_s(t) \in F(u_s(t), v_s(t)),\ g_s(t) \in G(u_s(t), v_s(t))$ a.e. in $(0, t_1)$, such that $(u_s, v_s)$ is a solution of the system:

$$\begin{cases} \frac{du_s}{dt} + A^s u_s = f_s & \text{in } (0, t_1), \\ \frac{dv_s}{dt} + B^s v_s = g_s & \text{in } (0, t_1), \\ u_s(0) = u_{0s}, v_s(0) = v_{0s}. \end{cases}$$

Taking the inner product of the first equation with $\frac{\partial u_s(t)}{\partial t}$, we obtain

$$\begin{aligned} \left\| \frac{\partial u_s(t)}{\partial t} \right\|_H^2 + \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) &= \left\langle f_s(t), \frac{\partial u_s(t)}{\partial t} \right\rangle \\ &\leq \|f_s(t)\|_H \left\| \frac{\partial u_s(t)}{\partial t} \right\|_H \\ &\leq \frac{1}{2}\|f_s(t)\|_H^2 + \frac{1}{2}\left\| \frac{\partial u_s(t)}{\partial t} \right\|_H^2 . \end{aligned}$$

In particular,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2. \tag{4.6}$$

Since $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$, using (2.6) we have that the initial values are in a bounded set of $H \times H$. Using Lemma 3.4 and the fact that $F$ and $G$ take bounded sets of $H \times H$ in bounded sets of $H$, it follows that there exists a positive constant $C$ such that $\|f_s(t)\|_H^2 \leq C$ for all $t \in [0, t_1]$ and $s \in \mathbb{N}$. Computing the integral from 0 to $\tau$, $\tau \in [0, t_1]$ in (4.6), we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + \frac{1}{2}Ct_1$$

for all $\tau \in [0, t_1]$ and $s \in \mathbb{N}$. Therefore,

$$\begin{aligned} &D_s \int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\tau)|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u_s(\tau)|^{p_s(x)}dx \\ &\leq D_s \int_\Omega \frac{1}{p_s(x)}|\nabla u_{0s}|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u_{0s}|^{p_s(x)}dx + \frac{1}{2}Ct_1, \end{aligned}$$

for all $\tau \in [0, t_1]$ and $s \in \mathbb{N}$. As a consequence,

$$\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx \leq \frac{L}{2}\int_\Omega |\nabla u_{0s}|^{p_s(x)}dx + \frac{L}{2D_s}\left(\int_\Omega |u_{0s}|^{p_s(x)}dx + Ct_1\right),$$

for all $\tau \in [0, t_1]$ and $s \in \mathbb{N}$. Analogously we prove that

$$\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx \leq \frac{L}{2}\int_\Omega |\nabla v_{0s}|^{q_s(x)}dx + \frac{L}{2D_s}\left(\int_\Omega |v_{0s}|^{q_s(x)}dx + Ct_1\right).$$

Since $p_s(x) \geq p$, $q_s(x) \geq q$ and $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$ the result follows by using (2.3). $\square$

Theorem 4.1 confirms that the equation (4.1) is a good candidate for the limit problem.

**Lemma 4.3** ([23]). *The problem* (4.1) *has a global solution.*

**Theorem 4.4** ([23]). *The problem* (4.1) *defines a generalized semiflow* $\mathbb{G}^{\infty}$ *which has a global B-attractor* $\mathcal{A}^{\infty}$.

The next result guarantees that (4.1) is in fact the limit problem for (1.1), as $s \to \infty$.

**Theorem 4.5.** *Let* $(u_s, v_s)$ *be a solution of the problem* (1.1). *Suppose that the initial values* $(u_s(0), v_s(0)) = (u_{0s}, v_{0s}) \to (u_0, v_0) \in \mathbb{R} \times \mathbb{R}$ *in the topology of* $H \times H$ *as* $s \to \infty$. *Then there exists a solution* $(u, v)$ *of the problem* (4.1) *satisfying* $(u(0), v(0)) = (u_0, v_0)$ *and a subsequence* $\{(u_{s_j}, v_{s_j})\}_j$ *of* $\{(u_s, v_s)\}_s$ *such that, for each* $T > 0$, $u_{s_j} \to u,\ v_{s_j} \to v$ *in* $C([0,T]; H)$ *as* $j \to \infty$.

