GEODESY AND CARTOGRAPHY
Vol. 65, No 2, 2016, pp. 283-312

DOI: 10.1515/geocart-2016-0017

Problems and methods of calculating the Legendre functions of arbitrary degree and order

Elena Novikova*, Alexander Dmitrenko

Kryvyi Rih National University
11 Vitaly Matusevich St., 50027 Kryvyi Rih, Ukraine
e-mail: novikova131254@mail.ua; alekc.dmitrenko@yandex.ua

* corresponding author: Elena Novikova

Received: 09 August 2016 / Accepted: 14 September 2016

Abstract: The known standard recursion methods of computing the full normalized associated Legendre functions do not give the necessary precision due to application of IEEE754-2008 standard, that creates a problems of underflow and overflow. The analysis of the problems of the calculation of the Legendre functions shows that the problem underflow is not dangerous by itself. The main problem that generates the gross errors in its calculations is the problem named the effect of "absolute zero". Once appeared in a forward column recursion, "absolute zero" converts to zero all values which are multiplied by it, regardless of whether a zero result of multiplication is real or not. Three methods of calculating of the Legendre functions, that removed the effect of "absolute zero" from the calculations are discussed here. These methods are also of interest because they almost have no limit for the maximum degree of Legendre functions. It is shown that the numerical accuracy of these three methods is the same. But, the CPU calculation time of the Legendre functions with Fukushima method is minimal. Therefore, the Fukushima method is the best. Its main advantage is computational speed which is an important factor in calculation of such large amount of the Legendre functions as 2 401 336 for EGM2008.

Keywords: Fully Normalized Associated Legendre Functions, recursion, underflow problem, overflow problem, Fukushima method

1. Introduction

One of the components of the Earth's gravity field is the gravitational potential V which is determined according to the following formula (Heiskanen and Moritz, 1967):

$$V = \frac{GM}{r}\left(1+\sum_{n=2}^{M}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n}\overline{P}_{nm}(\sin\phi)(\overline{C}_{nm}\cos m\lambda+\overline{S}_{nm}\sin m\lambda)\ \right), \qquad (1)$$

where GM is the product of the Universal gravitational constant and the mass of the Earth; r, ϕ, λ are geocentric spherical coordinates of a point in the space: radius, geocentric latitude and longitude respectively; $\overline{C}_{nm}$, $\overline{S}_{nm}$ are fully normalized spherical harmonic gravitational coefficients; $\overline{P}_{nm}$ are fully normalized Associated Legendre functions (fnALFs).

The value a in Eq. (1) is the equatorial radius of the Earth model. As a rule, this value is equated with the semi-major axis of the Earth ellipsoid (Moritz, 1980). Integer values n and m are degree and order respectively (Moritz, 1980). According to Eq. (1) these values vary from 0 to M which is the maximal degree of the model.

For global gravitational models EGM96 (Nima, 2000) and EGM2008 (Pavlis, 2012) maximal degrees are:

$$M_{EGM96} = 360, \quad M_{EGM2008} = 2190.$$

The total number of the fnALFs, necessary for computing the model, can be determined according to the formula:

$$N_{ALFs} = \frac{(M+1)(M+2)}{2}. \tag{2}$$

Thus, to calculate V using the model EGM96 it is necessary to calculate 65 341 Legendre functions, using the model EGM2008 – 2 401 336 functions.

Mathematical literature gives a great variety of recurrent formulas for computing fnALFs (Abramowitz, 1972; Olver, 2010). For the first global gravitational models the value of the maximal degree M did not exceed 30 (http://icgem.gfz-potsdam.de/ICGEM/ICGEM.html) that corresponded to the set of 496 fnALFs. With such comparatively small number of Legendre functions, the choice of a recurrent formula did not matter very much. However, gravitational models with the maximal degree of 300 and more had appeared, the majority of recurrent formulas proved to be unsuitable for computation because of their instability. Instability of the formulas becomes apparent when computing a number small in absolute value, intermediate calculations appear close to or greater than 10^{308}, that, at the best, results in a loss of precision and, at the worst, in overflow. The latter case is connected with application of IEEE (Institute of Electrical and Electronic Engineers) 754 standard (Kahan, 1996), according to which all calculations with double precision numbers with absolute values greater than $1{,}8 \times 10^{308}$ are considered equal to infinity and designated as NaN (Not a Number). As shown in (Holmes and Featherstone, 2002) the recurrent formulas called forward column recursion and forward row recursion are free from the overflow problem. But these recursions have the underflow problem, that is also associated with the standard IEEE754-2008, according to which, the number X, which satisfies the condition in the environment of double-precision:

$$4.9 \times 10^{-324} \le |X| < 2.2 \times 10^{-308}, \tag{3}$$

is called subnormal. The smaller absolute value of a subnormal number corresponds to less precision. As a result, the precision of fnALFs, the calculation of which involves subnormal numbers, also drops sharply. If the numbers in absolute value are smaller than the smallest subnormal number, i.e.

$$|X| < 4.9 \times 10^{-324}, \tag{4}$$

they are equal to zero. Forward column recursion is less susceptible to the underflow problem. It also completely has no the overflow problem. Therefore, it formed the basis of all modern methods for the calculation of fnALFs. According to (Holmes and Featherstone, 2002a), it can be written as:

$$\overline{P}_{nm} = \sin\phi \; a_{nm} \; \overline{P}_{n-1,m} - b_{nm} \; \overline{P}_{n-2,m}, \tag{5}$$

where

$$\left.\begin{aligned} a_{nm} &= \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}} \\ b_{nm} &= \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}} \end{aligned}\right\}, \tag{6}$$

for $2 \le n \le M$, $0 \le m \le n-2$.
Before using Eq. (5) for forward column recursion, values $\overline{P}_{mm}, \overline{P}_{m,m-1}$ must be calculated (Tschering et al., 1983; Holmes and Featherstone, 2002a):

$$\overline{P}_{0,0} = 1, \; \overline{P}_{1,1} = \sqrt{3}\cos\phi, \tag{7}$$

$$\overline{P}_{mm} = \sqrt{3}\prod_{i=2}^{m}\sqrt{\frac{2i+1}{2i}}\cos^{m}\phi, \tag{8}$$

$$\overline{P}_{m,m-1} = \sin\phi\sqrt{2m+1}\;\overline{P}_{m-1,m-1}, \tag{9}$$

Analysis of fnALFs, calculated by Eq. (5), shows (Holmes and Featherstone, 2002a) that with increasing latitude, the accuracy of the formula drops due to the underflow problem. There is no underflow problem at the equator.

To calculate fnALFs at the poles, Eqs. (5–9) with $\phi = \pm 90^0$ are reduced to the following simple form:

$$\overline{P}_{nm} = 0, \text{ when } m \ne 0, \; \overline{P}_{n,0} = \pm\sqrt{2n+1}. \tag{10}$$

There is no underflow problem for Eqs. (10). Thus, the underflow problem exists only for latitudes $\phi \neq 0$ or $\phi \neq \pm 90^0$.

2. Problems of Computing the Legendre Functions

2.1. Overflow problem

Because (Holmes and Featherstone, 2002a):

$$\sum_{n=0}^{M}\sum_{m=0}^{n}\overline{P}_{nm}^{2} = (M+1)^2 , \tag{11}$$

each member of this sum equals to or less than the sum, i.e.

$$\overline{P}_{nm}^{2} \leq (M+1)^2 , \tag{12}$$

or,

$$\left|\overline{P}_{nm}\right| \leq M+1 . \tag{13}$$

Thus, if the Legendre functions are calculated correctly, and if values appearing in the recursion formulas for their computation are of the same order as the functions themselves, the problem overflow appears only when the value M is overflow. Now there is no such models for which $M > 1.8 \times 10^{308}$. Therefore, the problem overflow correctly computed $\overline{P}_{nm}$ with Eq. (5) does not appear.

But this problem appeared in the modified recursion methods (Holmes and Featherstone, 2002a), when instead of the Legendre functions, functions depending on them, but artificially increased many times were calculated. A manifold increase in the absolute magnitude of the functions involved in recursion formulas removed the underflow problem, but created an overflow problem. Therefore, modified methods, suggested by (Holmes and Featherstone, 2002a), turned out to be impossible to use, when $M > 2700$.

