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Distributed optimal control problems driven by space-time fractional parabolic equations*

by

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Abstract: We study distributed optimal control problems, governed by space-time fractional parabolic equations (STFPEs) involving time-fractional Caputo derivatives and spatial fractional derivatives of Sturm-Liouville type. We first prove existence and uniqueness of solutions of STFPEs on an open bounded interval and study their regularity. Then we show existence and uniqueness of solutions to a quadratic distributed optimal control problem. We derive an adjoint problem using the right-Caputo derivative in time and provide optimality conditions for the control problem. Moreover, we propose a finite difference scheme to find the approximate solution of the considered optimal control problem. In the proposed scheme, the well-known L1 method has been used to approximate the time-fractional Caputo derivative, while the spatial derivative is approximated using the Grünwald-Letnikov formula. Finally, we demonstrate the accuracy and the performance of the proposed difference scheme via examples.

Keywords: space-time fractional parabolic equations, Caputo fractional derivative, distributed control, L1 method, Grünwald-Letnikov formula

1. Introduction

This article presents the analysis and discretization for a quadratic distributed optimal control problem, governed by space-time fractional parabolic equations.

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We consider the following optimal control problem

$$\min J(y,u)=\frac{1}{2}\int_0^T\int_a^b |y(x,t)-z_d(x,t)|^2dxdt+\frac{\nu}{2}\int_0^T\int_a^b |u(x,t)|^2dxdt, \quad (1.1)$$

where the minimum is sought with respect to the state variable y and the control u satisfying the following initial-boundary value space-time fractional parabolic equations

$$\begin{aligned} {}_C D^{\alpha}_{a,t} y(x,t) + {}_C D^{\beta}_{x,b}(\gamma(x)\, {}_{RL}D^{\beta}_{a,x} y(x,t)) + q(x)y(x,t) &= f(x,t)+u(x,t), \\ & \qquad (x,t)\in(a,b)\times(0,T), \\ y(x,0) &= y^0(x), \qquad x\in(a,b), \\ D^{-(1-\beta)}_{a,x} y(a,t) &= 0, \qquad t\in(0,T), \\ \gamma(b)\, {}_{RL}D^{\beta}_{a,x} y(b,t) &= 0, \qquad t\in(0,T). \end{aligned} \quad (1.2)$$

In the above problem, $z_d\in L^2((0,T);L^2(a,b))$ denotes a given target or an observed value of the state y, $\nu>0$ is a real number, u is the distributed control variable belonging to $L^2((0,T);L^2(a,b))$ and f is a source term. The real number $T>0$ denotes the final time, $a,b\in\mathbb{R}$ with $a<b$, ${}_C D^{\alpha}_{a,t}$ denotes the left Caputo fractional derivative of order $\alpha\in(0,1)$ with respect to the time variable t, ${}_C D^{\beta}_{x,b}$ and ${}_{RL}D^{\beta}_{a,x}$ represent the right Caputo, and the left Riemann-Liouville fractional derivatives of same order $\beta\in(0,1)$, respectively, with respect to the spatial variable x and $D^{-\beta}_{a,x}$ is the Riemann-Liouville fractional integral of order $\beta\in(0,1)$. We refer to Section 2 for the definitions of the fractional derivatives and fractional integral. The real valued functions $\gamma\in C[a,b]$ and $q\in L^\infty(a,b)$ satisfy suitable conditions (see Theorem 3.1).

Fractional calculus has been of great interest to researchers during the past decades. It has been widely used to describe various processes in different fields of science and engineering (see Hilfer, 2000; Magin and Ovadia, 2008; Patel and Mehra, 2020; Mehandiratta, Mehra and Leugering, 2020; Singh and Mehra, 2021; Singh, Mehra and Gulyani, 2021). Fractional derivatives have been proven to be more accurate in describing some physical phenomena than the classical (integer order) derivatives, see, in particular, Almeida, Bastos and Monteiro (2016), where experimental data have been used in a least square setting to show that in a variety of applications fractional derivatives are better suited than classical derivatives. Fractional derivatives of Sturm-Liouville type give rise to fractional Sturm-Liouville problems (FSLPs), derived by replacing the integer-order derivatives with fractional-order derivatives in the classical Sturm-Liouville problems. Many researchers in recent years have focused their attention on FSLPs. For instance, Klimek and Agrawal (2013) studied regular and singular fractional Sturm-Liouville eigenvalue problems involving two types of fractional Sturm-Liouville operators. The first consists of the composition of left-sided Riemann-Liouville fractional derivative and right-sided Caputo fractional derivative, and the second consists of the composition of right-sided Riemann-Liouville fractional derivative and left-sided Caputo fractional derivative. The authors have proven that the eigenvalues for both the regular and singular FSLPs are real, and the eigenfunctions, corresponding to different eigenvalues are orthogonal. Zayernouri and Kaniadakis (2013) considered fractional Sturm-Liouville eigenvalue problems involving the fractional Sturm-Liouville operator, consisting of the composition of right-sided Riemann-Liouville fractional derivative and left-sided Caputo fractional derivative. It has been proven that the eigenfunctions are non-polynomial functions and are orthogonal with respect to the weight function, associated with the problem. The authors in Klimek, Odzijewicz and Malinowska (2013) applied the fractional variational analysis to show the existence of discrete spectrum for a fractional Sturm-Liouville eigenvalue problem, involving left-sided and right-sided Caputo fractional derivatives of the same order. The exact and numerical solutions of the fractional Sturm-Liouville eigenvalue problem, involving right-sided Caputo fractional derivative and left-sided Riemann-Liouville fractional derivative have been studied in Klimek, Ciesielski and Błaszczyk (2018). Analytic and numerical results were obtained by transforming the differential form of FSLPs into an integral form. The abovementioned eigenvalue problems on an unbounded domain were considered in Arab, Dehghan and Eslahchi (2015). We also refer to the work of Idczak and Walczak (2013), where the authors introduced and characterized fractional Sobolev spaces via Riemann-Liouville fractional derivatives and exploited this to investigate the existence and uniqueness of the solution for a fractional boundary value problem associated with a Sturm-Liouville type equation, involving left-sided and right-sided Riemann-Liouville fractional derivatives of the same order.

On the other hand, fractional diffusion equations are of interest because of the close link between the phenomenon of anomalous diffusion (Luchko, 2012; Metzler and Klafter, 2000) and fractional derivatives. We refer to Sakamoto and Yamamoto (2011), Kubica and Yamamoto (2017), and McLean et al. (2019) and references therein for the well-posedness of some time-fractional parabolic equations. Space-time fractional diffusion equations have also been investigated by some authors. In Li and Xu (2010), the authors proved the existence and uniqueness of weak solutions for space-time fractional diffusion equations. They also developed an efficient spectral method for the approximate solutions of such equations, based on the weak formulation. Chen, Meerschaert and Nane (2012) obtained strong solutions of space-time fractional diffusion equations using the method of separation of variables. The theory of a probabilistic representation of these solutions, useful for particle tracking codes, was also developed. Klimek, Malinowska and Odzijewicz (2016) considered space-time fractional diffusion equations with fractional spatial derivatives of Sturm-Liouville type, consisting of left-sided and right-sided Caputo derivatives. In their work, these authors proved the existence of strong solutions for such equations by using the method of separation of variables and applying certain theorems that ensure the existence of solutions to the fractional Sturm-Liouville problem. We refer to Alvarez et al. (2019) and Gal and Warma (2020) and references therein for more results on space-time fractional parabolic equations.

Various results are also available in the area of fractional optimal control problems (FOCPs) on bounded domains, initiated by O. P. Agrawal in Agrawal (2002), where he formulated Euler-Lagrange equations for fractional variational problems. Later, various authors studied theoretical and numerical aspects of optimal control problems for fractional ordinary differential equations. We refer to Guo (2013), Sayevand and Rostami (2018), Kumar and Mehra (2021a,b) for some relevant articles regarding the underlying problems. For the control problems, governed by time-fractional partial differential equations on bounded domains, the work of Mophou and associates is worth mentioning. The authors considered distributed and boundary optimal control problems for time-fractional diffusion equations in Mophou (2011) and Dornville, Mophou and Valmorin (2011), respectively, where the fractional derivative was considered in Riemann-Liouville sense. The well-posedness of solutions to state equations has been studied, and an optimality system for the corresponding control problem was derived by the Lagrange method. For the sake of completeness, we also refer to the work by Bahaa (2017, 2018) for the abovementioned problems.

Recently, Leugering et al. (2021) studied the optimal control problems associated with space fractional parabolic problems of Sturm–Liouville type in an interval and on a star graph. The authors proved the existence and uniqueness of solutions for the governing equations and the fractional optimal control problems. Moreover, the characterization of the optimal control via Euler-Lagrange first-order optimality conditions was provided. However, the authors have not provided any numerical evidence for the approximate solution of such problems. Motivated by their work and the genuine interest in fractional Sturm-Liouville problems, outlined above, in this paper, we consider optimal control problems governed by space-time fractional parabolic equations of Sturm-Liouville type in an interval. The analysis and discretization of such problems on metric graphs, in the spirit of Mehandiratta, Mehra and Leugering (2019, 2021), Mehandiratta and Mehra (2020) and Kumar and Leugering (2021) will be considered in a forthcoming paper. To the authors' best knowledge, even the well-posedness and the numerical study of optimal control problems for STFPEs of Sturm-Liouville type on intervals has not been investigated yet. Hence, in this paper, we first prove the well-posedness for the weak solutions of the governing equations (STFPEs). For the corresponding control problem, we demonstrate that there exists a unique optimal solution (see Theorem 4.1) and derive an optimality system in terms of the right-sided time-fractional Caputo derivative. Finally, we propose a difference scheme based on the finite difference method for the approximate solution of the considered optimal control problem.

2. Notation and preliminaries

We now provide some notations, recall basic facts of fractional calculus and define appropriate function spaces.

2.1. Notation

If X and Y are normed spaces, then $X \hookrightarrow Y$ means that X is continuously embedded in Y and $X \overset{c}{\hookrightarrow} Y$ means that X is compactly embedded in Y. The dual of the normed space X is denoted by X' and $\|\cdot\|_X$ denotes the norm of X. $AC[0,T]$ denotes the space of all absolutely continuous functions, defined on $[0,T]$.

For $1 \leq p \leq \infty$, $L^p((0,T);X)$ denotes the space of X-valued functions, i.e.,

$$f : (0,T) \to X \quad \text{such that} \quad f(t) \in X,\ t \in (0,T),$$

whose norm with respect to X belongs to $L^p(0,T)$. This space forms a Banach space, endowed with the norm

$$\|f\|_{L^p((0,T);X)} = \left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}},\ 1 \leq p < \infty,$$
$$\|f\|_{L^\infty((0,T);X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_X.$$

For $f, g \in L^1(0,T)$, we denote by $f*g$ the convolution of f and g. Moreover, in what follows, we also use the result that if $f \in L^p(0,T)$, $g \in L^q(0,T)$, then $f*g \in L^r(0,T)$ and

$$\|f*g\|_{L^r(0,T)} \leq \|f\|_{L^p(0,T)} \|g\|_{L^q(0,T)}, \tag{2.1}$$

where $1 \leq p,\ q,\ r < \infty$ such that $(1/p) + (1/q) = (1/r) + 1$.

We denote by $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0,\ t > 0$, the kernel function. It is clear that $g_\alpha \in L^{1,loc}(\mathbb{R}_+)$, is nonincreasing and satisfies the property $g_\alpha * g_{1-\alpha} = 1$ in $(0,\infty)$.

2.2. Fractional integrals and derivatives

The left and right fractional integrals of order $\alpha > 0$ for a function $f \in L^1(0,T)$ are, respectively, defined by

$$D_{0,t}^{-\alpha} f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-\tau) f(\tau) d\tau, \quad D_{t,T}^{-\alpha} f(t) = \int_t^T g_\alpha(\tau - t) f(\tau) d\tau. \tag{2.2}$$

In view of (2.1), we have

$$\|D_{0,t}^{-\alpha} f\|_{L^1(0,T)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^1(0,T)}.$$

However, in general, we have the following result from Kilbas, Srivastava and Trujillo (2006), Lemma 2.1.

Proposition 2.1 *If $\alpha > 0$ and $1 \leq p \leq \infty$, then $D_{0,t}^{-\alpha}$ and $D_{t,T}^{-\alpha}$ are continuous from $L^p(0,T)$ into itself and*

$$\|D_{0,t}^{-\alpha} f\|_{L^p(0,T)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p(0,T)}, \quad \|D_{t,T}^{-\alpha} f\|_{L^p(0,T)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p(0,T)},$$

for all $f \in L^p(0,T)$.

