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SOME FORM OF OPEN SETS AND CONTINUITY IN IDEAL BITOPOLOGICAL SPACES

Abstract. We introduce the notion of (i, j)-mI-open sets as a unified form of (i, j)-*α-I-open* sets [4], (i, j)-*semi-I-open* sets [3], (i, j)-*pre-I-open* sets [1], (i, j)-*bI-open* sets [17] and (i, j)-*β-I-open* sets [2]. We show that properties of (i, j)-mI-open sets follow from the properties of minimal open sets in [14]. We introduce and investigate an (i, j)-mI-continuous function from an ideal bitopological space (X, τ_1, τ_2, I) to a bitopological space (Y, σ_1, σ_2).

Key words: minimal structure, m-continuous, ideal bitopological space, (i, j)mIO(X)-structure, $(i, j)mI$-continuous.

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1. Introduction

Kelly [6] introduced and investigated the notion of bitopological spaces. The notion of ideal topological spaces was introduced in [7] and [18]. In [5], the authors obtained a new topology $\tau^\star$ from a topology τ and an ideal $\mathcal{I}$.

The notion of minimal spaces is introduced in [14], as a generalization of topological spaces, and the notion of m-continuous functions on the space is introduced. Quite recently, (i, j)-*α-I-open* sets [4], (i, j)-*semi-I-open* sets [3], (i, j)-*pre-I-open* sets [1], (i, j)-*bI-open* sets [17] and (i, j)-*β-I-open* sets [2] in an ideal bitopological space have been introduced and investigated. And by using these open sets, some kind of continuous functions from an ideal bitopological space to a bitopological space are defined and investigated.

In this paper, we define the notion of (i, j)-mI-open sets as a unified form of (i, j)-*α-I-open* sets [4], (i, j)-*semi-I-open* sets [3], (i, j)-*pre-I-open* sets [1], (i, j)-*bI-open* sets [17] and (i, j)-*β-I-open* sets [2]. We show that properties of (i, j)-mI-open sets follow from the properties of minimal open sets in [14]. Moreover, we introduce and investigate an (i, j)-mI-continuous function from an ideal bitopological space (X, τ_1, τ_2, I) to a bitopological space (Y, σ_1, σ_2).

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall several properties of minimal structures and m-continuous functions.

Definition 1. *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a* minimal structure (*briefly m*-structure) *on X [13], [14] if $\emptyset \in m_X$ and $X \in m_X$.*

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be m_X-open (briefly m-open) and the complement of an m_X-open set is said to be m_X-closed (briefly m-closed).

Definition 2. *Let (X, m) be a minimal space. For a subset A of X, the* m_X-closure *of A and the* m_X-interior *of A are defined in [8] as follows:*

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Lemma 1 ([8])**.** *Let (X, m) be a minimal space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,

(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 3. *A minimal structure m_X on a nonempty set X is said to have* property $\mathcal{B}$ *[8] if the union of any family of subsets belonging to m_X belongs to m_X.*

Lemma 2 ([16])**.** *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,

(2) *A is m_X-closed if and only if* $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Lemma 3 ([13])**.** *Let (X, m_X) be a minimal space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x.*

Definition 4. *A function $f : (X, m_X) \to (Y, \sigma)$ is said to be m-continuous at $x \in X$ [14], where (Y, σ) is a topological space, if for each open set V containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be m-continuous if it has this property at each $x \in X$.*

By Theorem 3.1 of [14] and Lemma 2, we obtain the following theorem:

Theorem 1 ([14])**.** *For a function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is m-continuous;*
(2) *$f^{-1}(V)$ is m_X-open for every open set V of Y;*
(3) *$f^{-1}(F)$ is m_X-closed for every closed set F of Y;*
(4) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset B of Y;*
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset A of X;*
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ *for every subset B of Y.*