2.2. Underflow problem

Figure 1 shows the result of the calculation fnALFs with logarithmic method (Sect. 3.3) for geodetic latitude $\varphi = 68^0$,which is related to the geocentric latitude ϕ for points on ellipsoid with equation (Jekeli, 2006):

$$\tan\phi = \tan\varphi(1-e^2) , \tag{14}$$

where e^2 is the square of the first eccentricity. For WGS84 Ellipsoid this value equals (Nima, 2000):

$$e^2 = 6{,}69437999014 \times 10^{-3}$$

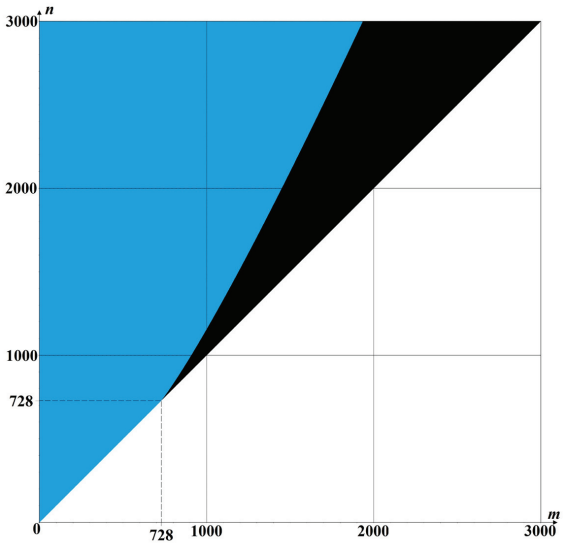

Fig. 1. The result of computation of fnALFs by logarithmic method for $\varphi = 68^0$. Blue pixels correspond to fnALFs, which are normal values, black pixels correspond to fnALFs, which are not normal values

For geodetic latitude $\varphi = 68^0$ geocentric latitude equals:

$$\phi = 1{,}18448528345954 \text{ (rad)}.$$

Every pixel of Figure 1 with the coordinates n, m corresponds to a specific function $\overline{P}_{nm}$, computed in the double precision environment. Black color in this figure shows the pixels representing fnALFs, the values of which satisfy the inequality:

$$\left|\overline{P}_{nm}\right| < 2.2 \times 10^{-308}. \tag{15}$$

The minimum values m, n for which fnALFs satisfy the inequality (15), according to Fig. 1, are equal to:

$$n = m = 728.$$

Blue color in Figure 1 shows the pixels corresponding to fnALFs, the absolute value of which does not satisfy the inequality (15). According to the standard IEEE754-2008 (Kahan, 1996), they are normal values. As can be seen from the

Figure 1, the number of fnALFs, satisfying the inequality (15) increases faster than the total number of fnALFs.

According to Figure 1, for $M = 3000$ and for latitude $\varphi = 68^0$, number of fnALFs, that satisfies the inequalities (15) and in the double-precision environment, practically equals to zero, is not less than 26% of the total number of fnALFs. Taking into consideration the fact that the number of Legendre functions satisfying the inequality (15) with a linear increase of M has non-linear increase, we can assume that at a certain value M the total number of fnALFs, that satisfy the inequality (15), reaches 99% of all fnALFs. However, they do not affect the accuracy of calculation of the gravitational potential. To prove this let's divide the expression by the sum included in Eq. (1) for calculating the gravitational potential, in two parts:

$$\sum_{n=2}^{M}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \overline{P}_{nm}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right) = E_1 + E_2 . \tag{16}$$

E_1 reserves the components with fnALFs, satisfying the condition (15):

$$E_1 = \sum_{n=2}^{M}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \overline{P}^*_{nm}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\right), \tag{17}$$

where $\overline{P}^*_{nm}$ are the fnALFs, satisfying the inequality (15).

The second part, E_2, gets components with fnALFs, not satisfying the condition (15).

According to the IEEE754 standard, in the double-precision environment E_1, practically equals to zero.