The left and right Caputo fractional derivatives of order $\alpha \in (0,1)$ for a function f on $[0,T]$ are, respectively, defined by

$$ {}_C D_{0,t}^\alpha f(t) = \int_0^t g_{1-\alpha}(t-\tau) f'(\tau) d\tau, \quad {}_C D_{t,T}^\alpha f(t) = -\int_t^T g_{1-\alpha}(\tau - t) f'(\tau) d\tau. \tag{2.3}$$

We also define the left and right Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ for a function f on $[0,T]$, respectively, by

$$\begin{aligned} {}_{RL}D_{0,t}^\alpha f(t) &= \frac{d}{dt}\left(\int_0^t g_{1-\alpha}(t-\tau) f(\tau) d\tau\right), \\ {}_{RL}D_{t,T}^\alpha f(t) &= -\frac{d}{dt}\left(\int_0^t g_{1-\alpha}(\tau-t) f(\tau) d\tau\right). \end{aligned} \tag{2.4}$$

The next result (Samko, Kilbas and Marichev 1993, Lemma 2.2) provides a sufficient condition for the existence of Riemann-Liouville fractional derivatives.

Lemma 2.1 *Let $\alpha \in (0,1)$ and $f \in AC[0,T]$, then the fractional derivatives ${}_{RL}D_{0,t}^\alpha$ and ${}_{RL}D_{t,T}^\alpha$ exist almost everywhere on $[0,T]$ and can be, respectively, expressed in the forms*

$$\begin{aligned} {}_{RL}D_{0,t}^\alpha f(t) &= \frac{y(0)}{\Gamma(1-\alpha)} t^{-\alpha} + \int_0^t g_{1-\alpha}(t-\tau) f'(\tau) d\tau, \\ {}_{RL}D_{t,T}^\alpha f(t) &= \frac{y(T)}{\Gamma(1-\alpha)} (T-t)^{-\alpha} - \int_t^T g_{1-\alpha}(\tau-t) f'(\tau) d\tau. \end{aligned} \tag{2.5}$$

Therefore, in view of (2.5), one can deduce that the Riemann-Liouville and the Caputo fractional derivatives of order $\alpha \in (0,1)$ are related as follows:

$$\begin{aligned} {}_{RL}D_{0,t}^\alpha f(t) &= {}_C D_{0,t}^\alpha f(t) + \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \\ {}_{RL}D_{t,T}^\alpha f(t) &= {}_C D_{t,T}^\alpha f(t) + \frac{f(T)}{\Gamma(1-\alpha)} (T-t)^{-\alpha}. \end{aligned} \tag{2.6}$$

Remark 2.1 *In view of Lemma* 2.1 *and the relation* (2.2)*, it is clear that for* $f \in AC[0,T]$*,* ${}_C D^{\alpha}_{0,t}$ *and* ${}_C D^{\alpha}_{t,T}$ *exist almost everywhere on* $[0,T]$*. However, using the fact that if* $f \in W^{1,1}(0,T)$*, then* f *is absolutely continuous, we deduce that the Caputo and Riemann-Liouville fractional derivatives exist almost everywhere on* $[0,T]$*, even for* $f \in W^{1,1}(0,T)$*.*

Remark 2.2 *In case of homogeneous boundary conditions, i.e.,* $f(0) = f(T) = 0$*, the Caputo and Riemann-Liouville fractional derivatives coincide.*

Next, we recall the fractional integration-by-parts formulas (Agrawal, 2007) that will be used later in order to obtain the optimality conditions for the optimal control problem.

Lemma 2.2 *Let* $f \in \mathbb{L}^{\alpha} := \{f \in C[0,T] : {}_{RL}D^{\alpha}_{0,t} f \in L^2(0,T)\}$ *and* $g \in \mathbb{R}^{\alpha} := \{g \in C[0,T] : {}_C D^{\alpha}_{t,T} g \in L^2(0,T)\}$*, then the following holds:*

$$\int_0^T {}_{RL}D^{\alpha}_{0,t} f(t) g(t) dt = \int_0^T f(t) {}_C D^{\alpha}_{t,T} g(t) dt + \Big[g(t) D^{-(1-\alpha)}_{0,t} f(t) \Big]_{t=0}^{t=T}, \quad (2.7)$$

$$\int_0^T {}_C D^{\alpha}_{t,T} g(t) f(t) dt = \int_0^T g(t) {}_{RL}D^{\alpha}_{0,t} f(t) dt - \Big[g(t) D^{-(1-\alpha)}_{0,t} f(t) \Big]_{t=0}^{t=T}. \quad (2.8)$$

Finally, we present the fractional Gronwall inequality (McLean et al., 2019) that will play an essential role in proving the uniqueness of the state equation.

Lemma 2.3 *Let* $\alpha > 0$ *and* $T > 0$*. Assume that* ξ *and* η *are non-negative and non-decreasing functions, defined on* $[0,T]$*. If* $q : [0,T] \to \mathbb{R}$ *is an integrable function satisfying*

$$0 \le q(t) \le \xi(t) + \eta(t) \int_0^t g_{\alpha}(t-\tau) q(\tau) d\tau, \qquad 0 \le t \le T,$$

then

$$q(t) \le \xi(t) E_{\alpha}(\eta(t) t^{\alpha}), \quad 0 \le t \le T,$$

where $E_{\alpha}(\cdot)$ *is the Mittag-Leffler function, defined by* $E_{\alpha}(z) = \displaystyle\sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}$*.*

The above definitions and results also hold for the spatial variable $x \in (a,b)$, $a > 0$, in place of temporal variable t.

2.3. Function spaces

Here, we shall define some function spaces and related known results corresponding to the spatial fractional derivatives that are required to study the considered problem. Let $0 < \beta < 1$ and c_0, $d_0 \in \mathbb{R}$. We define $AC_a^{\beta,2} = AC_a^{\beta,2}([a,b],\mathbb{R})$ as the set of all functions $f : [a,b] \to \mathbb{R}$ that have the representation

$$f(x) = \frac{c_0}{\Gamma(\beta)}(x-a)^{\beta-1} + D_{a,x}^{-\beta}\phi(x) \quad \text{for a.e. } x \in [a,b],$$

with $\phi \in L^2(a,b)$, and by $AC_b^{\beta,2} = AC_b^{\beta,2}([a,b],\mathbb{R})$ we mean the set of all functions $g : [a,b] \to \mathbb{R}$ that have the representation

$$g(x) = \frac{d_0}{\Gamma(\beta)}(b-x)^{\beta-1} + D_{x,b}^{-\beta}\psi(x) \quad \text{for a.e. } x \in [a,b],$$

with $\psi \in L^2(a,b)$.

Now, we have the following characterization from Idczak and Walczak (2013), Remark 8:

$$_{RL}D_{a,x}^{\beta} f \in L^2(a,b) \iff f \in AC_a^{\beta,2},$$

$$_{RL}D_{x,b}^{\beta} f \in L^2(a,b) \iff f \in AC_b^{\beta,2}.$$

Next, we set

$$H_a^{\beta}(a,b) = AC_a^{\beta,2} \cap L^2(a,b), \tag{2.9}$$

$$H_b^{\beta}(a,b) = AC_b^{\beta,2} \cap L^2(a,b). \tag{2.10}$$

Evidently, one can find

$$f \in H_a^{\beta}(a,b) \iff f \in L^2(a,b) \text{ and } {_{RL}D_{a,x}^{\beta}} f \in L^2(a,b), \tag{2.11}$$

$$f \in H_b^{\beta}(a,b) \iff f \in L^2(a,b) \text{ and } {_{RL}D_{x,b}^{\beta}} f \in L^2(a,b). \tag{2.12}$$

Moreover, the space $H_a^{\beta}(a,b)$ forms a Hilbert space (see, e.g., Idczak and Walczak, 2013) when equipped with the norm

$$\|\psi\|^2_{H_a^{\beta}(a,b)} = \|\psi\|^2_{L^2(a,b)} + \|_{RL}D_{a,x}^{\beta}\|^2_{L^2(a,b)}. \tag{2.13}$$

In order to obtain the well-posedness of the state equation (1.2), we introduce the following space:

$$\mathcal{V} = \{y \in H_a^{\beta}(a,b) \;:\; {_C D_{x,b}^{\beta}}(\gamma\, {_{RL}D_{a,x}^{\beta}} y) \in H_b^{1-\beta}(a,b)\}, \tag{2.14}$$

where $H_b^{1-\beta}(a,b)$, defined as in (2.10). Clearly, $\mathcal{V}$ is a closed subspace of $H_a^{\beta}(a,b)$ and thus, $\mathcal{V}$ equipped with the norm (2.13) is a Hilbert space. Moreover, we introduce the space

$$V := \{y \in \mathcal{V} \;:\; \gamma(b)\, {_{RL}D_{a,x}^{\beta}} y(b) = D_{a,x}^{-(1-\beta)} y(a) = 0\}, \tag{2.15}$$

which is closed in $\mathcal{V}$. Consequently, the space V, equipped with the norm (2.13), is also a Hilbert space.

In view of (2.9), (2.10), (2.14) and (2.15), we have the following continuous embeddings:

$$V \hookrightarrow H_a^\beta(a,b) \hookrightarrow L^2(a,b) \hookrightarrow (H_a^\beta(a,b))' \hookrightarrow V'.$$

Moreover, we have, see Brezis (1983),

$$\langle y, w\rangle_{L^2(a,b)} = \langle y, w\rangle_{V'\times V}, \quad y \in L^2(a,b),\ w \in V, \tag{2.16}$$

where $\langle \cdot,\cdot\rangle_{V'\times V}$ denotes the duality pairing between V' and V.

Further, we give the following result, which ensures the existence of the traces $D_{a,x}^{-(1-\beta)}y(a,t)$, $D_{a,x}^{-(1-\beta)}y(b,t)$, ${}_{RL}D_{a,x}^{\beta}y(a,t)$ and ${}_{RL}D_{a,x}^{\beta}y(b,t)$ for a.e. $t \in (0,T)$.

Lemma 2.4 *Let $\beta \in (0,1)$, $T > 0$ and $y \in L^2((0,T);\mathcal{V})$. Then, the following assertions hold.*

1. *For any $x_0 \in [a,b]$, the function $D_{a,x}^{-(1-\beta)}y(x_0,\cdot)$ exists and belongs to $L^2(0,T)$. Moreover, there exists a constant $C > 0$ such that*

$$\|D_{a,x}^{-(1-\beta)}y(x_0,\cdot)\|_{L^2(0,T)}^2 \le C\left(\frac{(b-a)^{1-\beta}}{\Gamma(2-\beta)}+1\right)\|y\|_{L^2((0,T);H_a^\beta(a,b))}^2.$$

2. *For any $x_0 \in [a,b]$, the function $[{}_{RL}D_{a,x}^{\beta}y(x_0,\cdot)]$ exists and belongs to $L^2(0,T)$. Moreover, there exists a constant $C > 0$ such that*

$$\|{}_{RL}D_{a,x}^{\beta}y(x_0,\cdot)\|_{L^2(0,T)}^2 \le C\|y\|_{L^2((0,T);H_a^\beta(a,b))}^2.$$

Proof The proof follows from the continuous embedding $H^1(a,b) \hookrightarrow C[a,b]$, Proposition 2.1 and the arguments provided in Leugering et al. (2021), Lemma 2.7.

Finally, we present the fractional integration-by-parts formula for the composite fractional derivatives, which directly follows from Lemma 2.2.