For a function $f : (X, m_X) \to (Y, \sigma)$, we define $D_m(f)$ as follows:

$$D_m(f) = \{x \in X : f \text{ is not } m\text{-continuous at } x\}.$$

Theorem 2 ([15])**.** *For a function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties hold:*

$$\begin{aligned} D_m(f) &= \bigcup_{G \in \sigma} \{f^{-1}(G) \setminus \mathrm{mInt}(f^{-1}(G))\} \\ &= \bigcup_{B \in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) \setminus \mathrm{mInt}(f^{-1}(B))\} \\ &= \bigcup_{B \in \mathcal{P}(Y)} \{\mathrm{mCl}(f^{-1}(B)) \setminus f^{-1}(\mathrm{Cl}(B))\} \\ &= \bigcup_{A \in \mathcal{P}(X)} \{\mathrm{mCl}(A) \setminus f^{-1}(\mathrm{Cl}(f(A)))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\mathrm{mCl}(f^{-1}(F)) \setminus f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

3. Ideal topological spaces

A subfamily I of the power set $\mathcal{P}(X)$ on a nonempty set X is called an *ideal* on X if it satisfies the following two conditions:

(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space and $\tau(x) = \{U \in \tau : x \in U\}$. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$ is called the local function of A with respect to τ and I [5]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. The set operator $Cl^\star$ called the $\star$-closure [5] is defined as follows: $\mathrm{Cl}^\star(A) = A \cup A^\star$

for every subset A of X. Let $\tau^\star = \{U \subset X : \mathrm{Cl}^\star(X \setminus U) = X \setminus U\}$. Then $\tau^\star$ is a topology which is finer than τ and is called the $\star$-topology.

Lemma 4 ([5]). *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset \mathrm{Cl}^\star(A)$,
(2) $\mathrm{Cl}^\star(X) = X$ *and* $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $A \subset B$ *implies* $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(4) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Let (X, τ_1, τ_2) be a bitopological space and I be an ideal of X. By (X, τ_1, τ_2, I), we denote an ideal bitopological space. Moreover, we set

$$jA^\star(I, \tau) = \{\, x \in X : U \cap A \notin I \text{ for every } U \in \tau_j(x) \,\}$$
$$\text{and } \mathrm{jCl}^\star(A) = jA^\star \cup A$$

for every subset A of X

Definition 5. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is said to be*
(1) *(i, j)-α-I-open [4] if* $A \subset \mathrm{iInt}(\mathrm{jCl}^\star(\mathrm{iInt}(A)))$, *where* $i \neq j, i, j = 1, 2$,
(2) *(i, j)-semi-I-open [3] if* $A \subset \mathrm{jCl}^\star(\mathrm{iInt}(A))$, *where* $i \neq j, i, j = 1, 2$,
(3) *(i, j)-pre-I-open [1] if* $A \subset \mathrm{iInt}(\mathrm{jCl}^\star(A))$, *where* $i \neq j, i, j = 1, 2$,
(4) *(i, j)-bI-open [17] if* $A \subset \mathrm{iInt}(\mathrm{jCl}^\star(A)) \cup \mathrm{jCl}^\star(\mathrm{iInt}(A))$, *where* $i \neq j, i, j = 1, 2$,
(5) *(i, j)-β-I-open [2] if* $A \subset \mathrm{jCl}(\mathrm{iInt}(\mathrm{jCl}^\star(A)))$, *where* $i \neq j, i, j = 1, 2$.

The family of all (i, j)-α-I-open (resp. (i, j)-semi-I-open, (i, j)-pre-I-open, (i, j)-bI-open, (i, j)-β-I-open) sets in an ideal bitopological space (X, τ_1, τ_2, I) is denoted by $(i, j)\alpha\mathrm{IO}(X)$ (resp. $(i, j)\mathrm{SIO}(X)$, $(i, j)\mathrm{PIO}(X)$, $(i, j)\mathrm{BIO}(X)$, $(i, j)\beta\mathrm{IO}(X)$).

Remark 1. By $(i, j)\mathrm{mIO}(X)$, we denote each one of the families $(i, j)\alpha\mathrm{IO}$ (X), $(i, j)\mathrm{SIO}(X)$, $(i, j)\mathrm{PIO}(X)$, $(i, j)\mathrm{BIO}(X)$, $(i, j)\beta\mathrm{IO}(X)$.