*Proof.* Let $T > 0$ be fixed arbitrarily large. Let $(u_s, v_s)$ be a solution of the problem (1.1) with $(u_s(0), v_s(0)) = (u_{0s}, v_{0s}) \to (u_0, v_0) \in \mathbb{R}\times\mathbb{R}$ in $H \times H$ as $s \to \infty$. Therefore, there are $f_s, g_s \in L^1(0,T;H)$, with

$$f_s(t) \in F(u_s(t), v_s(t)),\ g_s(t) \in G(u_s(t), v_s(t)) \text{ a.e. in } (0,T),$$

and such that $(u_s, v_s)$ is a solution of the system (4.7) below:

$$\begin{cases} \frac{du_s}{dt} + A^s u_s = f_s & \text{in } (0,T), \\ \frac{dv_s}{dt} + B^s v_s = g_s & \text{in } (0,T), \\ u_s(0) = u_{0s}, v_s(0) = v_{0s}. \end{cases} \tag{4.7}$$

We denote $I(u_{0s})f_s(\cdot) := u_s(\cdot)$ and $I(v_{0s})g_s(\cdot) := v_s(\cdot)$ and also denote by $I(u_0)f_s(\cdot) := z_s(\cdot)$ and $I(v_0)g_s(\cdot) := w_s(\cdot)$ being the corresponding solutions of the problems

$$\begin{cases} \frac{dz_s}{dt} + A^s z_s = f_s, \\ z_s(0) = u_0, \end{cases} \tag{4.8}$$

and

$$\begin{cases} \frac{dw_s}{dt} + B^s w_s = g_s, \\ w_s(0) = v_0, \end{cases}$$

respectively.

Taking the inner product of the first equation in (4.7) with $u_s$ and computing the integral from 0 to $t$, $t \le T$, we obtain

$$\frac{1}{2} \parallel u_s(t) \parallel_H^2 \le \frac{1}{2} \parallel u_{0s} \parallel_H^2 + \int_0^t \langle f_s(\tau), u_s(\tau)\rangle d\tau.$$

As $\{u_{0s}\}$ is a convergent sequence, we have that there exists a positive constant $R$ such that $\| u_{0s} \|_H^2 \leq R^2$. Thus,

$$\frac{1}{2} \| u_s(t) \|_H^2 \leq \frac{1}{2} R^2 + \int_0^t \langle f_s(\tau), u_s(\tau) \rangle d\tau.$$

Using the hypothesis that the couple $(F, G)$ is positively sublinear and Gronwall's inequality we obtain that there exist positive constants $\alpha, \beta$, $\gamma$ and $C$ such that

$$\| u_s(t) \|_H \leq C + \gamma T + \int_0^t [\alpha \| u_s(\tau) \|_H + \beta \| v_s(\tau) \|_H] d\tau.$$

So, there is a positive constant $M$ independent of $t \in [0, T]$ such that

$$\| u_s(t) \|_H \leq M + \int_0^t [\alpha \| u_s(\tau) \|_H + \beta \| v_s(\tau) \|_H] d\tau.$$

Analogously, there exists a positive constant $\widetilde{M}$ independent of $t \in [0, T]$ such that

$$\| v_s(t) \|_H \leq \widetilde{M} + \int_0^t [\beta \| u_s(\tau) \|_H + \alpha \| v_s(\tau) \|_H] d\tau.$$

Adding these two inequalities and denoting by $N := M + \widetilde{M}$ and $\rho := \alpha + \beta$ we have

$$\| u_s(t) \|_H + \| v_s(t) \|_H \leq N + \rho \int_0^t [\| u_s(\tau) \|_H + \| v_s(\tau) \|_H] d\tau$$

and so it follows by the Gronwall–Bellman inequality that

$$\| u_s(t) \|_H + \| v_s(t) \|_H \leq N e^{\rho T},$$

for all $t \in [0, T]$ and for all $s \in \mathbb{N}$.

As $F$ and $G$ map bounded sets of $H \times H$ into bounded sets of $H$, it follows by the inequality above that there exists $D > 0$ such that

$$\| f_s(t) \|_H \leq D \quad \text{and} \quad \| g_s(t) \|_H \leq D \quad \text{for all } t \in [0, T] \text{ and } s \in \mathbb{N}. \tag{4.9}$$

Consider $K := \{f_s; s \in \mathbb{N}\}$, $\widetilde{K} := \{g_s; s \in \mathbb{N}\}$, $M(K) := \{z_s; s \in \mathbb{N}\}$ and $M(\widetilde{K}) := \{w_s; s \in \mathbb{N}\}$. It follows by (4.9) that $K$ and $\widetilde{K}$ are uniformly integrable subsets of $L^1(0, T; H)$.