Lemma 2.5 *Let $\beta \in (0,1)$, $\gamma \in C[a,b]$ and $y, w \in \mathcal{V}$. Then*

$$\begin{aligned}
&\int_a^b {}_{C}D_{x,b}^{\beta}(\gamma(x)\,{}_{RL}D_{a,x}^{\beta}y(x))w(x)dx \\
&= \int_a^b \gamma(x)({}_{RL}D_{a,x}^{\beta}y(x))({}_{RL}D_{a,x}^{\beta}w(x))dx - \Big[(\gamma(x)\,{}_{RL}D_{a,x}^{\beta}y(x))D_{a,x}^{-(1-\beta)}w(x)\Big]_{x=a}^{x=b} \\
&= \int_a^b y(x)\,{}_{C}D_{x,b}^{\beta}(\gamma(x)\,{}_{RL}D_{a,x}^{\beta}w(x))dx + \Big[D_{a,x}^{-(1-\beta)}y(x)(\gamma(x)\,{}_{RL}D_{a,x}^{\beta}w(x))\Big]_{x=a}^{x=b} \\
&\quad - \Big[(\gamma(x)\,{}_{RL}D_{a,x}^{\beta}y(x))D_{a,x}^{-(1-\beta)}w(x)\Big]_{x=a}^{x=b}.
\end{aligned} \tag{2.17}$$

Remark 2.3 *In view of Lemma* 2.4, *one can deduce that the boundary terms involved in* (2.17) *are real.*

3. Well-posedness results for space-time fractional parabolic equation

In this section, we are going to prove the well-posedness for the space-time fractional parabolic equation of the following type:

$$\begin{cases} {}_C D^{\alpha}_{0,t} y(x,t) + {}_C D^{\beta}_{x,b}(\gamma(x)\,{}_{RL}D^{\beta}_{a,x} y(x,t)) + q(x)y(x,t) & = h(x,t), \\ & (x,t) \in (a,b)\times(0,T), \\ y(x,0) & = y^0(x), \quad x \in (a,b), \\ D^{-(1-\beta)}_{a,x} y(a,t) & = 0, \qquad t \in (0,T), \\ \gamma(b)\,{}_{RL}D^{\beta}_{a,x} y(b,t) & = 0, \qquad t \in (0,T), \end{cases} \tag{3.1}$$

where the functions $h \in L^2((0,T);L^2(a,b))$ and $y^0 \in L^2(a,b)$ are the right hand side and initial data, respectively. In what follows, the space $L^2((0,T);L^2(a,b))$ will be denoted by $L^2((a,b)\times(0,T))$.

We first derive the weak formulation of the problem (3.1). Hence, by multiplying $(3.1)_1$ with a test function $\psi \in \mathcal{V}$ and integrating over Ω with $\Omega := (a,b)$, we get

$$\int_\Omega {}_C D^{\alpha}_{0,t} y(x,t)\psi(x)dx + \int_\Omega \Big({}_C D^{\beta}_{x,b}(\gamma(x)\,{}_{RL}D^{\beta}_{a,x} y(x,t)) + q(x)y(x,t)\Big)\psi(x)dx = \\ = \int_\Omega h(x,t)\psi(x)dx.$$

Then, integration by parts, in view of Lemma 2.5, leads to

$$\langle {}_C D^{\alpha}_{0,t} y(t), \psi\rangle_{L^2(\Omega)} + a(t,y(t),\psi) - \Big[(\gamma(x)\,{}_{RL}D^{\beta}_{a,x} y(x)) D^{-(1-\beta)}_{a,x}\psi(x)\Big]_{x=a}^{x=b} = \\ = \langle h(t),\psi\rangle_{L^2(\Omega)}, \tag{3.2}$$

where $a(t,\cdot,\cdot)$ is the bilinear form, given by

$$a(t,y(t),\psi) = \int_\Omega \gamma(x)({}_{RL}D^{\beta}_{a,x} y(x,t))({}_{RL}D^{\beta}_{a,x}\psi(x))dx + \int_\Omega q(x)y(x,t)\psi(x)dx. \tag{3.3}$$

By choosing a more restrictive space for $\psi \in V$, we obtain from (3.2)

$$\langle {}_C D^{\alpha}_{0,t} y(t), \psi\rangle_{L^2(\Omega)} + a(t,y(t),\psi) = \langle h(t),\psi\rangle_{V'\times V}, \tag{3.4}$$

where we have replaced the inner product in $L^2(a,b)$ by the duality pairing in V to further reduce the regularity of $h(t)$.

Now, we rewrite the Caputo fractional derivative in terms of fractional integrals. Thus, from (2.2), (2.4) and (2.6), we get

$$\begin{aligned} {}_C D_{0,t}^{\alpha} y(x,t) &= {}_{RL} D_{0,t}^{\alpha} y(x,t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(x,0) \\ &= \frac{d}{dt}[D_{0,t}^{-(1-\alpha)} y(x,t)] - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(x,0) \\ &= \frac{d}{dt} D_{0,t}^{-(1-\alpha)} [y(x,t) - y(x,0)]. \end{aligned} \tag{3.5}$$

Therefore, we follow Zacher (2009) and seek a solution of (3.1) in the following space

$$W^{\alpha}(y^0, V, L^2(a,b)) := \left\{ y \in L^2((0,T);V) : \ D_{0,t}^{-(1-\alpha)}(y - y^0) \in {}_0H^1((0,T);V') \right\},$$

where the subscript 0 of ${}_0H^1$ means vanishing of the trace at $t = 0$.

Therefore, by incorporating the initial conditions $(3.1)_2$ in the form (3.5), we obtain the weak form of (3.1) as follows:

$$\left\{ \begin{array}{l} \text{Find } y \in W^{\alpha}(y^0, V, L^2(a,b)) \text{ such that } \forall \psi \in V \\ \frac{d}{dt} \left\langle [D_{0,t}^{-(1-\alpha)}(y - y^0)](t), \psi \right\rangle_{L^2(a,b)} + a(t, y(t), \psi) = \langle h(t), \psi \rangle_{V' \times V}, \\ \text{holds} \ \text{ a.e. } t \in (0,T). \end{array} \right. \tag{3.6}$$

We say that y is a weak solution of (3.1) if y solves (3.6). Finally, we note that (3.6) is equivalent to the operator equation

$$\frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(y - y^0)](t) + A(t)y(t) = h(t), \quad \text{for a.e. } t \in (0,T), \tag{3.7}$$

in V', where the operator $A(t) : V \to V'$ is given by

$$\langle A(t)y(t), \psi \rangle_{V' \times V} = a(t, y(t), \psi), \quad \text{a.e. } t \in (0,T). \tag{3.8}$$

THEOREM 3.1 *Let $\alpha, \beta \in (0,1)$, $y^0 \in L^2(a,b)$ and $h \in L^2((0,T);V')$. Also, assume that $\gamma \in C[a,b]$, $q \in L^{\infty}(a,b)$ and there exist two constants, $\gamma_0 > 0$ and $q_0 > 0$, such that $\gamma \geq \gamma_0 > 0$ and $q \geq q_0 > 0$. Then, the problem (3.1) has a unique weak solution $y \in W^{\alpha}(y^0, V, L^2(a,b))$, i.e., (3.6) holds and y satisfies the following estimate*

$$\|D_{0,t}^{-(1-\alpha)}(y - y^0)\|_{H^1((0,T);V')} + \|y\|_{L^2((0,T);V)} \leq C(\|y^0\|_{L^2(a,b)} + \|h\|_{L^2((0,T);V')}), \tag{3.9}$$

where C is a positive constant. Moreover, if $\alpha > \frac{1}{2}$, then $y \in C([0,T];V')$ and $y|_{t=0} = y^0$.

Before we embark on the proof of Theorem 3.1, we first recall a result from Leugering et al. (2021) regarding the continuity and coercivity of the bilinear form $a(t,\cdot,\cdot)$. Then, we discuss a method to approximate the kernel $g_{1-\alpha}$ of the Caputo fractional derivative, followed by the basic identity (Lemma 3.2) for kernels.

Lemma 3.1 *Let γ and q satisfy the assumption of Theorem 3.1. Then, for any $y,w \in H_a^\beta(a,b)$, the bilinear form $a(\cdot,\cdot): H_a^\beta(a,b)\times H_a^\beta(a,b)\to\mathbb{R}$, given by*

$$a(y,w)=\int_a^b \gamma(x)({}_{RL}D_{a,x}^\beta y(x))({}_{RL}D_{a,x}^\beta w(x))dx+\int_{a^b} q(x)y(x)w(x)dx,$$

is continuous and coercive. That is, we have

$$\textit{Continuity:}\quad a(y,w)\le(\|q\|_\infty+\|\gamma\|_\infty)\|y\|_{H_a^\beta(a,b)}\|w\|_{H_a^\beta(a,b)},\quad \forall y,w\in H_a^\beta(a,b). \tag{3.10}$$

$$\textit{Coercivity:}\quad a(t,y,y)\ge\min(\gamma_0,q_0)\|y\|^2_{H_a^\beta(a,b)}\quad \forall y\in H_a^\beta(a,b). \tag{3.11}$$

Approximation of the kernel $g_{1-\alpha}$: Since $g_{1-\alpha}\in L^{1,loc}(\mathbb{R}_+)$ and satisfies $g_\alpha * g_{1-\alpha}=1$, it belongs to a certain class of functions, see Zacher (2009), Definition 2.1. The kernels of this class can be approximated using the sequence of more regular kernel functions, which can be obtained from the approximation of the operator $\mathcal{T}$, defined by

$$\mathcal{T}w=\frac{d}{dt}(g_{1-\alpha}*w),\quad D(\mathcal{T})=\{w\in L^2((0,T);\mathcal{H}): g_{1-\alpha}*w\in {}_0H^1((0,T);\mathcal{H})\},$$

where $\mathcal{H}$ is a real Hilbert space. The Yosida approximation $\mathcal{T}_n$ of the operator $\mathcal{T}$ is defined by

$$\mathcal{T}_n=n\mathcal{T}(n+\mathcal{T})^{-1},\quad n\in\mathbb{N},$$

satisfying the property that for every $w\in D(\mathcal{T})$, one has $\mathcal{T}_nw\to\mathcal{T}w$ in $L^2((0,T);\mathcal{H})$ as $n\to\infty$. Furthermore, the operator $\mathcal{T}_n$ has the representation

$$\mathcal{T}_nw=\frac{d}{dt}(k_{n,\alpha}*w),\quad w\in L^2((0,T);\mathcal{H}),\ n\in\mathbb{N},$$

where $k_{n,\alpha}=ns_n$ with s_n solving the Volterra integral equation

$$s_n(t)+n(g_\alpha*s_n)(t)=1,\quad t>0,\ n\in\mathbb{N}.$$

The kernels $s_n\in W^{1,1}(0,T)$, $n\in\mathbb{N}$, are nonnegative and nonincreasing in $(0,\infty)$. Consequently, the kernel $k_{n,\alpha}$, $n\in\mathbb{N}$, satisfies the same properties as s_n. Moreover, one has $k_{n,\alpha}\to g_{1-\alpha}$ in $L^1(0,T)$ as $n\to\infty$. We refer to Vergara and Zacher (2008) for broader explanation.

Finally, we provide the following result from Vergara and Zacher (2008), which is of fundamental importance with respect to the existence and the priori estimates of (3.6).

LEMMA 3.2 *Let $\mathcal{H}$ be a real Hilbert space and $T > 0$. Then, for any $k \in W^{1,1}(0,T)$ and any $w \in L^2((0,T);\mathcal{H})$ there holds*

$$\left\langle \frac{d}{dt}(k * w)(t), w(t) \right\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\left(k * \|w\|^2_{\mathcal{H}}\right)(t) + \frac{1}{2}k(t)\|w(t)\|^2_{\mathcal{H}} + \frac{1}{2}\int_0^t [-\dot{k}(s)]\|w(t) - w(t-s)\|^2_{\mathcal{H}} ds, \quad a.e.\ t \in (0,T). \tag{3.12}$$

Now, we give the proof of Theorem 3.1, which mainly consists of four steps.