Lemma 5. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then $(i, j)mI$ $O(X)$ is a minimal structure on X and has property $\mathcal{B}$.*

Proof. By Lemmas 1(3) and 4(2), $(i, j)\mathrm{mIO}(X)$ is a minimal structure on X. It follows from Lemmas 1(4) and 4(3) that $(i, j)\mathrm{mIO}(X)$ has property $\mathcal{B}$. ■

Remark 2. It is shown in Theorem 3.17 of [4] (resp. Theorem 3.11 of [3], Theorem 2.15 of [1], Theorem 3.2 of [17], Theorem 1 of [2]) that $(i, j)\alpha\mathrm{IO}(X)$ (resp. $(i, j)\mathrm{SIO}(X)$, $(i, j)\mathrm{PIO}(X)$, $(i, j)\mathrm{BIO}(X)$, $(i, j)\beta\mathrm{IO}(X)$) has property $\mathcal{B}$.

Definition 6. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. For a subset A of X, $(i,j)\mathrm{mCl}_I(A)$ and $(i,j)\mathrm{mInt}_I(A)$ are defined as follows:*
(1) $(i,j)\mathrm{mCl}_I(A) = \cap\{F : A \subset F, X \setminus F \in (i,j)\mathrm{mIO}(X)\}$,
(2) $(i,j)\mathrm{mInt}_I(A) = \cup\{U : U \subset A, U \in (i,j)\mathrm{mIO}(X)\}$.

Lemma 6. *Let (X, τ_1, τ_2, I) an ideal bitopological space and A, B subsets of X. Then the following properties hold:*
(1) $(i,j)\mathrm{mInt_I}(A) \subset A$,
(2) $A \in (i,j)\mathrm{mIO}(X)$ *if and only if* $(i,j)\mathrm{mInt_I}(A) = A$,
(3) $(i,j)\mathrm{mInt_I}(\emptyset) = \emptyset$ *and* $(i,j)\mathrm{mInt_I}(X) = X$,
(4) *If* $A \subset B$, *then* $(i,j)\mathrm{mInt_I}(A) \subset (i,j)\mathrm{mInt_I}(B)$,
(5) $(i,j)\mathrm{mInt_I}((i,j)\mathrm{mInt_I}(A)) = (i,j)\mathrm{mInt_I}(A)$,
(6) $x \in (i,j)\mathrm{mInt_I}(A)$ *if and only if there exists an (i,j)-mI-open set U such that* $x \in U \subset A$.

Proof. Since $(i,j)\mathrm{mIO}(X)$ is a minimal structure with property $\mathcal{B}$, this follows easily from Lemmas 1 and 2. ■

Remark 3. By Lemma 6, we obtain Theorem 3.25 of [4], Theorems 3.20 of [3], Theorem 3.24 of [1], Theorem 3.7 of [17] and Theorem 7 of [2].

Lemma 7. *Let (X, τ_1, τ_2, I) be an ideal minimal space and A, B subsets of X. Then the following properties hold:*
(1) $A \subset (i,j)\mathrm{mCl_I}(A)$,
(2) $(X \setminus A) \in (i,j)\mathrm{mIO}(X)$ *if and only if* $(i,j)\mathrm{mCl_I}(A) = A$,
(3) $(i,j)\mathrm{mCl_I}(\emptyset) = \emptyset$, $(i,j)\mathrm{mCl_I}(X) = X$,
(4) *If* $A \subset B$, *then* $(i,j)\mathrm{mCl_I}(A) \subset (i,j)\mathrm{mCl_I}(B)$,
(5) $(i,j)\mathrm{mCl_I}((i,j)\mathrm{mCl_I}(A)) = (i,j)\mathrm{mCl_I}(A)$,
(6) $x \in (i,j)\mathrm{mCl_I}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in (i,j)\mathrm{mIO}(X)$ *containing* x.

Proof. This follows easily from Lemmas 1, 2 and 3. ■

Remark 4. By Lemma 7, we obtain Theorems 3.31 and 3.32 of [4], Theorems 3.22 and 3.23 of [3], Theorems 2.27 and 2.26 of [1], Theorem 3.7 of [17] and Theorems 8 and 9 of [2].

Lemma 8. *Let (X, τ_1, τ_2, I) be an ideal bitopological space and A be a subset of X.* $(i,j)\mathrm{mCl_I}(X \setminus A) = X \setminus (i,j)\mathrm{mInt_I}(A)$ *and* $(i,j)\mathrm{mInt_I}(X \setminus A) = X \setminus (i,j)\mathrm{mCl_I}(A)$.