By using compactness results we obtain that $M(K)$ is a relatively compact set in $C([0,T];H)$ and so there are $z \in C([0,T];H)$ and a subsequence $\{z_{s_j}\}$ of $\{z_s\}$ such that $z_{s_j} \to z$ in $C([0,T];H)$.

As each $z_{s_j}$ is a solution of (4.8) in $(0,T)$, then by Proposition 3.6 in [7], $z_{s_j}$ verifies

$$\frac{1}{2} \parallel z_{s_j}(t) - \theta \parallel^2 \leq \frac{1}{2} \parallel z_{s_j}(\ell) - \theta \parallel^2 + \int_{\ell}^{t} \langle f_{s_j}(\tau) - y_j, z_{s_j}(\tau) - \theta \rangle d\tau \tag{4.10}$$

for all $\theta \in \mathcal{D}(A^{s_j}) \subset W^{1,p_{s_j}(\cdot)}(\Omega) \subset H$, $y_j = A^{s_j}(\theta)$ and for all $0 \leq \ell \leq t \leq T$.

Analogously, we can show that there exists $w \in C([0,T];H)$ and there exists a subsequence $\{w_{s_j}\}$ of $\{w_s\}$ such that $w_{s_j} \to w$ in $C([0,T];H)$, verifying

$$\frac{1}{2} \parallel w_{s_j}(t) - \theta \parallel^2 \leq \frac{1}{2} \parallel w_{s_j}(\ell) - \theta \parallel^2 + \int_{\ell}^{t} \langle g_{s_j}(\tau) - y_j, w_{s_j}(\tau) - \theta \rangle d\tau \tag{4.11}$$

for all $\theta \in \mathcal{D}(B^{s_j}) \subset W^{1,q_{s_j}(\cdot)}(\Omega) \subset H$, $y_j = B^{s_j}(\theta)$ and for all $0 \leq \ell \leq t \leq T$.

As $\parallel f_{s_j}(\tau) \parallel_H \leq D$ and $\parallel g_{s_j}(\tau) \parallel_H \leq D$ for all $0 \leq \tau \leq T$ and for all $j \in \mathbb{N}$, we conclude that there exists a positive constant $\widetilde{D}$ such that $\parallel f_{s_j} \parallel_{L^2(0,T;H)} \leq \widetilde{D}$ and $\parallel g_{s_j} \parallel_{L^2(0,T;H)} \leq \widetilde{D}$, for all $j \in \mathbb{N}$.

As $L^2(0,T;H)$ is a reflexive Banach space, there are $f, g \in L^2(0,T;H)$ and subsequences, which we do not relabel, $\{f_{s_j}\}$ and $\{g_{s_j}\}$ such that $f_{s_j} \rightharpoonup f$ and $g_{s_j} \rightharpoonup g$ in $L^2(0,T;H)$. Consequently $f_{s_j} \rightharpoonup f$ and $g_{s_j} \rightharpoonup g$ in $L^1(0,T;H)$.

*Statement 1.* *$u_{s_j} \to z$ and $v_{s_j} \to w$ in $C([0,T];H)$. Moreover, $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e. in $[0,T]$.*

Indeed, let $t \in [0,T]$. We have

$$\parallel u_{s_j}(t) - z(t) \parallel_H \leq \parallel u_{s_j}(t) - z_{s_j}(t) \parallel_H + \parallel z_{s_j}(t) - z(t) \parallel_H .$$

Therefore,

$$\begin{aligned}\sup_{t\in[0,T]} \parallel u_{s_j}(t) - z(t) \parallel_H \leq & \sup_{t\in[0,T]} \parallel I(u_{0s_j}) f_{s_j}(t) - I(u_0) f_{s_j}(t) \parallel_H \\ & + \sup_{t\in[0,T]} \parallel z_{s_j}(t) - z(t) \parallel_H \\ \leq & \parallel u_{0s_j} - u_0 \parallel_H + \sup_{t\in[0,T]} \parallel z_{s_j}(t) - z(t) \parallel_H \to 0\end{aligned}$$

as $j \to \infty$. Thus $u_{s_j} \to z$ in $C([0,T];H)$ as $j \to \infty$. Analogously we show that $v_{s_j} \to w$ in $C([0,T];H)$ as $j \to +\infty$. Then, by Theorem 3.3 in [13], $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e. in $[0,T]$.