Step 1: Galerkin approach

The space V is a separable Hilbert space as it is a closed subspace of a separable Hilbert space H^β_a. Indeed, the mapping

$$\begin{aligned} F\ &: H^\beta_a \to L^2 \times L^2 \\ &u \to (u, {}_{RL}D^\beta_{a,x} u) \end{aligned}$$

is an isometry and $F(H^\beta_a)$ is the closed subspace of the separable space $L^2 \times L^2$. Consequently, H^β_a is the separable space with the norm $\|\cdot\|_{H^\beta_a}$. Hence, there exists a countable orthonormal basis of V, say $(w_1, w_2, \ldots, w_m, \ldots)$. Now, we introduce the Galerkin ansatz

$$y_m(x,t) = \sum_{i=1}^m d_{i,m}(t) w_i(x). \tag{3.13}$$

For the initial condition, we have

$$y_{m,0}(x) = \sum_{i=1}^m c_{i,m} w_i(x) \quad \text{with } c_{i,m} = \langle y^0, w_i \rangle_{L^2(a,b)} \text{ and}$$

$$\sum_{i=1}^m c_{i,m} w_i \to y^0 \text{ in } L^2(a,b) \text{ as } m \to \infty. \tag{3.14}$$

Since $\{w_m\}_{m\in\mathbb{N}}$ are the orthonormal basis of V, upon replacing y, y_0 and ψ in the weak form (3.6) by y_m, $y_{m,0}$ and w_i respectively, we obtain the system of Galerkin equations

$$\left\langle \frac{d}{dt}[D^{-(1-\alpha)}_{0,t}(y_m - y_{m,0})](t), w_i \right\rangle_{L^2(a,b)} + a\left(t, y_m(t), w_i\right) = \langle h(t), w_i \rangle_{V' \times V}, \quad i = 1, 2, \ldots, m. \tag{3.15}$$

Using the fact that vectors $w_1, w_2, \ldots, w_m$ are orthonormal in V, we can rewrite the system (3) into an equivalent system of fractional ODEs as

$$\frac{d}{dt}[D^{-(1-\alpha)}_{0,t}(d_m - c_m)](t) + A_m(t) d_m(t) = H_m(t), \tag{3.16}$$

where the matrix $A_m \in L^\infty((0,T);\mathbb{R}^{m\times m})$ is given by $(A_m(t))_{i,j} := a(t, w_i, w_j)$, and the vector $H_m \in L^2((0,T);\mathbb{R}^m)$ is given by $(H_m(t))_i := \langle h(t), w_i\rangle_V$. Now, in order to solve (3.16), we apply $D_{0,t}^{-\alpha}$ to both sides of it and then, using the semigroup property for fractional integrals,

$$D_{0,t}^{-\alpha}(D_{0,t}^{-\beta}h(t)) = D_{0,t}^{-(\alpha+\beta)}h(t), \quad \text{a.e. } t \in (0,T) \text{ for } h \in L^p(0,T),\ 1 \le p < \infty,$$

we convert it into an equivalent system of integral equations as

$$d_m(t) = c_m - D_{0,t}^{-\alpha}(A_m d_m)(t) + D_{0,t}^{-\alpha}H_m(t),$$

which has a unique solution $d_m \in L^2((0,T);\mathbb{R}^m)$, see, e.g., Gripenberg, Londen and Staffans (1990). But then $d_m \in W^\alpha(c_m, \mathbb{R}^m, \mathbb{R}^m)$, which shows that (3.16) holds for almost all $t \in (0,T)$ and therefore for each $m \in \mathbb{N}$, the Galerkin equation (3) has a unique solution $y_m \in W^\alpha(y_{m,0}, V, L^2(a,b))$.

Step 2: Priori estimates for the Galerkin solutions

We begin with the Galerkin equation (3), multiply it with $d_{i,m}(t)$ and sum over $i = 1, 2, \dots, m$. This leads to

$$\left\langle \frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(y_m - y_{m,0})](t), y_m(t)\right\rangle_{L^2(a,b)} + a\left(t, y_m(t), y_m(t)\right) = \\ = \langle h(t), y_m(t)\rangle_{V'\times V}. \tag{3.17}$$

Now, in order to employ the approximation technique for the kernel function $g_{1-\alpha}$, we rewrite (3.17) in terms of the convolution as

$$\left\langle \frac{d}{dt}[g_{1-\alpha} * (y_m - y_{m,0})](t), y_m(t)\right\rangle_{L^2(a,b)} + a\left(t, y_m(t), y_m(t)\right) = \langle h(t), y_m(t)\rangle_{V'\times V}. \tag{3.18}$$

Let $k_{n,\alpha} \in W^{1,1}(0,T)$, $n \in \mathbb{N}$, be the kernel corresponding to the Yosida approximation $\mathcal{T}_n$ of the operator

$$\mathcal{T}w = \frac{d}{dt}(g_{1-\alpha} * w), \quad D(\mathcal{T}) = \{w \in L^2((0,T);V) : g_{1-\alpha} * w \in {}_0H^1((0,T);V')\}. \tag{3.19}$$

Then $k_{n,\alpha} * y_m \in H^1((0,T);V')$, and using (2.16), we get from (3.18)

$$\left\langle \frac{d}{dt}(k_{n,\alpha} * y_m)(t), y_m(t)\right\rangle_{L^2(a,b)} + a\left(t, y_m(t), y_m(t)\right) \\ = k_{n,\alpha}(t)\langle y_{m,0}, y_m(t)\rangle_{L^2(a,b)} + \langle h(t), y_m(t)\rangle_{V'\times V} + r_{m,n}(t), \text{ a.e. } t \in (0,T),$$

where

$$r_{m,n}(t) = \left\langle [k_{n,\alpha} * (y_m - y_{m,0})]'(t) - [g_{1-\alpha} * (y_m - y_{m,0})]'(t), y_m(t) \right\rangle_{V' \times V}.$$

Further, using Lemma 3.2 and (3.11), one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(k_\alpha^n * \|y_m\|^2_{L^2(a,b)}\right)(t) + \frac{1}{2}k_{n,\alpha}(t)\|y_m(t)\|^2_{L^2(a,b)} + \min(\gamma_0, q_0)\|y_m(t)\|^2_V \\
&\le k_{n,\alpha}(t)\langle y_{m,0}, y_m(t)\rangle_{L^2(a,b)} + \langle h(t), y_m(t)\rangle_{V'\times V} + r_{m,n}(t),
\end{aligned}$$

which, by applying Young's inequality, gives the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(k_{n,\alpha} * \|y_m\|^2_{L^2(a,b)}\right)(t) + \min(\gamma_0, q_0)\|y_m(t)\|^2_V \\
&\le \frac{1}{2}k_{n,\alpha}(t)\|y_{m,0}\|^2_{L^2(a,b)} + \epsilon\|y_m(t)\|^2_V + \frac{1}{4\epsilon}\|h(t)\|^2_{V'} + r_{m,n}(t),\ \epsilon > 0.
\end{aligned}$$

By taking $\epsilon = \frac{\min(\gamma_0,q_0)}{2}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(k_{n,\alpha} * \|y_m\|^2_{L^2(a,b)}\right)(t) + \min(\gamma_0, q_0)\|y_m(t)\|^2_V \\
&\le k_{n,\alpha}(t)\|y_{m,0}\|^2_{L^2(a,b)} + \frac{1}{\min(\gamma_0, q_0)}\|h(t)\|^2_{V'} + 2r_{m,n}(t).
\end{aligned} \tag{3.20}$$

Since $r_{m,n} \to 0$ in $L^1(0,T)$, we have from Proposition 2.1 that $D^{-\alpha}_{0,t} r_{m,n}(t) \to 0$ in $L^1(0,T)$ as $n \to \infty$. Moreover, we have

$$\begin{aligned}
&D^{-\alpha}_{0,t}\left[\frac{d}{dt}\left(k_{n,\alpha} * \|y_m\|^2_{L^2(a,b)}\right)(t)\right] \\
&= \left[g_\alpha * \frac{d}{dt}\left(k_{n,\alpha} * \|y_m\|^2_{L^2(a,b)}\right)\right](t) \\
&= \frac{d}{dt}\left(k_{n,\alpha} * g_\alpha * \|y_m\|^2_{L^2(a,b)}\right)(t) \\
&\to \frac{d}{dt}\left(g_{1-\alpha} * g_\alpha * \|y_m\|^2_{L^2(a,b)}\right)(t) = \|y_m(t)\|^2_{L^2(a,b)}
\end{aligned}$$

in $L^1(0,T)$ as $n \to \infty$. Therefore, by applying the fractional integral $D^{-\alpha}_{0,t}$ in (3.20) and letting $n \to \infty$, we obtain

$$\|y_m(t)\|^2_{L^2(a,b)} \le \|y_{m,0}\|^2_{L^2(a,b)} + \frac{1}{\min(\gamma_0, q_0)} D^{-\alpha}_{0,t}\|h(t)\|^2_{V'}, \tag{3.21}$$

$\forall m \in \mathbb{N}$, and for a.e. $t \in (0,T)$. Finally, using the positivity of the kernel g_α and the convolution identity (2.1), it follows from (3.21) that

$$\|y_m\|_{L^2((a,b)\times(0,T))} \le C\left(\|y_{m,0}\|_{L^2(a,b)} + \|h\|_{L^2((0,T);V')}\right), \tag{3.22}$$

where C is a constant, which depends on α, $\min(\gamma_0, q_0)$ and T.

Now, we return to (3.20), and integrate it from 0 to T and then letting $n \to \infty$. This results in

$$\min(\gamma_0, q_0)\|y_m(t)\|_V^2 \leq \|g_{1-\alpha}\|_{L^1(0,T)}\|y_{m,0}\|_{L^2(a,b)}^2 + \frac{1}{\min(\gamma_0, q_0)}\|h\|_{L^2((0,T);V')}^2,$$

where we have used the fact that $\left(k_{n,\alpha} * \|y_m(t)\|_{L^2(a,b)}^2\right)(0) = 0$. Finally, using (3.14) and (3.22), we obtain a priori bound

$$\|y_m\|_{L^2((0,T);V)} \leq C_1\left(\|y^0\|_{L^2(a,b)} + \|h\|_{L^2((0,T);V')}\right), \quad m \in \mathbb{N}, \tag{3.23}$$

where C_1 is a positive constant which is independent of m.

Step 3: Existence of solution and norm estimate

By (3.23), we see that y_m is bounded in $L^2((0,T);V)$. Thus, we may extract a subsequence of $\{y_m\}$ (still denoted by $\{y_m\}$) such that

$$y_m \overset{w}{\rightharpoonup} y \quad \text{in } L^2((0,T);V) \quad \text{as } m \to \infty, \tag{3.24}$$

for some $y \in L^2((0,T);V)$. Additionally, we have

$$\|y\|_{L^2((0,T);V)} \leq \liminf_{m\to\infty} \|y_m\|_{L^2((0,T);V)}. \tag{3.25}$$

We shall prove that $y \in W^\alpha(y^0, V, L^2(a,b))$, and that y is a solution of (3.6).

Let $\phi \in C^\infty([0,T];\mathbb{R})$ be some test function with $\phi(T) = 0$. Multiplying the Galerkin equation (3.15) by ϕ and using integration by parts in time, we get

$$\begin{aligned} -\int_0^T \left\langle [D_{0,t}^{-(1-\alpha)}(y_m - y_{m,0})](t), w_i\right\rangle_{L^2(a,b)} \phi'(t)dt + \int_0^T a(t, y_m(t), w_i)\phi(t)dt = \\ = \int_0^T \langle h(t), w_i\rangle_{V'\times V}\phi(t)dt \end{aligned} \tag{3.26}$$

for all $i = 1, 2, \ldots, m$, since $[D_{0,t}^{-(1-\alpha)}(y_m - y_{m,0})](0) = 0$. Now, we apply the weak limits (3.24) and the convergence of $y_{m,0}$ to y^0 in $L^2(a,b)$, given in (3.14), to equation (3.26). This leads to

$$\begin{aligned} -\int_0^T \left\langle [D_{0,t}^{-(1-\alpha)}(y - y^0)](t), w_i\right\rangle_{L^2(a,b)} \phi'(t)dt + \int_0^T a(t, y(t), w_i)\phi(t)dt = \\ = \int_0^T \langle h(t), w_i\rangle_{V'\times V}\phi(t)dt \end{aligned} \tag{3.27}$$

for all $i \in \mathbb{N}$, where we have used the fact that the integrals in (3.26) are continuous linear functionals with respect to y_m.

Since $\{w_i\}_{i\in\mathbb{N}}$ forms a basis of V, there exists a $\psi \in V$ and a sequence ψ_n, consisting of finite linear combinations of $\{w_i\}_{i=1}^n$, such that $\lim_{n\to\infty} \psi_n = \psi$. Moreover, all the terms in (3) are continuous linear functionals on the space V with respect to w_i. Therefore, replacing w_i by ψ and using (2.16), one gets from (3.27)

$$
\begin{aligned}
&-\int_0^T \left\langle [D_{0,t}^{-(1-\alpha)}(y-y^0)](t), \psi \right\rangle_{V'\times V} \phi'(t)dt + \int_0^T a(t, y(t), \psi)\phi(t)dt = \\
&= \int_0^T \langle h(t), \psi\rangle_{V'\times V}\phi(t)dt, \ \forall \psi \in V.
\end{aligned} \tag{3.28}
$$

Since (3.28) holds, in particular, for all $\phi \in C_c^\infty(0,T)$, thus, we deduce that $D_{0,t}^{-(1-\alpha)}(y-y^0)$ has a generalized weak derivative on $(0,T)$ with

$$
\frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(y-y^0)](t) = h(t) - A(t)y(t), \tag{3.29}
$$

where the operator $A(t) : V \to V'$ is defined as in (3.8). From $y \in L^2((0,T); \mathcal{V}_0)$ and (3.10), we have

$$
\|A(t)y(t)\|_{V'} \le (\|q\|_\infty + \|\gamma\|_\infty)\|y(t)\|_V \quad \text{for a.e. } t \in (0,T).
$$

Therefore, we find that $Ay \in L^2((0,T); V')$. Since $h \in L^2((0,T); V^{'})$, it follows from (3.29) that $[D_{0,t}^{-(1-\alpha)}(y-y^0)]^{'} \in L^2((0,T); V^{'})$.

