Extended Models of Sedimentation in Coastal Zone

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Abstract

Construction of a generalized hyperbolic model of sediment dynamics predicting a sediment evolution on the bottom surface with a finite velocity is presented. The transport equation is extended with introducing a generalized operator of flux change and a generalized operator of gradient. Passing to the convenient model is a singular degeneration of extended model. In this case the results are obtained in the class of generalization solutions. Some expressive examples of constructions of hyperbolic models predicting a finite velocity of disturbance propagation are presented. This problem is developed starting from Maxwell (1861). His approach in the theory of electromagnetism and the kinetic theory of gases is commented. A brief review on propagation of heat and diffusive waves is presented. The similar problems in the theory of probability and diffusion waves are considered. In particular, it was shown on the microscopic level for metals that the conservation law can be violated.

Keywords: sediment dynamics, hyperbolic equation, finite velocity, disturbance propagation

1. Introduction

The problem of sediment reformation under the effect of water waves is referred directly to ocean lithodynamics [1, 2]. Thus interconnected processes of hydro- and lithomass transport occur under the wave action. Mathematical modeling of such processes in the shoreline zone on the basis of the classical approach is presented in [3].

We consider the generalized hyperbolic model of sediment evolution which predicts a finite velocity of sediment transport [3] unlike the convenient model of parabolic type predicting an infinite velocity of propagation of small perturbations.

It is known that a real process of sediment transport occurs with a finite speed [4]. As it is shown in the natural observations, a velocity of transport of the energy and substance mass in the coastal zone is a finite magnitude. It can be noted that some investigations have been conducted for aggraded channels in [5] and for the channel degradation which fits to the observed degradation in [6].

The question, which is of great interest, is a comparison of possibilities and a physical content of "parabolic" and "hyperbolic" models of sediment dynamics in a coastal zone. A generalized hyperbolic model was firstly proposed in [7].

Some examples are presented in this paper but the problems of thermoelasticity are not considered.

2. Mathematical model

Wave motion of the inviscid incompressible fluid of the variable depth in the rectangular Cartesian co-ordinate system (x, y, z) is considered. A plane $z = 0$ coincides with the

undisturbed free surface and an axis Oz is directed upwards. The ground surface can be deformed and it is described by the equation

$$z = -H_d(x, y, t).$$

So the depth $H(x, y, z, t)$ varies in time due to the sediment transport. Hereinafter a plane problem is considered corresponding to frontal incoming waves.

The mathematical problem is formulated as follows: to determine the fluid depth $H(x, z, t)$ and the energy flux vector $\vec{Q} = \vec{Q}(x, z, t)$ in the area $\Omega = \Sigma \times T$, where $\Sigma \subset R^3$, $T = \{t \mid \in [0, t_1]\}$, as solutions of equations (1) and (2), which satisfy corresponding boundary and initial conditions.

The conservation law is written in the form

$$\frac{\partial H}{\partial t} + \vec{\nabla} \cdot \vec{Q} = 0. \tag{1}$$

The transport equation for the closure of the system, unlike the previous researches, is postulated in the generalized form [7]

$$L\vec{Q} = -\vec{M}H, \tag{2}$$

where the scalar operator L characterizes a flux change in time:

$$L \equiv \gamma_0 + \gamma_1 \partial_t + \gamma_3 \partial_{ttt} + \cdots + \gamma_{2n+1} \underset{(2n+1)\text{ times}}{\partial \partial_{tt\ldots t}}, \tag{3}$$

with coefficients γ_0, γ_1, $\gamma_3, \ldots$, and a vector operator $\vec{M}$ is represented by the operator of gradient type:

$$\vec{M} \equiv \vec{k}_0 + k_1 \vec{\nabla} + k_3 \vec{\nabla} \nabla^2 + \cdots + k_{2n+1} \vec{\nabla} \nabla^{2n} \tag{4}$$

with coefficients $\vec{k}_0$, k_1, $k_3 \ldots$.