Now consider $\overline{\theta} \in \mathbb{R} \subset H$ and let $\overline{h} := \phi_p(\overline{\theta}) \in \mathbb{R} \subset H$. We consider

$$y_j := A^{s_j}(\overline{\theta}) = -\mathrm{div}(D_{s_j} |\nabla \overline{\theta}|^{p_{s_j}(x)-2} \nabla \overline{\theta}) + |\overline{\theta}|^{p_{s_j}(x)-2} \overline{\theta}.$$

Note that $D(A^{s_j}) \supset \mathbb{R}$ for all $j \in N$ and since $\overline{\theta}$ is a constant function $\nabla\overline{\theta} = 0$, so $y_j = |\overline{\theta}|^{p_{s_j}(x)-2}\overline{\theta}$. By (4.10), we know that

$$\begin{aligned} \frac{1}{2} \parallel z_{s_j}(t) - \overline{\theta} \parallel^2 &\leq \frac{1}{2} \parallel z_{s_j}(\ell) - \overline{\theta} \parallel^2 + \int_\ell^t \langle f_{s_j}(\tau) - y_j, z_{s_j}(\tau) - \overline{\theta} \rangle d\tau \\ &= \frac{1}{2} \parallel z_{s_j}(\ell) - \overline{\theta} \parallel^2 + \int_\ell^t \langle f_{s_j}(\tau) - \overline{h}, z_{s_j}(\tau) - \overline{\theta} \rangle d\tau \\ &\quad + \int_\ell^t \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta} \rangle d\tau \end{aligned} \tag{4.12}$$

for all $0 \leq l \leq t \leq T$ and for all $j \in \mathbb{N}$. We claim that $\int_\ell^t \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta} \rangle d\tau \to 0$ as $j \to \infty$. In fact, for $\overline{\theta} = 0$ this is immediate and if $\overline{\theta} \neq 0$ then for each $\tau > 0$

$$|\langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta} \rangle| \leq \int_\Omega \Big( \Big| \, |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1} \Big| \Big) |z_{s_j}(\tau)| dx + \int_\Omega \Big| \, |\overline{\theta}|^p - |\overline{\theta}|^{p_{s_j}(x)} \Big| dx.$$

Since $p_{s_j}(x) \to p$ in $L^\infty(\Omega)$ as $j \to \infty$ it follows by the Dominated Convergence Theorem that

$$\int_\Omega \Big| \, |\overline{\theta}|^p - |\overline{\theta}|^{p_{s_j}(x)} \Big| dx \to 0 \ \text{ as } \ j \to \infty.$$

On the other hand, using the Mean Value Theorem we obtain

$$\int_\Omega \Big( \Big| \, |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1} \Big| \Big) |z_{s_j}(\tau)| dx \leq \int_\Omega |\overline{\theta}|^{\tau(s_j,x)} \ln(|\overline{\theta}|)(p_{s_j}(x) - p)|z_{s_j}(\tau)| dx$$

where $p < \tau(s_j, x) < p_{s_j}(x)$. Thus, considering $p'_{s_j}(\cdot)$ such that $\frac{1}{p_{s_j}(x)} + \frac{1}{p'_{s_j}(x)} = 1$ for all $x \in \Omega$, we have

$$\begin{aligned} &\int_\Omega \Big( \Big| \, |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1} \Big| \Big) |z_{s_j}(\tau)| dx \\ &\leq \|p_{s_j} - p\|_\infty \int_\Omega |\overline{\theta}|^{\tau(s_j,x)+1} |z_{s_j}(\tau)| dx \\ &\leq \|p_{s_j} - p\|_\infty \left[ \int_\Omega \frac{1}{p'_{s_j}(x)} |\overline{\theta}|^{(\tau(s_j,x)+1)p'_{s_j}(x)} dx + \int_\Omega \frac{1}{p_{s_j}(x)} |z_{s_j}(\tau)|^{p_{s_j}(x)} dx \right] \\ &\leq \|p_{s_j} - p\|_\infty \left[ \int_\Omega |\overline{\theta}|^{(\tau(s_j,x)+1)p'_{s_j}(x)} dx + \frac{1}{2} \int_\Omega |z_{s_j}(\tau)|^{p_{s_j}(x)} dx \right]. \end{aligned}$$

By Lemma 3.7 there exists a constant $C > 0$ such that $\int_\Omega |z_{s_j}(\tau)|^{p_{s_j}(x)}dx \leq C$ for every $\tau \in (\ell, t)$ and $j \in \mathbb{N}$.