It remains to show that $[D_{0,t}^{-(1-\alpha)}(y-y^0)](0) = 0$. Let us set $z := D_{0,t}^{-(1-\alpha)}(y-y^0)$. Then, $z \in H^1((0,T); V^{'}) \hookrightarrow C([0,T]; V^{'})$, and from (3.28) and (3.29), it holds that

$$
-\int_0^T \langle z(t), \psi\rangle_{V'\times V}\phi'(t)dt = \int_0^T \langle z'(t), \psi\rangle_{V'\times V}\phi(t)dt \tag{3.30}
$$

for all $\psi \in V$ and $\phi \in C^\infty([0,T]; \mathbb{R})$ with $\phi(T) = 0$.

In particular, by choosing ϕ such that $\phi(0) = 1$, approximating z by a sequence of functions $z_n \in C^\infty([0,T]; V^{'})$ and using integration by parts, it follows from (3.30) that $\langle z(0), \psi\rangle_{V'\times V} = 0$ for all $\psi \in V$. Hence, $z(0) = 0$.

Thus, we have $y \in W^\alpha(y^0, V, L^2(a,b))$ that solves (3.6), due to the equivalence of the equation (3.7). For the norm estimate, we have from (3.23) and (3.25) that

$$
\|y\|_{L^2((0,T);V)} \le C_1 \left(\|y^0\|_{L^2(a,b)} + \|h\|_{L^2((0,T);V')} \right). \tag{3.31}
$$

Moreover, using (3.29) along with the estimate

$$
\|A(t)y(t)\|_{V'} \le (\|q\|_\infty + \|\gamma\|_\infty)\|y(t)\|_V
$$

and integrating in time, we obtain

$$\|[D_{0,t}^{-(1-\alpha)}(y-y^0)](t)\|_{H^1((0,T)V')} \leq \|h\|_{L^2((0,T);V')} + \int_0^T \|A(t)y(t)\|_{V'}^2\,dt \\ \leq C\left[\|h\|_{L^2((0,T);V')} + \|y\|_{L^2((0,T);V)}\right]. \tag{3.32}$$

By combining (3.32) with (3.31), we get (3.9).

Step 4: Uniqueness of solution

Let y_1 and y_2 be two different solutions of (3.6), and $w = y_1 - y_2$. Then, $w \in W^\alpha(0, V, L^2(a,b))$ and satisfies the equation

$$\left\langle \frac{d}{dt}[D_{0,t}^{-(1-\alpha)}w(t)], \psi \right\rangle_{V'\times V} + a(t, w(t), \psi) = 0, \quad \psi \in V, \text{ a.e. } t \in (0,T).$$

By taking $\psi = w(t)$, we get

$$\left\langle \frac{d}{dt}[D_{0,t}^{-(1-\alpha)}w(t)], w(t) \right\rangle_{V'\times V} + a(t, w(t), w(t)) = 0, \quad \text{a.e. } t \in (0,T). \tag{3.33}$$

Let $k_{n,\alpha}$, $n \in \mathbb{N}$ be the kernel as in the existence part above (Step 2). Then (3.33) can be equivalently written as

$$\left\langle \frac{d}{dt}(k_{n,\alpha} * w)(t), w(t) \right\rangle_{L^2(a,b)} + a\left(t, w(t), w(t)\right) = r_n(t), \quad \text{a.e. } t \in (0,T), \tag{3.34}$$

for all $n \in \mathbb{N}$, where

$$r_n(t) = \langle (k_{n,\alpha} * w)'(t) - (g_{1-\alpha} * w)'(t), w(t) \rangle, \quad \text{a.e. } t \in (0,T).$$

Now, using Lemma 3.2 and (3.11), we obtain from (3.34)

$$\frac{d}{dt}\left(k_\alpha^n * \|w\|_{L^2(\mathcal{G})}^2\right)(t) + 2\min(\gamma_0, q_0)\|w(t)\|_V^2 \leq 2r_n(t), \quad \text{a.e. } t \in (0,T). \tag{3.35}$$

Similarly, as in the existence part, we find that $D_{0,t}^{-(1-\alpha)} r_n(t) \to 0$ and

$$D_{0,t}^{-\alpha}\left[\frac{d}{dt}\left(k_{n,\alpha} * \|w\|_{L^2(a,b)}^2\right)(t)\right] \to \|w(t)\|_{L^2(a,b)}^2$$

in $L^1(0,T)$ as $n \to \infty$. Therefore, on applying the fractional integral $D_{0,t}^{-\alpha}$ in (3.35) and letting $n \to \infty$, we get

$$\|w(t)\|_{L^2(a,b)}^2 + 2\min(\gamma_0, q_0) D_{0,t}^{-\alpha}\|w(t)\|_{L^2(a,b)}^2 \leq 0, \quad \text{a.e. } t \in (0,T). \tag{3.36}$$

Finally, in view of Lemma 2.3 (the fractional Gronwall lemma), inequality (3.36) implies that $\|w(t)\|_{L^2(a,b)}^2 = 0$, i.e., $w = 0$. Therefore, $y_1 = y_2$. □

Remark 3.1 *The assertion $y \in C([0,T]; V')$ with $y|_{t=0} = y^0$ for $\alpha > \frac{1}{2}$ in Theorem* 3.1 *follows from Kubica and Yamamoto (2017), Proposition 6.7.*

4. Fractional control problem

This section is devoted to finding the existence of optimal solution, which minimizes the cost functional (1.1), and to deriving the optimality conditions.

Let $u \in L^2((a,b)\times(0,T)) \hookrightarrow L^2((0,T);V')$. Then, in view of Theorem 3.1, the solution $y(u)$ of (1.1)-(1.2) belongs to $W^\alpha(y^0, V, L^2(a,b))$. Moreover, since $W^\alpha(y^0, V, L^2(a,b)) \hookrightarrow L^2((a,b)\times(0,T))$, the functional

$$\begin{aligned} J(y,u) &= \frac{1}{2}\int_0^T\int_a^b |y(x,t)-z_d(x,t)|^2 dxdt + \frac{\nu}{2}\int_0^T\int_a^b |u(x,t)|^2 dxdt \\ &= \frac{1}{2}\|y-z_d\|^2_{L^2((a,b)\times(0,T))} + \frac{\nu}{2}\|u\|^2_{L^2((a,b)\times(0,T))}, \end{aligned} \tag{4.1}$$

is well defined, where $z_d \in L^2((a,b)\times(0,T))$ and $\mu > 0$. Now, we define with $u \in \mathscr{U}_{ad} := L^2((a,b)\times(0,T))$,

$$\mathscr{A}_{ad} := \Big\{(y,u) : y \text{ is a unique weak solution to (1.2) with control } u \in \mathscr{U}_{ad}\Big\}, \tag{4.2}$$

and then the optimal control problem (1.1)-(1.2) can be interpreted as follows:

$$\begin{aligned} &\min_{(y,u)\in\mathscr{A}_{ad}} J(y,u) \\ &\text{subject to (1.2).} \end{aligned} \tag{4.3}$$

A solution to the problem (4.3) is called an optimal solution, denoted by $(\bar{y},\bar{u})$, and the corresponding control is called an optimal control. The following theorem states the existence of an optimal pair.

Theorem 4.1 *Let $y^0 \in L^2(a,b)$. Then there exists a unique optimal pair $(\bar{y},\bar{u}) \in \mathscr{A}_{ad}$ such that (4.3) holds.*

Proof Let us first define

$$M := \inf_{u\in\mathscr{U}_{ad}} J(y,u).$$

As $J(y,u) \geq 0$, there exists a minimizing sequence $\{u_n\} \in \mathscr{U}_{ad}$ such that

$$J(y_n,u_n) \to M, \text{ as } n\to\infty, \tag{4.4}$$

where $y_n = y(u_n)$ is a solution of (1.2), i.e., y_n satisfies:

$$\begin{aligned} {}_CD^\alpha_{0,t} y_n(x,t) + {}_CD^\beta_{x,b}(\gamma(x)\,{}_{RL}D^\beta_{a,x} y_n(x,t)) + q(x)y_n(x,t) = f(x,t) + u_n(x,t),& \\ (x,t)\in(a,b)\times(0,T),& \\ y_n(x,0) = y^0(x), \quad x\in(a,b),& \\ D^{-(1-\beta)}_{a,x} y_n(a,t) = 0 \quad t\in(0,T),& \\ \gamma(b)\,{}_{RL}D^\beta_{a,x} y_n(b,t) = 0, \quad t\in(0,T).& \end{aligned} \tag{4.5}$$

In view of the weak formulation (3.6) and its equivalence with the operator equation (3.7), we find that the pair (y_n, u_n) satisfies

$$\frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(y_n - y^0)](t) + A(t)y_n(t) = f(t) + u_n(t), \qquad \text{a.e. } t \in (0,T), \tag{4.6}$$

where the operator $A(t) : V \to V^{'}$ is given by

$$\langle A(t)y_n(t), \psi\rangle_{V'\times V} = a(t, y_n(t), \psi), \quad \text{a.e. } t \in (0,T).$$

Since $0 \in \mathscr{U}_{ad}$, we may assume that $J(y_n, u_n) \leq J(y, 0) < \infty$, where $(y, 0) \in \mathscr{A}_{ad}$. In particular, there exists a $C > 0$, such that

$$\int_0^T \|u_n(t)\|^2_{L^2(a,b)} = \|u_n\|^2_{L^2((a,b)\times(0,T))} \leq C.$$

Moreover, from (3.31), we have

$$\|y_n\|_{L^2((0,T);H_a^\beta(a,b))} \leq C.$$

Therefore, there exist subsequences of $\{u_n\}$ and $\{y_n\}$ (still denoted by $\{u_n\}$ and $\{y_n\}$), such that

$$u_n \xrightarrow{w} \bar{u} \quad \text{in } L^2((a,b)\times(0,T)), \tag{4.7}$$

$$y_n \xrightarrow{w} \bar{y} \quad \text{in } L_2((0,T); H_a^\beta(a,b)), \tag{4.8}$$

$$y_n \xrightarrow{w} \bar{y} \quad \text{in } L_2((a,b)\times(0,T)), \tag{4.9}$$

$${}_{RL}D_{a,x}^\beta y_n \xrightarrow{w} {}_{RL}D_{a,x}^\beta \bar{y} \quad \text{in } L_2((a,b)\times(0,T)). \tag{4.10}$$

Since $\{y_n\} \subseteq L_2((0,T);V)$ is bounded, then $\{Ay_n\}$ is bounded in $L_2((0,T);V^{'})$. Indeed,

$$\int_0^T \|Ay_n(t)\|^2_{V'}dt \leq (\|q\|_\infty + \|\gamma\|_\infty)\int_0^T \|y_n(t)\|^2_V dt.$$

Hence, in view of (4.9) and (4.10), by choosing another subsequence, one can get

$$Ay^n \xrightarrow{w} A\bar{y} \quad \text{in } L_2((0,T);V^{'}). \tag{4.11}$$

Let $\psi \in V$. Taking the scalar product of (4.6) with ψ and multiplying the resulting equation by the test function $\phi \in C^\infty([0,T];\mathbb{R})$ with $\phi(T) = 0$, we obtain

$$\int_0^T \left\langle \frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(y_n - y^0)](t), \psi \right\rangle_{V'\times V} \phi(t)dt + \int_0^T \langle A(t)y_n(t), \psi\rangle_{V'\times V}\phi(t)dt =$$
$$= \int_0^T \langle h_n(t), \psi\rangle_{L^2(a,b)}\phi(t)dt,$$

where $h_n(x,t) = u_n(x,t) + f(x,t)$. The integration by parts leads to

$$-\int_0^T \left\langle [D_{0,t}^{-(1-\alpha)}(y_n - y^0)](t), \psi \right\rangle_{V'\times V} \phi'(t)dt + \int_0^T \langle A(t)y_n(t), \psi\rangle_{V'\times V}\phi(t)dt =$$
$$= \int_0^T \langle h_n(t), \psi\rangle_{L^2(a,b)}\phi(t)dt, \tag{4.12}$$

because $[D_{0,t}^{-(1-\alpha)}(y_n - y^0)](0) = 0$. By means of (3.10), and Young's and Hölder's inequality, it is easy to see that the integrals in (4.12) are continuous linear functional on the space V and $L^2(a,b)$ with respect to y_n and u_n, respectively. Therefore, on applying the weak limits (4.7)-(4.11), we get from (4.12)

$$-\int_0^T \left\langle [D_{0,t}^{-(1-\alpha)}(\bar{y} - y^0)](t), \psi \right\rangle_{V'\times V} \phi'(t)dt + \int_0^T \langle A(t)\bar{y}(t), \psi\rangle_{V'\times V}\phi(t)dt =$$
$$= \int_0^T \langle h(t), \psi\rangle_{V'\times V}\phi(t)dt. \tag{4.13}$$

Finally, using a similar argument to that of Step 3 in Theorem 3.1, we have from (4.13) that

$$\frac{d}{dt}[D_{0,t}^{-(1-\alpha)}(\bar{y} - y^0)](t) + A(t)\bar{y}(t) = f(t) + \bar{u}(t), \qquad \text{a.e. } t \in (0,T).$$

Since $(\bar{y}, \bar{u})$ satisfies (4.6), we conclude that $\bar{y}$ is a weak solution of (1.2), corresponding to the control $\bar{u}$. Therefore, $(\bar{y}, \bar{u}) \in \mathscr{A}_{ad}$.