Since

$$a \le r \text{ и } \left|\overline{C}_{nm}\right| < 1,\ \left|\overline{S}_{nm}\right| < 1, \tag{18}$$

then:

$$\left|\left(\frac{a}{r}\right)^n \left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\right| < \sqrt{2} . \tag{19}$$

For this reason:

$$\left|E_1\right| = \left|\sum_{n=2}^{M}\sum_{m=0}^{n}\overline{P}^*_{nm}\left(\frac{a}{r}\right)^n \left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\right|,$$

or,

$$\left|E_1\right| \le \sum_{n=2}^{M}\sum_{m=0}^{n}\left|\overline{P}^*_{nm}\right| \cdot \left|\left(\frac{a}{r}\right)^n \left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\right| < \sqrt{2}\sum_{n=0}^{M}\sum_{m=0}^{n}\left|\overline{P}^*_{nm}\right|. \tag{20}$$

Since the sum E_1 includes the members with fnALFs, satisfying the condition (15), inequality (20) can be written as:

$$\left|E_1\right| < \sqrt{2} \cdot \sum_{n=0}^{M} \sum_{m=0}^{n} 2.2 \times 10^{-308},$$

or with formula (2):

$$\left|E_1\right| < \sqrt{2} \cdot 2.2 \times 10^{-308} \frac{(M+1)(M+2)}{2}. \tag{21}$$

The sum E_2 is about 0.48×10^{-3}, because extreme value of fully normalized spherical harmonic gravitational coefficients equals (Nima, 2000):

$$\overline{C}_{2,0} = -0.484166774985 \times 10^{-3}.$$

The other fully normalized spherical harmonic gravitational coefficients are much less.

In the double precision environment in order for E_1 was affected E_2, E_1 in absolute value must be more than 10^{-20}. The maximum degree of the gravitational potential model, for which the right part of inequality (21) reaches the size of $10^{-20,}$, is equal:

$$M \approx 8.0 \times 10^{143}.$$

Now there are no such models for which M is comparable with the above recorded value.

For maximal degree, applied in (Fukushima, 2012a):

$$M = 2^{32} = 4\ 294\ 967\ 296,$$

there is the following estimation for the sum of E_1

$$\left|E_1\right| < 2.9 \times 10^{-289}.$$

For EGM2008 it is possible to record such an inequality for the sum E_1:

$$\left|E_1\right| < 7.6 \times 10^{-302}.$$

So, the value E_1 satisfying the condition (21), will not affect the accuracy of total sum Eq. (16). Thus, the underflow problem itself does not affect the accuracy of the gravitational potential calculation.

2.3. The effect of "absolute zero"

Figure 1 shows the result of fnALFs calculation with the classical method using recursions Eqs. (5–9) for $\varphi = 68^0$. In this figure blue color pixels correspond to the fnALFs, the values of which are normal numbers, dark blue – the values of which are subnormal numbers, grey – the values of which are normal numbers, but in their calculations a subnormal number was involved. As a result, they were calculated with gross errors. The black color pixels correspond to the fnALFs, the values of which in calculations are less than the minimum subnormal numbers. That's why they are equal to zero. Figure 3 shows a magnified fragment of Figure 2.

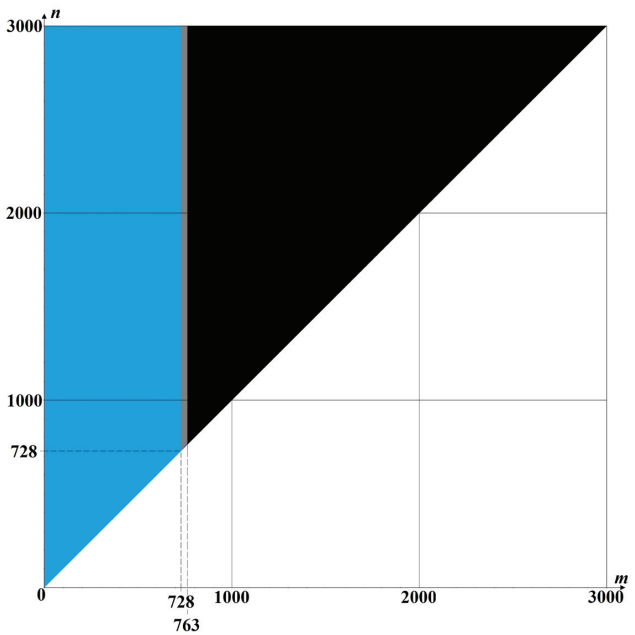

Fig. 2. The result of calculation the fnALFs by the classical method using recursion formulas (5-9) for $\varphi = 68^0$. Blue pixels correspond to the fnALFs, which are the normal numbers, dark blue pixels correspond to the fnALFs, which are the subnormal numbers, the gray pixels correspond to the fnALFs which are the normal numbers, but subnormal numbers are used in its calculation, and black pixels correspond to the fnALFs that are the "absolute zero"

As can be seen from the Figures 2–3, all the fnALFs from $m = 763$ are equal to zero.