On the other hand, as $p+1 < \tau(s_j, x)+1 < p_{s_j}(x)+1 < L+1$ and $1 < q_{s_j}(x) < 2$ we obtain $\int_\Omega |\overline{\theta}|^{(\tau(s_j,x)+1)p'_{s_j}(x)}dx \leq \tilde{C}$ for all $j \in \mathbb{N}$. Thus, considering $\bar{C} := C+\tilde{C} > 0$, we have

$$\int\limits_\Omega \Big(\Big| \, |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1}\Big|\Big)|z_{s_j}(\tau)|dx \leq \|p_{s_j} - p\|_\infty \bar{C} \to 0 \quad \text{as} \quad j \to \infty,$$

and we conclude that

$$\int\limits_\ell^t \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta}\rangle d\tau \to 0 \quad \text{as} \quad j \to \infty.$$

Thus, taking the limit as $j \to \infty$, in (4.12) we obtain

$$\frac{1}{2} \parallel z(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel z(\ell) - \overline{\theta} \parallel^2 + \int\limits_\ell^t \langle f(\tau) - \overline{h}, z(\tau) - \overline{\theta}\rangle d\tau \tag{4.13}$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

In the same way we can show that

$$\frac{1}{2} \parallel w(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel w(\ell) - \overline{\theta} \parallel^2 + \int\limits_\ell^t \langle g(\tau) - \overline{h}, w(\tau) - \overline{\theta}\rangle d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_q(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

*Statement 2. $z(t)$ and $w(t)$ are independent of $x$, for each $t > 0$.*

Indeed, let $t > 0$. We already know that $z_{s_j}(t) \to z(t)$ in $H$. Since $z_{s_j}(0) = u_0$ for all $j \in \mathbb{N}$, then by Proposition 4.2 and Theorem 4.1 we have that $\|\nabla z_{s_j}(t)\|_H \to 0$ as $j \to \infty$. We also have that $z_{s_j}(t) \in \mathcal{D}(A^{s_j}) \subset W^{1,p_{s_j}}(\Omega) \subset W^{1,2}(\Omega)$. Then, by the Poincaré–Wirtinger inequality (see [8])

$$\|z_{s_j}(t) - \overline{z_{s_j}(t)}\|_H \leq C\|\nabla z_{s_j}(t)\|_H \to 0 \quad \text{as} \quad j \to \infty,$$

where $\overline{z_{s_j}(t)} := \frac{1}{|\Omega|}\int_\Omega z_{s_j(t)}(x)\, dx$. Then,

$$\begin{aligned}\|z(t) - \overline{z(t)}\|_H &\leq \|z(t) - z_{s_j}(t)\|_H + \|z_{s_j}(t) - \overline{z_{s_j}(t)}\|_H \\ &\quad + \|\overline{z_{s_j}(t)} - \overline{z(t)}\|_H \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Thus $z(t) = \overline{z(t)}$. Analogously, we show that $w(t) = \overline{w(t)}$ and the assertion follows.

We already showed in the Statement 1 that $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e. in $(0, T)$. Therefore $f(t)$ and $g(t)$ are independents on $x$, $t$-a.e. in $(0, T)$.

Thus, from (4.13)

$$\frac{1}{2}|z(t)-\overline{\theta}|^2|\Omega| \leq \frac{1}{2}|z(\ell)-\overline{\theta}|^2|\Omega| + \int_\ell^t \int_\Omega (f(\tau)-\overline{h})(z(\tau)-\overline{\theta})\,dx\,d\tau.$$

Hence

$$\frac{1}{2}|z(t)-\overline{\theta}|^2 \leq \frac{1}{2}|z(\ell)-\overline{\theta}|^2 + \int_\ell^t (f(\tau)-\overline{h})(z(\tau)-\overline{\theta})\,d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell < t \leq T$.