Next, we show that $(\bar{y}, \bar{u})$ is a minimizer, i.e., $M = J(\bar{y}, \bar{u})$. Since the cost function $J(\cdot,\cdot)$ is continuous and convex on $L^2((a,b)\times(0,T))\times L^2((a,b)\times(0,T))$, it follows that $J(\cdot,\cdot)$ is weakly lower semi-continuous. Therefore, we obtain

$$J(\bar{y}, \bar{u}) \le \liminf_{n\to\infty} J(y_n, u_n).$$

Hence, in view of (4.4), we deduce that

$$M \le J(\bar{y}, \bar{u}) \le \liminf_{n\to\infty} J(y_n, u_n) = \lim_{n\to\infty} J(y_n, u_n) = M,$$

and thus $(\bar{y}, \bar{u})$ is a minimizer of the problem (4.3). The uniqueness follows from the strict convexity of the cost functional $J(\cdot,\cdot)$. $\square$

Now, we derive the optimality system for the considered optimal control problem (1.1)-(1.2). Before stating the result, we define the following space:

$$W^\alpha(V, L^2(a,b)) := \left\{ p \in L^2((0,T); V) : \; D_{0,t}^{-(1-\alpha)} p \in H^1((0,T); V') \right\}.$$

Theorem 4.2 *Let $(\bar{y},\bar{u})$ be an optimal solution for the problem* (1.1)-(1.2), *i.e.,* $(\bar{y},\bar{u})$ *satisfies* (4.3). *Then there exists a unique $p \in W^{\alpha}(V, L^2(a,b))$ such that* $(\bar{y},\bar{u},p)$ *satisfies the following optimality system:*

$$\begin{cases} {}_{C}D^{\alpha}_{0,t}\bar{y}(x,t) + {}_{C}D^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}\bar{y}(x,t)) + q(x)\bar{y}(x,t) & = f(x,t) + \bar{u}(x,t), \\ & (x,t) \in (a,b)\times(0,T), \\ \bar{y}(x,0) & = y^0(x), \quad x \in (a,b), \\ D^{-(1-\beta)}_{a,x}\bar{y}(a,t) & = 0, \quad t \in (0,T), \\ \gamma(b)_{RL}D^{\beta}_{a,x}\bar{y}(b,t) & = 0, \quad t \in (0,T), \end{cases} \tag{4.14}$$

$$\begin{cases} {}_{C}D^{\alpha}_{t,T}p(x,t) + {}_{C}D^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}p(x,t)) + q(x)p(x,t) & = \bar{y}(x,t) - z_d(x,t), \\ & (x,t) \in (a,b)\times(0,T), \\ p(x,T) & = 0, \quad x \in (a,b), \\ D^{-(1-\beta)}_{a,x}p(a,t) & = 0, \quad t \in (0,T), \\ \gamma(b)_{RL}D^{\beta}_{a,x}p(b,t) & = 0, \quad t \in (0,T), \end{cases} \tag{4.15}$$

and

$$\bar{u}(x,t) = -\frac{p(x,t)}{\nu}, \quad (x,t) \in (a,b)\times(0,T). \tag{4.16}$$

Proof Theorem 4.1 implies that $\bar{y}$ is a unique solution of (1.2), corresponding to the control $\bar{u}$. Consequently, (4.14) follows. In order to prove (4.15) and (4.16), we proceed via the formal Lagrange method. To this end, we define the Lagrangian function as

$$\begin{aligned} &\mathscr{L}(y,u,p) \\ &= J(y,u) - \left[\int_0^T \int_a^b \left({}_{C}D^{\alpha}_{0,t}y + {}_{C}D^{\beta}_{x,b}(\gamma_{RL}D^{\beta}_{a,x}y) + qy - f - u\right) p \; dxdt\right] \end{aligned}$$

where p is the Lagrange multiplier function, defined on $(a,b)\times(0,T)$.

Since $(\bar{y},\bar{u})$ is an optimal solution, it follows that

$$D_y\mathscr{L}(\bar{y},\bar{u},p)(\hat{y}) = 0, \tag{4.17}$$

$$D_u\mathscr{L}(\bar{y},\bar{u},p)(\hat{u}) = 0, \quad \forall\ \hat{u} \in L^2((a,b)\times(0,T)), \tag{4.18}$$

where the state $\hat{y}$, associated to control $\hat{u}$, is a solution to STFPE

$$\begin{aligned}
{}_CD^{\alpha}_{0,t}\hat{y}(x,t) + {}_CD^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}\hat{y}(x,t)) + q(x)\hat{y}(x,t) = f(x,t) + \hat{u}(x,t),\\
(x,t) \in (a,b)\times(0,T),\\
\hat{y}(x,0) = 0, \quad x \in (a,b),\\
D^{-(1-\beta)}_{a,x}\hat{y}(a,t) = 0, \quad t \in (0,T),\\
\gamma(b)_{RL}D^{\beta}_{a,x}\hat{y}(b,t) = 0, \quad t \in (0,T).
\end{aligned} \tag{4.19}$$

Now, using (4.17) and the fact that the derivative of a linear and continuous mapping is the mapping itself, we get

$$\begin{aligned}
&D_y\mathscr{L}(\bar{y},\bar{u},p)(\hat{y}) = \Big[\int_0^T\int_a^b (\bar{y}(x,t) - z_d(x,t))\hat{y}(x,t)dxdt\\
&-\int_0^T\int_a^b \Big({}_CD^{\alpha}_{0,t}\hat{y}(x,t) + {}_CD^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}\hat{y}(x,t)) + q(x)\hat{y}(x,t)\Big)p(x,t)dxdt\Big]\\
&= 0.
\end{aligned} \tag{4.20}$$

Using the fractional integration by parts formula for left Caputo fractional derivative, given in Antil, Otarola and Salgado (2016), Lemma 3 and Lemma 2.17, it follows that

$$\begin{aligned}
&-\int_0^T\int_a^b \Big({}_CD^{\alpha}_{0,t}\hat{y}(x,t) + {}_CD^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}\hat{y}(x,t)) + q(x)\hat{y}(x,t)\Big)p(x,t)dxdt\\
&= -\int_0^T\int_a^b\Big({}_CD^{\alpha}_{t,T}p(x,t) + {}_CD^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}p(x,t)) + q(x)p(x,t)\Big)\hat{y}(x,t)dxdt\\
&\quad -\int_a^b p(x,T)D^{-(1-\alpha)}_{0,t}\hat{y}(x,T)dx - \int_0^T D^{-(1-\beta)}_{0,t}\hat{y}(b,t)(\gamma(b)_{RL}D^{\beta}_{a,x}p(b,t))dt\\
&\quad -\int_0^T \gamma(a)_{RL}D^{\beta}_{a,x}\hat{y}(a,t)D^{-(1-\beta)}_{0,t}p(a,t) = 0, \quad \forall \hat{y} \in W^{\alpha}(V, L^2(a,b)).
\end{aligned} \tag{4.21}$$

In view of (4.21) and previous computations, one can rewrite (4) as follows:

$$\begin{aligned}
&D_y\mathscr{L}(\bar{y},\bar{u},p)(\hat{y})\\
&= \langle \bar{y} - z_d, \hat{y}\rangle_{L^2((a,b)\times(0,T))} - \Big\langle {}_CD^{\alpha}_{t,T}p + {}_CD^{\beta}_{x,b}(\gamma(x)_{RL}D^{\beta}_{a,x}p) + qp, \hat{y}\Big\rangle_{L^2((a,b)\times(0,T))}\\
&\quad -\Big\langle p(T), (D^{-(1-\alpha)}_{0,t}\hat{y})(T)\Big\rangle_{L^2(a,b)} - \int_0^T D^{-(1-\beta)}_{0,t}\hat{y}(b,t)(\gamma(b)_{RL}D^{\beta}_{a,x}p(b,t))dt\\
&\quad -\int_0^T \gamma(a)_{RL}D^{\beta}_{a,x}\hat{y}(a,t)D^{-(1-\beta)}_{0,t}p(a,t) = 0.
\end{aligned} \tag{4.22}$$

Now, for $p(t) \in V$ and choosing, in particular, $\hat{y} \in C_c^\infty\big((a,b)\times(0,T)\big)$ with $\hat{y}(x,t) = \varphi(x)\phi(t)$, where $\phi(t)$ solves the equation $D_{0,t}^{-(1-\alpha)}\phi(t) = \eta(t)$ with $\eta \in C_c^\infty(0,T)$, we obtain from (4.22)

$$\int_0^T \Big\langle {}_C D_{t,T}^{\alpha} p + {}_C D_{t,T}^{\alpha} p + {}_C D_{x,b}^{\beta}(\gamma(x)_{RL} D_{a,x}^{\beta} p) + qp - (\bar{y} - z_d), \varphi \Big\rangle_{L^2(a,b)} \phi(t)dt = 0.$$

Then, according to Samko, Kilbas and Marichev (1993), Theorems 13.2 and 13.5, the above relation must hold for all $\phi \in C_c^\infty(0,T)$ and, thus, it follows from the density arguments that

$$ {}_C D_{t,T}^{\alpha} p(x,t) + {}_C D_{x,b}^{\beta}(\gamma(x)_{RL} D_{a,x}^{\beta} p(x,t)) + q(x)p(x,t) = \bar{y}(x,t) - z_d(x,t), \quad \text{in } (a,b)\times(0,T). \tag{4.23}$$

It remains to show that $p(x,T) = 0$. To do so, we notice that we are left with

$$\Big\langle p(T), (D_{0,t}^{-(1-\alpha)}\hat{y})(T)\Big\rangle_{L^2(a,b)} = 0. \tag{4.24}$$

If we now consider $\hat{y}$ as a function, which is constant in time, then from (4.24) we have $p(T) = 0$. However, since $\hat{y}(x,0) = 0$, the only acceptable constant-in-time function would be $\hat{y} \equiv 0$. To avoid this, we take $\hat{y} = l_\epsilon(t)\zeta(x)$, where $\zeta \in C_c^\infty(a,b)$ is arbitrary, and $l_\epsilon(t)$ is given by

$$l_\epsilon(t) = \begin{cases} \epsilon^{-\alpha}T^{-\alpha}t^{\alpha}, & 0 < t \le \epsilon T, \\ 1, & \epsilon T < t \le T \end{cases}.$$

In view of this particular $\hat{y}$, (4.24) becomes

$$\left[\int_a^b p(x,T)\zeta(x)dx\right](D_{0,t}^{-(1-\alpha)}l_\epsilon)(T) = 0. \tag{4.25}$$

Since $\lim_{\epsilon\to 0}(D_{0,t}^{-(1-\alpha)}l_\epsilon)(T) = \lim_{\epsilon\to 0}(D_{0,t}^{-(1-\alpha)}1)(T) > 0$, it follows from (4.25) that

$$p(x,T) = 0, \quad x \in (a,b). \tag{4.26}$$

Moreover, using (4.18), one gets

$$\int_0^T\int_a^b \big[\mu\bar{u}(x,t) + p(x,t)\big]\hat{u}(x,t)dxdt = 0 \quad \forall\ \hat{u} \in L_2((a,b)\times(0,T)),$$

which implies

$$\bar{u}(x,t) = -\frac{p(x,t)}{\nu}, \quad \text{in } (a,b)\times(0,T),$$

which, in turn, is the required optimality condition (4.16).