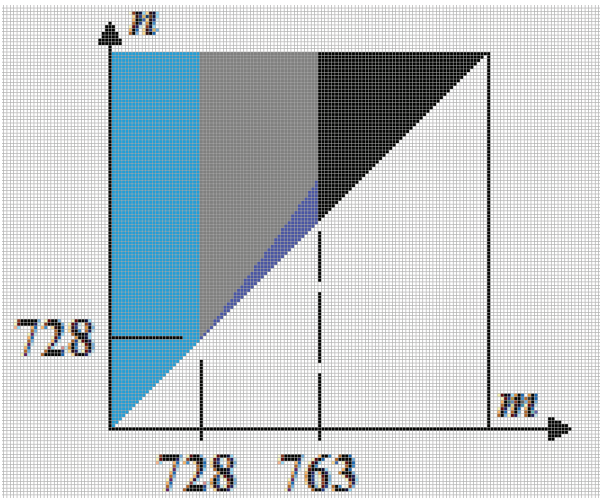

Fig. 3. The magnified fragment of Figure 2

For $\varphi = 68^0$:

$$\ln\left|\overline{P}_{763,763}\right| = -742{,}7\,.$$

At the same time the natural logarithm of the minimal subnormal number equals to:

$$-744{,}0$$

Therefore the value $\overline{P}_{763,763}$ must be subnormal. The first fnALFs which is valid underflow and the absolute value of which is less than the minimum subnormal number for latitude $\varphi = 68^0$ is a the function $\overline{P}_{765,765}$, the natural logarithm of which is equal:

$$\ln\left|\overline{P}_{765,765}\right| = -744{,}7\,.$$

So, theoretically, absolute zero (the first black pixel) must be in Figures 1–3 not for the fnALFs degree and order of which are equal to the number 763, and for the fnALFs, the degree and order of which are equal to the number 765.

However, when calculating the value $\overline{P}_{763,763}$ the effect of "absolute zero" starts to work.

The first part of Eq. (8) for $m = 763$ gives:

$$\sqrt{3}\prod_{i=2}^{m}\sqrt{\frac{2i+1}{2i}} = 2{,}0665.$$

But the logarithm of the second part of this Eq., i.e. the value $\cos^m\phi$, is equal to:

$$\ln(\cos^m\phi) = -744{,}8, \text{ for } m = 763, (\phi = 1{,}18448528345954 \text{ (rad)}).$$

Thus, the value $\cos^m\phi$ is less than the minimum subnormal number. Therefore, the result of its computation in the double-precision environment is equal to zero:

$$\cos^m\phi \equiv 0.$$

The result of multiplying any number by zero is zero. So the value $\overline{P}_{763,763}$ is equal to absolute zero. Similarly, the function $\overline{P}_{764,763}$, calculated by the formula (9), gives the absolute zero. The following function $\overline{P}_{765,763}$, and all subsequent functions computed with recursion formula (5), are also zero. Therefore, once appeared in calculations of the value $\overline{P}_{mm}$ the absolute zero like a virus spreads on all values $\overline{P}_{nm}$ for which $n \geq m$. No matter what values the coefficients a_{nm}, b_{nm} in Eq. (5) take, the result is only zero. This effect can be called as effect of "absolute zero".

The schematic representation of computing fnALFs using recursions (5–9), that shows the extension of absolute zero from one value $\overline{P}_{nm}$ to another is given in Figure 4. The schematic representation is similar to the ones presented in (Holmes and Featherstone, 2002a; Fantino and Casoto, 2005; Šprlák, 2011).