Keeping operators to a certain order generates a set of the generalized hyperbolic models [8].

In the case when all the terms in (3) are equal to zero except γ_1, i.e. $\gamma_0 = 0$, $\gamma_1 \neq 0$, $\gamma_3 - 0, \ldots$, $\gamma_{2n+1} = 0$, and all the terms in (4) are equal to zero except k_1, i.e. $\vec{k}_0 = 0$, $k_1 \neq 0$, $k_3 = 0, \ldots$, $k_{2n+1} = 0$ $(n = 1, 2, \ldots)$, we obtain the known parabolic model of sediment evolution. However if all the operators remain to a certain order s, in (3), (4), in that case we obtain a set of the generalized hyperbolic models [7].

For the case $n = 1$ from relations (3), (4) the elementary hyperbolic model can be derived in the form

$$\nabla^2 H - \frac{1}{c_1^2} \frac{\partial^2 H}{\partial t^2} - \frac{1}{k_1} \frac{\partial H}{\partial t} = 0, \tag{5}$$

where c_1 is the speed of propagation of disturbance, which is defined as $c_1 = \sqrt{k_1 / \eta}$, η is the relaxation parameter, k_1 is the kinematic viscosity.

In the classical case, when relaxation parameter η tends to zero, equation (5) is degenerates into (6) and for the depth $H(x, y, t)$ we obtain the equation of parabolic type

$$\nabla^2 H - \frac{1}{k_1}\frac{\partial H}{\partial t} = 0 , \tag{6}$$

which satisfies the conservation law and is used in all traditional studies [3]. In what follows consider the case of frontal approach waves (plane problem).

3. Singular degeneration

Statement of the initial boundary value (IBV) problem for the equation (5) has the form

$$\varepsilon H_{tt} + H_t = k_1 H_{xx} , \tag{7}$$

$$H\big|_{t=0} = u_0\left(x\right), \quad H_t\big|_{t=0} = H_1\left(x\right), \quad H\big|_{x=0} = H\big|_{x=1} = 0 , \tag{8}$$

where ε is the small parameter, $\varepsilon = \eta$.

We investigate the singular degeneracy problem (7), (8) with $\varepsilon \to 0$. Called a generalized solution of problem (7), (8) the function H из $W_{20}^{11}\left(Q_T\right)$ of satisfying to integral identity

$$-\varepsilon\int_0^T \left(H_t, \Phi_t\right)_\Omega dt + k_1\int_0^T \left(H_x, \Phi_x\right)_\Omega dt +$$

$$+\int_0^T \left(H_t, \Phi\right)_\Omega dt + \varepsilon\int_0^T \left(H_1\left(x\right), \Phi\left(0\right)\right)_\Omega = 0, \tag{9}$$

$$\forall\Phi \in W_2^{1,1}\left(Q_T\right), \quad \Phi\left(T\right) = 0, \quad \left(H, \Phi\right)_\Omega = \int_\Omega H\,\Phi\,dx,$$

where $\Omega = \left(0, 1\right), \; Q_T = \left[0, T\right]\times\Omega$.

For the generalized solution of the problem (7), (8) the theorem is true: if $H_1\left(x\right) \in L_2\left(\Omega\right), \; u_0\left(x\right) \in \overset{0}{W_2^1}\left(\Omega\right)$, then for the problem (7), (8) there exists a unique generalized solution. The proof of solvability is given in [9].

Passing in (7) to the limit at $\varepsilon \to 0$, we can obtain the identity that is a solution of the problem

$$H_t = k_1 H_{xx} ,$$

$$H\big|_{t=0} = u_0\left(x\right), \quad H\big|_{x=0} = H_{x=1} = 0. \tag{10}$$

Thus, we arrive to the following theorem: a Generalized solution of the problem (7), (8) passes at $\varepsilon \to 0$ to the generalized solution of the problem (6).