Finally, we prove the well-posedness of the adjoint system (4.15) by converting the backward in time problem with the right Caputo derivative into a forward in time problem with left Caputo derivative. Indeed, by making the change of variable $t \to T-t$ in (4.15) with setting $\mathscr{T}_T p(t) = p(T-t)$, $t \in [0,T]$, one could obtain, see Mophou (2011), Proposition 3.6,

$$\begin{cases} {}_C D^{\alpha}_{0,t} \mathscr{T}_T p(x,t) + {}_C D^{\beta}_{x,b}(\gamma(x)_{RL} D^{\beta}_{a,x} \mathscr{T}_T p(x,t)) + q(x)\mathscr{T}_T p(x,t) \\ = \mathscr{T}_T(\bar{y}(x,t) - z_d(x,t)),\ x \in (a,b),\ T-t \in (0,T), \\ \mathscr{T}_T p(x,0) = 0, \qquad\qquad x \in (a,b), \\ D^{-(1-\beta)}_{a,x} \mathscr{T}_T p(a,t) = 0, \quad T-t \in (0,T), \\ \gamma(b)_{RL} D^{\beta}_{a,x} \mathscr{T}_T p(b,t) = 0,\ T-t \in (0,T). \end{cases} \tag{4.27}$$

Hence, by letting $T-t = \tau \in [0,T]$ in (4.27), we deduce that solution of (4.15) is equivalent to solution of (3.1) with homogeneous initial condition and the right hand side

$$\bar{y} - z^d \in L^2((a,b)\times(0,T)) \hookrightarrow L^2((0,T);V').$$

Thus, in view of Theorem 3.1, we infer that (4.15) has a unique solution $p \in W^{\alpha}(V, L^2(a,b))$. $\square$

Remark 4.1 *Since the cost functional $J(\cdot,\cdot)$ is convex, the converse of Theorem 4.2 is also true, i.e., any pair $(\bar{y},\bar{u})$ and the solution p of (4.15), which satisfies (4.16), is an optimal solution. Therefore, a control u, along with the optimal state y and the adjoint state p is optimal for (1.1)-(1.2) if and only if the triplet (u,y,p) satisfies the following optimality system:*

$$\boxed{\begin{array}{rr} & {}_C D^{\alpha}_{0,t} y + {}_C D^{\beta}_{x,b}(\gamma(x)_{RL} D^{\beta}_{a,x} y) + qy = f + u = f - \dfrac{p}{\nu} \\ & {}_C D^{\alpha}_{t,T} p + {}_C D^{\beta}_{x,b}(\gamma(x)_{RL} D^{\beta}_{a,x} p) + qp = y - z_d \\ & \textit{in } (a,b)\times(0,T) \\ y(x,0) = y^0(x) & p(x,T) = 0, \quad x \in (a,b), \\ D^{-(1-\beta)}_{a,x} y(a,t) = 0 & D^{-(1-\beta)}_{a,x} p(a,t) = 0, \quad t \in (0,T), \\ \gamma(b)_{RL} D^{\beta}_{a,x} y(b,t) = 0 & \gamma(b)_{RL} D^{\beta}_{a,x} p(b,t) = 0, \quad t \in (0,T). \end{array}}. \tag{4.28}$$

Remark 4.2 *For the sake of completeness, we add that control constraints can be handled in this context with little extra work. Let us, therefore, introduce the set of constraints $U_d := \{u : a(x,t) \le u(x,t) \le b(x,t),\ (x,t) \in (a,b)\times(0,T),\ i = 1,2,\ldots,k\}$, where both the function a and b belongs [????] to $L^2((a,b)\times(0,T))$. Indeed, instead of equality in (4.18), using standard variational theory, we obtain the following variational inequality*

$$D_u \mathcal{L}(\bar{y},\bar{u},p)(u-\bar{u}) \ge 0 \quad \forall\ u \in U_d,$$

which gives

$$\int_0^T \int_a^b \left[\nu \bar{u}(x,t) + p(x,t)\right] (u(x,t) - \bar{u}(x,t)) dx dt \geq 0, \quad \forall u \in U_d.$$

Equivalently, we have

$$\langle \nu \bar{u} + p, u - \bar{u} \rangle_{L^2((a,b)\times(0,T))} \geq 0, \quad \forall u \in U_d. \tag{4.29}$$

5. Finite difference approximation

In this section, we propose a finite difference scheme for the approximation of the considered FOCP (1.1)-(1.2) while assuming the sufficient regularity of the solution in both the space and time variables. In view of Remark 4.1, it follows that the solution of the optimal control problem (1.1)-(1.2) requires the solvability of the optimality system (4.28). Therefore, we first develop a difference scheme for the STFPE (3.1), which then leads to the approximation of the optimality system.

We begin with the time discretization of the problem. Let $N \in \mathbb{N}$ be the number of time steps. Define the uniform time step $\Delta t = \frac{T}{N} > 0$, and set $t_n = n\Delta t$ for $0 \leq n \leq N$ with $t_0 = 0$ and $t_N = T$. Moreover, we denote $y^n := y(t_n)$ for $y \in C[0,T]$. We recall the well-known L1 method for the discrete approximation of left Caputo fractional derivative (see Lin and Xu, 2007; Alikhanov, Beshtokov and Mehra, 2021),

$$\begin{aligned}
{}_C D_{0,t}^{\alpha} y(t_{n+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{-\alpha} y'(\tau) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \left(\int_{t_s}^{t_{s+1}} (t_{n+1} - \tau)^{-\alpha} y'(\tau) d\tau \right) \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \frac{y^{s+1} - y^s}{\Delta t} \left(\int_{t_s}^{t_{s+1}} (t_{n+1} - \tau)^{-\alpha} d\tau \right) \\
&= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{n} b_{s,\alpha} \left[y^{n+1-s} - y^{n-s} \right],
\end{aligned}$$

where $b_{s,\alpha} = (s+1)^{1-\alpha} - s^{1-\alpha}$, $0 \leq s \leq n$, $0 < \alpha < 1$ and $1 \leq i \leq k$, and provided the sum for $n = 0$ is defined to be zero. Furthermore, if we define the discrete fractional differential operator as

$$L_t^{\alpha} y(t_{n+1}) := \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{n} b_{s,\alpha} \left[y^{n+1-s} - y^{n-s} \right], \tag{5.1}$$

then we have the following result:

LEMMA 5.1 (LIN AND XU, 2007) *Suppose* $0 < \alpha < 1$ *and* $y \in C^2[0, t_n]$*, then*

$$ {}_C D^{\alpha}_{0,t} y(t_n) = L^{\alpha}_t y(t_n) + \mathcal{O}\left(\Delta t^{2-\alpha}\right). $$

REMARK 5.1 *Before the spatial discretization of the problem, we first analyze the fractional boundary conditions* $(3.1)_3$ *and* $(3.1)_4$ *for sufficiently smooth functions. For any function* $d \in C[a,b]$ *and* $\beta \in (0,1)$*, we can find that* $D^{-(1-\beta)}_{a,x} d(a) = 0$*. Indeed,*

$$ \begin{aligned} |D^{-(1-\beta)}_{a,x} d(x)| &\leq \frac{1}{\Gamma(1-\beta)} \int_a^x (x-\tau)^{-\beta} |d(\tau)| d\tau \\ &\leq \frac{1}{\Gamma(1-\beta)} \|d\|_\infty \int_a^x (x-\tau)^{-\beta} d\tau \\ &= \frac{1}{\Gamma 2-\beta)} \|d\|_\infty (x-a)^{1-\beta}, \end{aligned} $$

which gives $D^{-(1-\beta)}_{a,x} d(a) = 0$*. Therefore, one does not require the discrete approximation for* $(3.1)_3$*. Moreover, using* (2.2)*, one can observe that*

$$ \begin{aligned} & {}_C D^{\beta}_{x,b} (\gamma(x) {}_{RL} D^{\beta}_{a,x} y(x,t)) \\ &= {}_{RL} D^{\beta}_{x,b} (\gamma(x) {}_{RL} D^{\beta}_{a,x} y(x,t)) - \gamma(b) \frac{{}_{RL} D^{\beta}_{a,x} y(b,t)}{\Gamma(1-\beta)} (b-x)^{-\beta}, \quad \beta \in (0,1), \end{aligned} $$

which, on applying boundary condition $(3.1)_4$*, gives*

$$ {}_C D^{\beta}_{x,b} (\gamma(x) {}_{RL} D^{\beta}_{a,x} y(x,t)) = {}_{RL} D^{\beta}_{x,b} (\gamma(x) {}_{RL} D^{\beta}_{a,x} y(x,t)), \ (x,t) \in (a,b) \times (0,T). \tag{5.2} $$

Now, in order to do the space discretization, we discretize the interval (a,b) as $x_m = m\Delta x$, where Δx denotes the spatial discretization step, given by $\Delta x = (b-a)/M$, $m = 0, 1, \ldots, M$, $x_0 = a$ and $x_M = b$. The approximation of the exact solution y at a mesh point (x_m, t_{n+1}) is denoted by Y^{n+1}_m.

We employ the standard Grünwald-Letnikov formula for the approximation of Riemann-Liouville fractional derivatives, given as

$$ \begin{aligned} [{}_{RL} D^{\beta}_{a,x} d(x)]_{x=x_m} &= \frac{1}{\Delta x^\beta} \sum_{k=0}^{m} w^{(\beta)}_k d(x_{m-k}) + \mathcal{O}(\Delta x), \\ [{}_{RL} D^{\beta}_{x,b} d(x)]_{x=x_m} &= \frac{1}{\Delta x^\beta} \sum_{k=0}^{M-m} w^{(\beta)}_k d(x_{m+k}) + \mathcal{O}(\Delta x), \end{aligned} $$

where $w^{(\beta)}_k = (-1)^k \begin{pmatrix} \beta \\ k \end{pmatrix}$.

Hence, based on above formula, the spatial fractional derivative in the considered problem can be approximated as

$$
\begin{aligned}
&[{}_C D^{\beta}_{x,b}(\gamma(x)\, {}_{RL}D^{\beta}_{a,x} y(x,t))]_{(x_m,t_{n+1})} \\
&= [{}_{RL} D^{\beta}_{x,b}(\gamma(x)\, {}_{RL}D^{\beta}_{a,x} y(x,t))]_{(x_m,t_{n+1})} \\
&\approx \frac{1}{\Delta x^{\beta}} \sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \left(\frac{1}{\Delta x^{\beta}} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y^{n+1}_{m+k-\ell} \right) \\
&= \frac{1}{\Delta x^{2\beta}} \sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y^{n+1}_{m+k-\ell}.
\end{aligned} \tag{5.3}
$$

Therefore, using (5.1) and (5.3), we obtain the following difference scheme for (3.1):

$$
\begin{aligned}
&L_t^{\alpha} Y_m^{n+1} + \frac{1}{\Delta x^{2\beta}} \sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y^{n+1}_{m+k-\ell} + q_m Y_m^{n+1} = h_m^{n+1}, \\
&\qquad m = 0,1,\dots,M, \quad n = 0,1,2,\dots,N-1,
\end{aligned} \tag{5.4}
$$

which further can be expressed as follows:

$$
Y_m^1 + r \sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y^1_{m+k-\ell} + \left[\Delta t^{\alpha}\Gamma(2-\alpha)\right] q_m Y_m^1 = \Delta t^{\alpha}\Gamma(2-\alpha) h_m^1, \tag{5.5}
$$

$$
\begin{aligned}
&Y_m^{n+1} + r \sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y^{n+1}_{m+k-\ell} + \left[\Delta t^{\alpha}\Gamma(2-\alpha)\right] q_m Y_m^{n+1} \\
&= (1-b_{1,\alpha}) Y_m^n + \sum_{s=1}^{n-1} (b_{s,\alpha} - b_{s+1,\alpha}) Y_m^{n-s} + b_{n,\alpha} Y_m^0 + \Delta t^{\alpha}\Gamma(2-\alpha) h_m^{n+1}, \\
&\qquad m = 0,1,\dots,M, \quad n = 1,2,\dots,N-1,
\end{aligned} \tag{5.6}
$$

where

$$
r = \frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{\Delta x^{2\beta}}.
$$

We note that the inclusion of $m = 0$ and $m = M$ is necessary in the difference scheme since we need to compute the approximate value of the functions at the boundary as well.