If $t = \ell = 0$, we have $z(0) = \lim_{j\to\infty} z_{s_j}(0) = \lim_{j\to\infty} u_0 = u_0$, and, therefore, $\frac{1}{2}|z(0)-\overline{\theta}|^2 = \frac{1}{2}|u_0-\overline{\theta}|^2$. Thus

$$\frac{1}{2}|z(t)-\overline{\theta}|^2 \leq \frac{1}{2}|z(\ell)-\overline{\theta}|^2 + \int_\ell^t (f(\tau)-\overline{h})(z(\tau)-\overline{\theta})\,d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

In the same way,

$$\frac{1}{2}|w(t)-\overline{\theta}|^2 \leq \frac{1}{2}|w(\ell)-\overline{\theta}|^2 + \int_\ell^t (g(\tau)-\overline{h})(w(\tau)-\overline{\theta})\,d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_q(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

So by the Proposition 3.6 in [7], we conclude that $(z, w)$ is a weak solution of problem (4.1) with $(z(0), w(0)) = (u_0, v_0)$, but as $f, g \in L^2(0, T; H)$ we have in fact that $(z, w)$ is a strong solution of problem (4.1). □

**Remark 4.6.** The Theorem 4.5 remains valid without the hypothesis $(u_0, v_0) \in \mathbb{R}\times\mathbb{R}$, whenever $(u_{0s}, v_{0s}) \in \mathcal{A}_s$ for all $s \in \mathbb{N}$, because in this case we prove, analogously as was done in Lemma 4.1 in [24], that $u_0$ and $v_0$ are independent of $x$.

The proof of the next result is completely analogous as in [23], but for convenience of the reader we give the proof.

**Theorem 4.7.** *The family of attractors* $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ *associated with the problem* (1.1) *is upper semicontinuous at infinity in the topology of* $H \times H$.

*Proof.* Let $\{(u_{0s}, v_{0s})\}_{s\in\mathbb{N}}$ be an arbitrary sequence with

$$(u_{0s}, v_{0s}) \in \mathcal{A}_s \quad \text{for all } s \in \mathbb{N} \text{ and } D_s \to \infty \ \text{ as } s \to \infty.$$

By Corollary 3.6(c), there exists a subsequence, that we still denote the same, such that $(u_{0s}, v_{0s}) \to (u_0, v_0)$ in $H \times H$ as $s \to \infty$. By [11], it is enough to prove that $(u_0, v_0) \in \mathcal{A}^\infty$.

Using the invariance of the attractors, Theorem 4.1 and Poincaré–Wirtinger's inequality, we can prove analogously to Lemma 4.1 in [24], that $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$.

For each $s \in \mathbb{N}$, consider $t_s > s$, $t_1 < t_2 < \ldots < t_s < \ldots$. By invariance of the attractors, there are $(x_s, y_s) \in \mathcal{A}_s$ and solutions $\varphi^s = (\varphi_1^s, \varphi_2^s) \in \mathbb{G}_s$ with $\varphi^s(0) = (x_s, y_s)$ such that $\varphi^s(t_s) = (u_{0s}, v_{0s}) \to (u_0, v_0)$ in $H \times H$ as $s \to \infty$. Note that

$$\varphi^s(t_s) \in T_s(t_s)(x_s, y_s) \in \mathcal{A}_s, \quad s \in \mathbb{N}.$$

By the definition of generalized semiflow, for each $s \in \mathbb{N}$, the translates $(\varphi^s)^{t_s}$ also are solutions, and we have $(\varphi^s)^{t_s}(0) \to (u_0, v_0)$ in $H \times H$ as $s \to \infty$.

Using Theorem 4.5, we obtain that there exists a solution $g_0$ of the limit problem (4.1) with $g_0(0) = (u_0, v_0)$ and a subsequence of $\left\{(\varphi^s)^{t_s}\right\}_s$, that we still denote the same, such that

$$(\varphi^s)^{t_s}(t) \to g_0(t) \text{ in } H \times H \text{ as } s \to \infty, \text{for all } t \geq 0.$$

Now we consider the sequence $\{\varphi^s(t_s - 1)\}$. Note that

$$\varphi^s(t_s - 1) \in T_s(t_s - 1)(x_s, y_s) \subset \bigcup_s \mathcal{A}_s$$

which is a precompact subset of $H \times H$, then, passing to a subsequence if necessary,

$$(\varphi^s)^{(t_s-1)}(0) = \varphi^s(t_s - 1) \to z_1 \text{ in } H \times H \text{ as } s \to \infty.$$