4. Some examples

A finite propagation velocity of electromagnetic waves. A development of the generalized models originates from the works of Maxwell. He was the first who realized the

FV-principle (the principle of the propagation velocity finiteness of perturbation) at the development of the electromagnetic field model (1861-1864), and then he has generalized this principle to the theory of gases [10].

We consider Maxwell's equations for the nonconducting homogeneous isotropic medium. Before the Maxwell work the following system of equations described a perturbation propagation with the infinite velocity

$$\vec{\nabla} \times \vec{H} = \boxed{0}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\vec{\nabla} \times \vec{B} = 0, \quad \vec{\nabla} \times \vec{D} = 0, \tag{11}$$

$$\vec{B} = \mu\vec{H}, \quad \vec{D} = \varepsilon\vec{E}.$$

After the Maxwell work the system of equations describes a perturbation propagation with the finite velocity, which equals to the speed of light

$$\vec{\nabla} \times \vec{H} = \boxed{\frac{\partial \vec{D}}{\partial t}}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\vec{\nabla} \times \vec{B} = 0, \quad \vec{\nabla} \times \vec{D} = 0, \tag{12}$$

$$\vec{B} = \mu\vec{H}, \quad \vec{D} = \varepsilon\vec{E}.$$

The system (12) can be reduced to the hyperbolic equation (wave equation)

$$\left(\nabla^2 - \frac{1}{c_e^2}\frac{\partial^2}{\partial t^2} \right) \left\{ \begin{matrix} \vec{H} \\ \vec{E} \end{matrix} \right\} = 0, \tag{13}$$

where $c_e = \sqrt{1/\mu\varepsilon}$ is the speed of light.

A comparison of (11) and (12) shows that (12) differs from (11) by a symmetry. From the mathematical point of view this procedure is an expansion of the nonhyperbolic differential operator to the hyperbolic one [11].

Heat propagation [10]. The transport equation following from the kinetic theory of gases is represented as

$$\left(1 + \xi\frac{\partial}{\partial t} \right)\vec{q} = -k\vec{\nabla}\theta, \tag{14}$$

and the conservation equation can be taken in the form

$$\gamma m \frac{\partial \theta}{\partial t} = -\vec{\nabla} \cdot \vec{q}. \tag{15}$$

The resolving equations for the flux $\vec{q}$ and heat θ, following from (14) and (15), take the form

$$\left[\left(\vec{\nabla} \times \vec{\nabla} \times \nabla^2 \right) - \xi\gamma\frac{m}{k}\frac{\partial^2}{\partial t^2} \right]q - \gamma\frac{m}{k}\frac{\partial}{\partial t}\vec{q} = 0, \tag{16}$$

$$\left(\nabla^2 - \xi\gamma\frac{m}{k}\frac{\partial^2}{\partial t^2}\right)\theta - \gamma\frac{m}{k}\theta = 0 \,, \tag{17}$$

where the propagation speed is equal to $c_m = \sqrt{\xi\gamma m/k}$, for example, $c_m = 150$ m/s for nitrogen.

Fock (1926) [12] considers probabilities $u(x,t)$ of light particles to be at time t in the point x and to move upwards, and probabilities $v(x,t)$ of light particles to be at the same place, but to move downwards. As a result functions u and v satisfy to the hyperbolic equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} + \frac{1}{D}\frac{\partial U}{\partial t} \,. \tag{18}$$

Presence of the term $\dfrac{1}{c^2}\dfrac{\partial^2 U}{\partial t^2}$ in (18) shows, that any perturbation and concentration

inhomogeneities are spread with a finite velocity c. But after these inhomogeneities have smoothed out (that happens quickly, if a velocity c is large), the further process differs a little from the process, which is described by the usual diffusion equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{D}\frac{\partial U}{\partial t} \,. \tag{19}$$

Later on for more extended discussion this result considered by Kac (1956) [13] and announced again in [14].