Further, if we denote the column vectors in the following manner:

$$\begin{aligned}
\mathbf{Y}^n &= (Y_0^n, Y_1^n, \ldots, Y_M^n)^T\,,\\
\mathbf{Q}^{n-1} &= \left(Y_0^{n-1}, Y_1^{n-1}, \ldots, Y_M^{n-1}\right),\\
\mathbf{H}^n &= \left(\Delta t^\alpha \Gamma(2-\alpha) h_1^n, \Delta t^\alpha \Gamma(2-\alpha) h_2^n, \ldots, \Delta t^\alpha \Gamma(2-\alpha) h_M^n\right),
\end{aligned}$$

for $m = 0, 1, \ldots, M$, then the difference scheme (5.5)-(5.6) can be rewritten in the matrix form as follows

$$\begin{aligned}
&A\mathbf{Y}^1 = \mathbf{Q}^0 + \mathbf{H}^1,\\
&A\mathbf{Y}^{n+1} = (1 - b_{1,\alpha})\mathbf{Q}^n + \sum_{s=1}^{n-1}(b_{s,\alpha} - b_{s+1,\alpha})\mathbf{Q}^{n-s} + b_{n,\alpha}Q^0 + \mathbf{H}^{n+1},
\end{aligned}$$

where $n = 1, 2, \ldots, N-1$ and the coefficient matrix $A = (a_{i,j})_{i,j=1}^{M+1}$ is a symmetric matrix with the entries

$$a_{i,j} = \begin{cases} 1 + r\displaystyle\sum_{k=0}^{M-i+1}\left(w_k^{(\alpha)}\right)^2 \gamma_{k+i-1} + \Delta t^\alpha \Gamma(2-\alpha) q_{i-1} & \text{if } i = j,\\ r\displaystyle\sum_{k=0}^{M-i+1} w_k^{(\alpha)} w_{k+i-j}^{(\alpha)} \gamma_{k+i-1} & \text{if } i < j. \end{cases}$$

Now, to obtain the difference scheme for the optimality system, we first discretize the right Caputo fractional derivative, present in the adjoint equation (4.15). The discrete approximation of the right Caputo fractional derivative by the L1 method is given by

$$\begin{aligned}
L_T^\alpha p^n &= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=n}^{N-1} b_{s-n,\alpha}\left[p^s - p^{s+1}\right]\\
&= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{N-n-1} b_{s,\alpha}\left[p^{s+n} - p^{s+n+1}\right],
\end{aligned} \tag{5.7}$$

where L_T^α denotes the discrete right fractional differential operator.

Hence, in view of (5.4) and (5.7), we obtain the discrete version of the optimality system, given as

$$\begin{cases} L_t^\alpha Y_m^{n+1} + \frac{1}{\Delta x^{2\beta}} \displaystyle\sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} Y_{m+k-\ell}^{n+1} + q_m Y_m^{n+1} = f_m^{n+1} - \frac{P_m^{n+1}}{\nu},\\ L_T^\alpha P_m^n + \frac{1}{\Delta x^{2\beta}} \displaystyle\sum_{k=0}^{M-m} w_k^{(\beta)} \gamma_{m+k} \sum_{\ell=0}^{m+k} w_\ell^{(\beta)} P_{m+k-\ell}^n + q_m P_m^n = Y_m^n - z_{d,m}^n,\\ Y_m^0 = y^0(x_m), \qquad P_m^N = 0, \quad m = 0, 1, \ldots, M, \quad n = 0, 1, 2, \ldots, N-1, \end{cases} \tag{5.8}$$

where P_m^n denotes the approximation of $p(x_m, t_n)$.

Equation (5.8) represents a linear system of $2N(M+1)$ equations in $2N(M+1)$ unknowns (Y_m^{n+1} and P_m^n, $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N-1$). The solution of the linear system gives the value of Y and P at all the grid points. Once the value of P_m^n is known, the approximated control values U_m^n can be obtained using $U_m^n = -\frac{P_m^n}{\nu}$.

6. Numerical results

In this section, we shall demonstrate the accuracy and performance of the difference scheme, proposed in the previous section, via examples. The maximum absolute error between the exact and the approximate solution for the STFPE (3.1) is given by

$$E_y(M, N) := \max_{(x_m, t_n) \in \bar{\Omega}} |y(x_m, t_n) - Y_m^n|,$$

where $\bar{\Omega} = [0, 1] \times [0, T]$. Moreover, we define the maximum spatial error at a fixed time $t = t_n$ between the exact solution and the approximated solution for the state variable and control variable as

$$\begin{aligned} E_{y,\infty}(M, t_n) = \max_{0 \le m \le M} |y(x_m, t_n) - Y_m^n| \quad \text{and} \quad & E_{u,\infty}(M, t_n) \\ := \max_{0 \le m \le M} |u(x_m, t_n) - U_m^n|, & \end{aligned}$$

respectively.

Example 1 *In the first example, we only consider the governing equations, i.e., STFPEs* (3.1)*, with the following data:*

$$\gamma(x) = x^2 + e^x, \ q(x) = \frac{1}{4}\sin(\pi x), \ y^0(x) = x^2(1-x)^2,$$

$$\begin{aligned} h(x,t) = & \left[\frac{\Gamma(3+\alpha)}{2} t^2 + \frac{1}{4}\sin(\pi x)(t^{2+\alpha}+1)\right] x^2(1-x)^2 \\ & -(t^{2+\alpha}+1)\left[(x^2+e^x)(12x^2-12x+2) + (2x+e^x)(2x+4x^3-6x^2)\right], \end{aligned}$$

and $T = 1$ *. It can be checked that* $y(x,t) = (t^{2+\alpha}+1)x^2(1-x)^2$ *is the exact solution of the problem in case of fractional order* $\beta = 1$*.*

We investigate the maximum absolute error for the above example when $\beta = 1$. The error for different values of α is shown in Table 1. It can be observed that as the number of subintervals is increased (increment in the value of M and N), a reduction in the error occurs, which ensures the convergence of the method.

Table 1. The maximum absolute error for Example 1 for different values of α

	$\alpha = 0.2,\ \beta = 1$	$\alpha = 0.5,\ \beta = 1$	$\alpha = 0.8,\ \beta = 1$
$N = M$	$E_y(M,N)$	$E_y(M,N)$	$E_y(M,N)$
64	$5.352e-2$	$5.256e-2$	$5.14e-2$
128	$2.652e-2$	$2.608e-2$	$2.55e-2$
256	$1.320e-2$	$1.299e-2$	$1.272e-2$
512	$6.586e-3$	$6.484e-3$	$6.35e-3$
1024	$3.289e-3$	$3.239e-3$	$3.173e-3$
2048	$1.643e-3$	$1.618e-3$	$1.586e-3$
4096	$8.216e-4$	$8.092e-4$	$7.931e-4$

EXAMPLE 2 *In this example, we consider the fractional optimal control problem* (1.1)-(1.2) *with the same data as those considered in Example* 1, *i.e.,*

$$\gamma(x) = x^2 + e^x,\ q(x) = \frac{1}{4}\sin(\pi x),\ y^0(x) = x^2(1-x)^2,$$

$$f(x,t) = h(x,t) = \left[\frac{\Gamma(3+\alpha)}{2}t^2 + \frac{1}{4}\sin(\pi x)(t^{2+\alpha}+1)\right]x^2(1-x)^2$$
$$-(t^{2+\alpha}+1)\left[(x^2+e^x)(12x^2-12x+2)+(2x+e^x)(2x+4x^3-6x^2)\right],$$

$T = 1$ *and* $\nu = 1$. *The observed value* $z_d(x,t)$ *is chosen equal to the exact solution of the Example* 1 *(for* $\beta = 1$*), i.e., we take*

$$z_d(x,t) = (t^{2+\alpha}+1)x^2(1-x)^2.$$

It can be easily seen that $(y(x,t), u(x,t)) = ((t^{2+\alpha}+1)x^2(1-x)^2, 0)$ *is the exact solution of the problem in the case of the fractional order* $\beta = 1$.

The discrete optimality system (5.8) has been solved for the data considered in Example 2 for different values of M with $\Delta t = 0.02$ and $\alpha = 0.5$. We examine the maximum spatial error at $t = 0.5$, in the case of $\beta = 1$, for the state variable and the control variable as this is shown in Table 2. It can be observed that as the number of subintervals is increased (increment in the value of M), a reduction in the error occurs for both variables, which ensures the convergence of the method. Moreover, we have plotted the numerical solution for both variables in Fig. 1. And so, Fig. 1 (a) depicts the approximate solution of the state variable, while Figure 1 (b) shows the approximate solution for the control variable for different values of M at $\alpha = 1/2$, $\beta = 1/2$ and $t = 0.4$. From the figures, one can observe that if the approximation of the state variable and the control variable at $M = 128$ is considered to be the reference solution, then the approximate solution for both the variables is converging to the reference solution as the value of M increases.

Table 2. The maximum spatial error for Example 2 at $t = 0.5$ for $\alpha = 0.5$ and $\beta = 1$.

M	$E_{y,\infty}(M, t_{N/2})$	$E_{u,\infty}(M, t_{N/2})$
32	$6.32e-2$	$1.23e-2$
64	$3.07e-2$	$5.60e-3$
128	$1.51e-2$	$2.54e-3$
256	$7.54e-3$	$1.09e-3$
512	$3.79e-3$	$3.91e-4$

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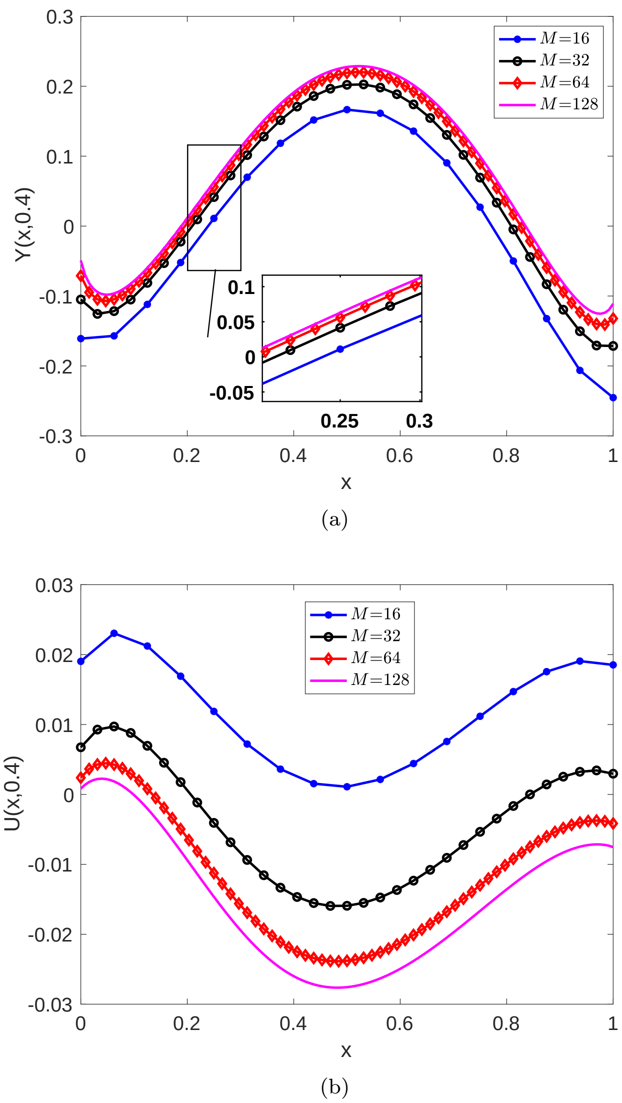

Figure 1. Approximate solution for the state variable y (top) and control variable u (bottom) for different values of M for the optimal control problem, defined in Example 2 at $t = 0.4$ with $\Delta t = 0.02$, $\alpha = 0.5$ and $\beta = 0.5$

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