As for each $s \in \mathbb{N}$, $\varphi^s$ is a solution starting on the attractor $\mathcal{A}_s$, we obtain by the invariance of the attractors that the sequence of initial values

$$\varphi^s(t_s - 1) \in \mathcal{A}_s \quad \text{for all } s \in \mathbb{N}.$$

Thus, using Remark 4.6 and Theorem 4.5, we obtain that there exists a solution $g_1$ of the limit problem (4.1) with $g_1(0) = z_1$ and a subsequence of $\left\{(\varphi^s)^{(t_s-1)}\right\}_s$, that we still denote in the same way, such that

$$(\varphi^s)^{(t_s-1)}(t) \to g_1(t) \text{ in } H \times H \text{ as } s \to \infty, \text{for all } t \geq 0.$$

Now note that $g_1^1 = g_0$, since for each $t \geq 0$, we have

$$g_1^1(t) = g_1(t+1) = \lim_{s \to \infty} (\varphi^s)^{(t_s-1)}(t+1) = \lim_{s \to \infty} (\varphi^s)^{t_s}(t) = g_0(t).$$

Proceeding inductively, we find for each $r = 0, 1, 2, \ldots$, a solution $g_r \in \mathbb{G}^\infty$ with $g_r(0) = z_r$ such that $g_{r+1}^1 = g_r$. Given $t \in \mathbb{R}$, we define $g(t)$ as the common value of $g_r(t + r)$ for $r > -t$. Then we have that $g$ is a complete orbit for $\mathbb{G}^\infty$ with $g(0) = g_0(0) = (u_0, v_0)$.

Note that for each $t \geq 0$, $r = 0, 1, 2, \ldots$, we have that each

$$g_r(t) = \lim_{s \to \infty} (\varphi^s)^{(t_s-r)}(t) \text{ and } (\varphi^s)^{(t_s-r)}(t) \in \mathcal{A}_s \quad \text{for all } s \in \mathbb{N}.$$

Working with the coordinated functions and using the invariance of the attractors, Lemma 4.1 and the Poincaré–Wirtinger inequality, we can prove, analogously to Lemma 4.1 in [24], that each $g_r(t)$ is independent on $x$. Consequently, we obtain that $g(t)$ is a constant function in $x$. As $\mathcal{A}_s \subset \overline{\bigcup_s \mathcal{A}_s}$, for all $s \in \mathbb{N}$, we obtain that there exists a constant $C > 0$ such that $\|g_r(t)\|_{H\times H} \leq C$ for all $t \geq 0$ and $r = 0, 1, 2, \ldots$ So, in particular, we have that $g(t)$ is bounded in $H \times H$. Then, there exists a constant $\tilde{C} > 0$ such that

$$|g(t)|_{\mathbb{R}\times\mathbb{R}} = \frac{1}{|\Omega|^{1/2}} \|g(t)\|_{H\times H} \leq \tilde{C} \quad \text{for all } t \in \mathbb{R}.$$

So, we conclude that $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a complete bounded orbit for $\mathbb{G}^\infty$ through $(u_0, v_0)$.

Using Theorem 15 in [21], we obtain that $(u_0, v_0) \in \mathcal{A}^\infty$. □

The next result is a direct consequence of Theorem 4.7 and Corollary 3.6.

**Corollary 4.8.** *The family of attractors* $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ *associated with the problem* (1.1) *is upper semicontinuous at infinity in the topology of* $L^p(\Omega) \times L^q(\Omega)$.

*Proof.* Let $\{a_s\}_{s\in\mathbb{N}}$ be an arbitrary sequence with $a_s \in \mathcal{A}_s$ for each $s \in \mathbb{N}$. We will prove that there exists $a_\infty \in \mathcal{A}^\infty$ and a subsequence $\{a_{s_k}\}_{k\in\mathbb{N}}$ such that $a_{s_k} \to a_\infty$ in the topology of $L^p(\Omega) \times L^q(\Omega)$.

By Theorem 4.7, $\text{dist}_{H\times H}(\mathcal{A}_s, \mathcal{A}^\infty) \to 0$ as $s \to \infty$. Therefore, we have the existence of a subsequence $\{a_{s_k}\}_{k\in\mathbb{N}}$ and $a_\infty \in \mathcal{A}^\infty$ such that $a_{s_k} \to a_\infty$ in the topology of $H \times H$.