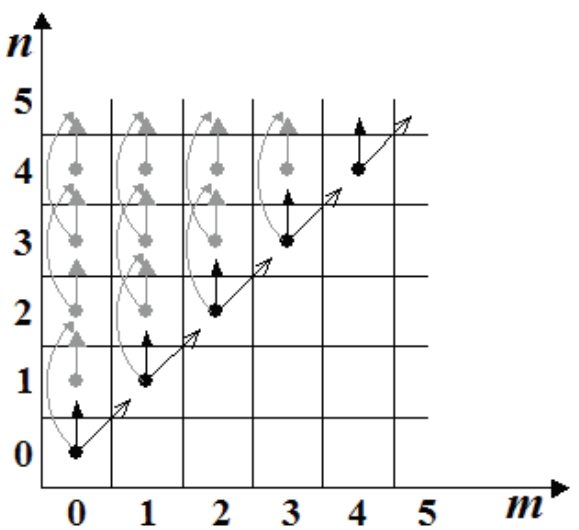

Fig. 4. Schematic representation for computing fnALFs with forward column recursion by Eqs. (5-9)

According to the standard IEEE754-2008 (Kahan, 1996), the subnormal values involved in calculations have the smallest possible exponent and the significand, first number of which is 0. The smaller a subnormal number, in absolute value, i.e. the closer it is to zero on the numerical axis, the less significant digits/bits are in its

significand. As the result, the subnormal numbers lose accuracy when approaching to zero. For double-precision environment the minimal positive subnormal number is (IEEE, 2008):

$$2^{-1074}$$

In binary record its significand has only one non-zero bit.

Figure 5 shows a fragment of the graph fnALFs depending on the order m for $n = 2200$ and latitude $\varphi = 68^0$. The solid line on the Figure 5 represents the fnALFs, computed according to classical recursion formulas (5–9), dashed line represents the fnALFs, calculated using the logarithmic method (Sect. 3.3). The gray area of the graph represents the effect of subnormal numbers on the result of the fnALFs calculation using the classic recursion. As can be seen from Figure 5, initially both graphs are almost identical. This means that subnormal values involved in the calculation of fnALFs are still quite accurate. However, when approaching to $m = 763$ the accuracy of fnALFs decreases rapidly and for $m \geq 763$ the values of all the Legendre functions, calculated with Eqs (5-9), become zero.

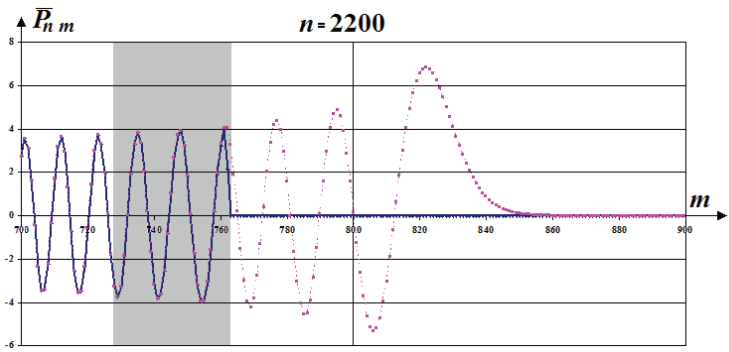

Fig. 5. Fragment of the graph where the fnALFs, depending on m for $n = 2200$ and for geodetic latitude 68^0, were calculated with the standard column recursion (solid line) and the logarithmic method (dashed line)

Therefore, it can be concluded that the use of subnormal numbers does not improve situation with the computation of Legendre functions and only moves the problem of "absolute zero" for $\varphi = 68^0$ from $m = 728$ to $m = 763$.

All above mentioned refers to the geodetic latitude 68^0. The limit value m_0, by which $\overline{P}_{m0,m0}$ is less in absolute value minimum subnormal number, depends on latitude. Substitution of the condition

$$\left|\overline{P}_{m0,m0}\right| < 4.9 \times 10^{-424}. \qquad (22)$$

in Eq. 8 and taking the logarithm gives

$$m_0 > \frac{\lg 4{,}9 - \lg\sqrt{3} - 424 - \lg\prod_{i=2}^{m0}\sqrt{\frac{2i+1}{2i}}}{\lg\cos\phi}. \tag{23}$$

Inequality (23) is a transcendental in relation to m_0. For its solution (Wittwer et al., 2007) have introduced the approximate formula. A more accurate value of m_0 one can get with the program presented by (Dmitrenko, 2012), which uses the method of successive approximations. In Figure 6 is a graph of m_0 depending on latitude obtained using this program.