Remark 5. By Lemma 8, we obtain Theorem 3.33 of [4], Theorem 3.24 of [3], Theorem 2.28 of [1], Theorem 3.9 of [17] and Theorem 10 of [2].

4. (i,j)-mI-continuous functions

Definition 7. *A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-α-$\mathcal{I}$-continuous [4] (resp. (i,j)-semi-$\mathcal{I}$-continuous [3], (i,j)-pre-$\mathcal{I}$-continuous [1], (i,j)-bI-continuous [17], (i, j)-β-$\mathcal{I}$-continuous [2]) at $x \in X$ if for each σ_i-open set V of Y containing $f(x)$, there exists an (i,j)-α-I-open (resp. (i,j)-semi-I-open, (i,j)-pre-I-open, (i,j)-bI-open, (i,j)-β-I-open) set U in (X, τ_1, τ_2, I) containing x such that $f(U) \subset V$. And f is said to be (i, j)-α-$\mathcal{I}$-continuous [4] (resp. (i,j)-semi-$\mathcal{I}$-continuous [3], (i, j)-pre-$\mathcal{I}$-continuous [1], (i,j)-bI-continuous [17], (i,j)-β-$\mathcal{I}$-continuous [2]) (on X) if it has this property at each point $x \in X$.*

Definition 8. *A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-mI-continuous at $x \in X$ (resp. on X) if $f : (X, (i,j)mIO(X)) \to (Y, \sigma_i)$ is m-continuous at $x \in X$ (resp. on X). And f is said to be* pairwise mI-continuous *if f is (i,j)-mI-continuous and (j,i)-mI-continuous, where $i,j = 1,2$ and $i \neq j$.*

By Theorem 1, we obtain the following characterizations of (i,j)-mI-continuous functions.

Theorem 3. *For a function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*
(1) *f is (i,j)-mI-continuous;*
(2) *$f^{-1}(V)$ is (i,j)-mI-open for every σ_i-open set V of Y;*
(3) *$f^{-1}(F)$ is (i,j)-mI-closed for every σ_i-closed set F of Y;*
(4) $(i,j)\mathrm{mCl_I}(f^{-1}(B)) \subset f^{-1}(\mathrm{iCl}(B))$ *for every subset B of Y;*
(5) $f((i,j)\mathrm{mCl_I}(A)) \subset \mathrm{iCl}(f(A))$ *for every subset A of X;*
(6) $f^{-1}(\mathrm{iInt}(B)) \subset (i,j)\mathrm{mInt_I}(f^{-1}(B))$ *for every subset B of Y.*

Remark 6. If $(i,j)\mathrm{mIO}(X)$ is $(i,j)\alpha\mathrm{IO}(X)$ (resp. $(i,j)\mathrm{SIO}(X)$, (i,j) $\mathrm{PIO}(X)$, $(i,j)\mathrm{BIO}(X)$, $(i,j)\beta\mathrm{IO}(X)$), then by Theorem 3 we obtain Theorem 4.4 of [4] (resp. Theorem 4.3 of [3], Theorem 3.5 of [1], Theorem 4.1 of [17], Theorem 11 of [2]).

For a function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$, we define $D_{(i,j)mI}(f)$ as follows:

$$D_{(i,j)mI}(f) = \{x \in X : f \text{ is not } (i,j)\text{-}mI\text{-continuous at } x\}.$$

Theorem 4. *For a function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \tau_2)$, the following properties hold:*

$$D_{(i,j)mI}(f) = \bigcup_{G \in \sigma_i} \{f^{-1}(G) \setminus (i,j)\mathrm{mInt_I}(f^{-1}(G))\}$$

$$= \bigcup_{B\in\mathcal{P}(Y)} \{f^{-1}(\mathrm{iInt}(B)) \setminus (i,j)\mathrm{mInt}_{\mathrm{I}}(f^{-1}(B))\}$$

$$= \bigcup_{B\in\mathcal{P}(Y)} \{(i,j)\mathrm{mCl}_{\mathrm{I}}(f^{-1}(B)) \setminus f^{-1}(\mathrm{iCl}(B))\}$$

$$= \bigcup_{A\in\mathcal{P}(X)} \{(i,j)\mathrm{mCl}_{\mathrm{I}}(A) \setminus f^{-1}(\mathrm{iCl}(f(A)))\}$$

$$= \bigcup_{F\in\mathcal{F}} \{(i,j)\mathrm{mCl}_{\mathrm{I}}(f^{-1}(F)) \setminus f^{-1}(F)\},$$

where $\mathcal{F}$ is the family of σ_i-closed sets of (Y,σ_1,σ_2).