By Corollary 3.6 item (b), $\|u_s(t)\|_X + \|v_s(t)\|_Y < r_2$, for all $t \geq t_1$ and $s \in \mathbb{N}$, where $X = W^{1,p}(\Omega)$, $Y = W^{1,q}(\Omega)$ and $(u_s, v_s)$ is any solution of problem (1.1). So, using the invariance of the global attractors, we have that there exists a bounded set $B \subset X \times Y$ such that $\mathcal{A}_s \subset B$ for all $s \in \mathbb{N}$. (Observe that as $\Omega$ is a bounded domain, we can also consider, without loss of generality, that $\mathcal{A}^\infty \subset B$).

So, the subsequence $\{a_{s_k}\}_{k\in\mathbb{N}}$ is bounded in $X \times Y$ and therefore, since $X \times Y \hookrightarrow L^p(\Omega) \times L^q(\Omega)$ is compact, there exists another subsequence, that we still denote by $\{a_{s_k}\}_{k\in\mathbb{N}}$ and $\overline{a_\infty} \in L^p(\Omega) \times L^q(\Omega)$ such that $a_{s_k} \to \overline{a_\infty}$ in $L^p(\Omega) \times L^q(\Omega)$. Since $L^p(\Omega) \times L^q(\Omega) \hookrightarrow H \times H$, we also have $a_{s_k} \to \overline{a_\infty}$ in $H \times H$. By the uniqueness of the limit, we have $\overline{a_\infty} = a_\infty \in \mathcal{A}^\infty$ and the convergence is also in $L^p(\Omega) \times L^q(\Omega)$. This implies that

$$\text{dist}_{L^p(\Omega)\times L^q(\Omega)}(\mathcal{A}_s, \mathcal{A}^\infty) \to 0 \quad \text{as } s \to \infty. \qquad \square$$

## 5. FINAL REMARKS

**Remark 5.1.** Note that if $p_s(\cdot) \equiv p$ and $q_s(\cdot) \equiv q$ the family of attractors is also lower semicontinuous since each solution of (4.1) is also a solution of (1.1). For the general case of a variable exponent, lower semicontinuity is an open problem.

**Remark 5.2.** In the works [6, 32], the authors considered families of equations with $p_s \to 2$. This is also a point that could be considered for the coupled system in future works. Note that $p > 2$ was essential to guarantee that the limit problem is dissipative. Even for the simple case of one equation

$$\begin{cases} \dot{u}(t) + u(t) = \alpha u(t), & t > 0, \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{5.1}$$

with $\alpha > 1$ a real number, the solution is $u(t) = u_0 e^{(\alpha-1)t}$ and we have $|u(t)| \to \infty$ as $t \to \infty$. In this case a global $B$-attractor for the problem (5.1) does not exist.

**Remark 5.3.** By using Faedo–Galerkin method and working with a space $W$ of functions in $L^2(\Omega)$ having gradients in $L^{p(x)}(\Omega)$ one could try to prove existence of solution for the following system:

$$\begin{cases} \frac{\partial u_s}{\partial t} - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + |u_s|^{\sigma_s(x)-2}u_s \in F(u_s, v_s), & t > 0,\ x \in \Omega, \\ \frac{\partial v_s}{\partial t} - \operatorname{div}(D_s|\nabla v_s|^{q_s(x)-2}\nabla v_s) + |v_s|^{\mu_s(x)-2}v_s \in G(u_s, v_s), & t > 0,\ x \in \Omega, \\ \frac{\partial u_s}{\partial n}(t,x) = \frac{\partial v_s}{\partial n}(t,x) = 0, & t \geq 0,\ x \in \partial\Omega, \\ u_s(0,x) = u_{0s}(x),\ v_s(0,x) = v_{0s}(x), & x \in \Omega, \end{cases} \tag{5.2}$$

with $p_s^-, q_s^-, \sigma_s^-, \mu_s^- \geq 1$ and $p_s^+, q_s^+, \sigma_s^+, \mu_s^+ \leq L$, for all $s \in \mathbb{N}$. It is worth emphasizing that this space $W$ was used in [19] for a problem with one equation where we had uniqueness of solution while the coupled system of inclusions is a worse case, we have no guaranty of uniqueness of solution. This idea could be explored in further works.

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