It is well known that the classical theory of the thermoconductivity is based on the Fick law. According to this law the heat flux q is directly proportional to the gradient of temperature T:

$$q = -k\frac{\partial T}{\partial x} \,, \tag{20}$$

where k is the heat conductivity coefficient. This law leads to the heat conduction equation of parabolic type

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \,. \tag{21}$$

It follows from (20) and (21) that the heat flux is directed from areas with a high temperature to areas with a low temperature. Thus, the infinite large propagation speed of temperature perturbations is postulated.

The hyperbolic heat conduction equation is investigated in [15] according to [16]. On the basis of the works [17] and [18] it is shown that a time of the mean free part of an electron with a velocity 10^6 m/s in metals is on 2-4 order less than the electron - photon relaxation time $t_p \sim 10^{-11}$ s [19]. This value coincides well with a magnitude t_p for aluminium [16] and the velocity

$$c_t = \sqrt{a/t_p} \tag{22}$$

has a value of some kilometers per second, i.e. in the order of magnitude it is equal to the sound velocity (22). It leads to the hyperbolic heat conduction equation

$$\frac{\partial T}{\partial t} + \frac{a}{c_t^2} \cdot \frac{\partial^2 T}{\partial t^2} = a\Delta T \ .$$
(23)

In the work of [20] the estimation of the geometrical area has been carried out, when a thermal process is described by the hyperbolic equation (23) at the given initial and boundary conditions. It was shown that for a copper $l = 0.0002$ mm, for a cork $l < 2$ mm. This effect can be important for the armor penetration or for explosion.

At the same time it should be noted that the analysis of propagation of thermal waves in metals at the microscopic level was conducted in the work [21]. It has been shown that generalization of the transport equation (Fourier's law) by taking into account the relaxation time is inadmissible, as it leads to a violation of the fundamental law of energy conservation.

For the case of the heat propagation within very short time intervals the classical equation should be substituted by the more general equation of hyperbolic type [22]

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \ .$$
(24)

The calculations on the basis of the equation (24) and comparisons with the data of experiments [22] have shown that in many cases, which are important for modern applications, the diffusion equation (21) leads to the rather underestimated values of temperature at the wave front. The qualitative effect consisting in the strong concentration of energy in a peak zone, which appears at the wave front, according to the hyperbolic equation, is also has been discovered. In the diffusion theory the energy is always "spread" on the whole area. Mc Nelly (1970) [23] obtained experimental results for dielectric crystals of sodium fluoride NaF. Distribution of thermal impulse (pulse height) as a function of arrival time (μs) shows clearly the presence of two front zones.

There are some considerations about diffusion waves which lack wave fronts and don't travel very far [24]. These considerations are based on revolutionary measurement thechnologies.

Useful contribution to the study of wave propagation with a finite speed is presented in the works [25-32]. Recently, a numerical simulation of hyperbolic heat equation has been presented in [33]. It is remarkable that the similar situation also takes place in the classical mechanics. As it is known, the classical Galilei-Newton mechanics is a special case of the Einstein mechanics when the speed of light is considered as an infinite big magnitude. Such an analogy gives the grounds to pay much attention to the problem of the group transformations of dependent and independent variables [34], which considers diffusion and hyperbolic heat conduction equations as invariants. Similarly it could be expected, that the equations of "diffusion" and "hyperbolic" heat conduction could assume various groups of transformations.

5. Conclusions

A new generalized hyperbolic model for the evolution of sediment is presented. It predicts a finite speed of formation of bottom sediments unlike the traditional model of parabolic type , predicting infinite speed of propagation of small disturbances . This is consistent with field observations from which it follows that the rate of transport of energy and mass of the substance in the coastal zone is a finite quantity [4]. On the basis of the corresponding initial-boundary value problem, a singular degeneration of the generalized hyperbolic model into traditional parabolic ones is carried out. The existence of generalized solutions is demonstrated. Some examples of generalization of parabolic models into hyperbolic ones starting from Maxwell (1867) are considered.

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