5. Properties of (i,j)-mI-continuous functions

Definition 9. *A bitopological space (X,τ_1,τ_2) is said to be* pairwise-T_2 *[6] if for each pair of distinct points $x,y \in X$, there exist a τ_i-open set U and a τ_j-open set V containing x and y, respectively, such that $U \cap V = \emptyset$.*

Definition 10. *Let (X,τ_1,τ_2,I) be an ideal bitopological space and $(i,j)m$ $IO(X)$ be the family of (i,j)-mI-open sets. Then the space (X,τ_1,τ_2,I) is said to be* pairwise mI-T_2 *if for each pair of distinct points $x,y \in X$, there exist $U \in (i,j)mIO(X)$ and $V \in (j,i)mIO(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$, where $i,j = 1,2$ and $i \neq j$.*

Theorem 5. *If $f : (X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2)$ is a pairwise mI-continuous injection and Y is pairwise-T_2, then (X,τ_1,τ_2,I) is pairwise mI-T_2.*

Proof. Let x and y be any pair of distinct points of X. Then $f(x) \neq f(y)$. Since Y is pairwise-T_2, there exist a σ_i-open set U and a σ_j-open set V containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is pairwise mI-continuous, there exist $G \in (i,j)mIO(X)$ and $H \in (j,i)mIO(X)$ such that $x \in G$, $y \in H$, $f(G) \subset U$ and $f(H) \subset V$. This implies that $G \cap H = \emptyset$. Hence (X,τ_1,τ_2) is pairwise mI-T_2. ■

Definition 11. *A function $f : (X,m_X) \to (Y,\sigma)$ is said to have an* m-closed graph *[14] if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $(U\times V)\cap G(f) = \emptyset$.*

Definition 12. *A function $f : (X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2)$ is said to have an (i,j)-mI-*closed graph *if a function $f : (X,(i,j)mIO(X)) \to (Y,\sigma_i)$ has an m-closed graph.*

Theorem 6. *If $f : (X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2)$ is (i,j)-mI-continuous and (Y,σ_i) is Hausdorff, then f has an (i,j)-mI-closed graph.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since (Y, σ_i) is Hausdorff, there exist disjoint σ_i-open sets V and W in Y containing y and $f(x)$, respectively. Since f is (i, j)-m-continuous, there exists $U \in (i, j)mIO(X)$ containing x such that $f(U) \subset W$. This implies that $f(U) \cap V = \emptyset$ and hence the function $f : (X, (i, j)mI(X)) \to (Y, \sigma_i)$ has an m-closed graph. Hence f has an (i, j)-mI-closed graph. ■

Definition 13. *A bitopological space* (X, τ_1, τ_2) *is said to be* pairwise connected *[12] if* X *cannot be expressed as the union of two nonempty disjoint sets* U *and* V *such that* U *is* τ_i*-open and* V *is* τ_j*-open for* $i \neq j$ *and* $i, j = 1, 2$.

Definition 14. *An ideal bitopological space* (X, τ_1, τ_2, I) *is said to be* pairwise mI-connected *if* X *cannot be expressed as the union of two disjoint nonempty sets* U *and* V *such that* $U \in (i, j)mIO(X)$ *and* $V \in (j, i)mIO(X)$ *for* $i \neq j$ *and* $i, j = 1, 2$.

Theorem 7. *If an ideal bitopological space* (X, τ_1, τ_2, I) *is pairwise* mI*-connected and* $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ *is a pairwise* mI*-continuous surjection, then* (Y, σ_1, σ_2) *is pairwise connected.*

Proof. Let (Y, σ_1, σ_2) be not pairwise connected. Then there exist a nonempty σ_i-open set V_1 and a nonempty σ_j-open set V_2 such that $Y = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Hence $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty since f is surjective, $f^{-1}(V_1) \cup f^{-1}(V_2) = X$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Since f is pairwise mI-continuous, $f^{-1}(V_1)$ is (i, j)-mI-open and $f^{-1}(V_2)$ is (j, i)-mI-open. This contradicts that X is pairwise mI-connected. Therefore, (Y, σ_1, σ_2) is pairwise connected. ■

Definition 15. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space. A subset* A *of* X *is said to be* (i,j)-mI-compact relative to X *if for every cover* $\{U_\alpha : \alpha \in \Delta\}$ *of* K *by* (i, j)*-*mI*-open sets of* X*, there exists a finite subset* Δ_0 *of* Δ *such that* $K \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$*. The space* (X, τ_1, τ_2, I) *is said to be* (i,j)-mI-compact *if* X *is* (i, j)*-*mI*-compact relative to* X.

Definition 16. *Let* $(Y, \sigma_1, \sigma_2, J)$ *be an ideal bitopological space. A subset* K *of* Y *is said to be* $\sigma_i J$-compact relative to Y *if for every cover* $\{U_\alpha : \alpha \in \Delta\}$ *of* K *by* σ_i*-open sets of* Y*, there exists a finite subset* Δ_0 *of* Δ *such that* $K \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in J$*. The space* $(Y, \sigma_1, \sigma_2, J)$ *is said to be* $\sigma_i J$-compact *if* Y *is* $\sigma_i J$*-compact relative to* Y.

It is known in [9] that if $f : X \to Y$ is a function and I is an ideal on X then $f(I)$ is an ideal on Y.

Theorem 8. *If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is an (i, j)-mI-continuous function and K is (i, j)-mI-compact relative to X, then $f(K)$ is $\sigma_i f(I)$-compact relative to Y.*

Proof. Let K be (i, j)-mI-compact relative to X and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by σ_i-open sets of Y. For each $x \in K$, there exists an $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is (i, j)-mI-continuous, there exists an (i, j)-mI-open set $U(\alpha(x))$ containing x such that $f(U(\alpha(x))) \subset V_{\alpha(x)}$. Since $\{U(\alpha(x)) : x \in K\}$ is a cover of K by (i, j)-mI-open sets of X, there exists a finite subset K_0 of K such that $K \setminus \cup\{U(\alpha(x)) : x \in K_0\} = I_0$, where $I_0 \in I$; hence

$$f(K) \subset \cup\{f(U(\alpha(x))) : x \in K_0\} \cup f(I_0) \subset \cup\{V_{\alpha(x)} : x \in K_0\} \cup f(I_0).$$

Therefore, we obtain $f(K) \setminus \cup\{V_{\alpha(x)} : x \in K_0\} \in f(I_0)$. This shows that $f(K)$ is $\sigma_i f(I)$-compact relative to Y. ■

Corollary 1. *If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is an (i, j)-mI-continuous surjective function and (X, τ_1, τ_2, I) is (i, j)-mI-compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is $\sigma_i f(I)$-compact.*

Remark 7. If $(i, j)\mathrm{mIO}(X) = (i, j)\mathrm{BIO}(X)$, then by Corollary 1 we obtain Theorem 4.6 of [17].

6. Other forms of open sets in an ideal bitopological space

We shall obtain similar open sets with those in Definition 5 and other. For example, we have the following:

Definition 17. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is said to be*

(1) *weakly (i, j)-semi-I-open if $A \subset \mathrm{jCl}^\star(\mathrm{iInt}(\mathrm{jCl}(A)))$,*

(2) *weakly (i, j)-bI-open if $A \subset \mathrm{jCl}(\mathrm{iInt}(\mathrm{jCl}^\star(A))) \cup \mathrm{jCl}^\star(\mathrm{iInt}(\mathrm{jCl}(A)))$,*

(3) *strongly (i, j)-β-I-open if $A \subset \mathrm{jCl}^\star(\mathrm{iInt}(\mathrm{jCl}^\star(A)))$.*

The family of all weakly (i, j)-semi-I-open (resp. weakly (i, j)-bI-open, strongly (i, j)-β-I-open) sets in an ideal bitopological space space (X, τ_1, τ_2, I) is denoted by $w(i, j)\mathrm{SIO}(X)$ (resp. $w(i, j)\mathrm{BIO}(X)$, $s(i, j)\beta\mathrm{IO}(X)$). For these families, we obtain the similar properties with those in Sections 3, 4 and 5.

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