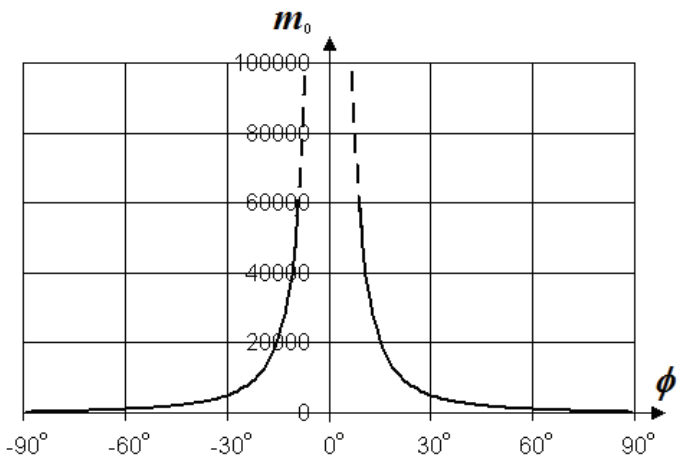

Fig. 6. Change m_0 depending on latitude ϕ

As can be seen from the graph in Figure 6, the effect of "absolute zero" is almost absent in areas close to the equator. At the equator the effect of "absolute zero" is completely absent. With increasing latitude the value m_0 decreases exponentially and continuously, reaching a minimum at the points of poles. Therefore, the effect of "absolute zero" must increase with increasing latitude. This means that fnALFs calculations errors also have to increase continuously with latitude increasing from the equator to the poles. However, as proved by (Dmitrenko, 2012), for the forward column recursion method the fnALFs calculating errors increase from the equator to the geodetic latitude $\pm 68^0$. Further the errors of calculating fnALFs decrease very rapidly. The value NA'_ϕ changing graph calculated according to the formula (62) (see Sect. 4) is shown in Figure 7. On the graph there are two maximums at latitude $\pm 68^0$.

This maximum of function errors in the calculation of fnALFs with standard method of forward column recursion depending on latitude can be seen on Figure 5 for M = 2700 (Fukushima, 2012a), Figure 2 for M = 3000 (Fukushima, 2014a) and

with Wenzel method on Figure 8 for M = 2160, Figure 8 for M = 2700 (Wittwer et al., 2008).

This extreme behavior of calculation errors of fnALFs is related to behavior peculiarities of these functions near the poles. Figure 8 presents two graphs of fnALFs depending on m with n = 2200 for two geodetic latitudes: $\varphi = 68^0$ and $\varphi = 89^0$.

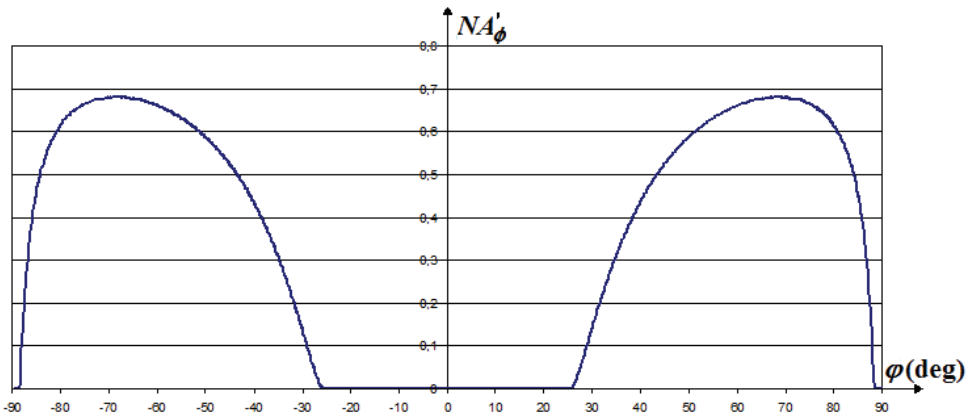

Fig. 7. The graph of NA'_ϕ (Sect. 4), calculated for the standard forward column recursion for M = 8000

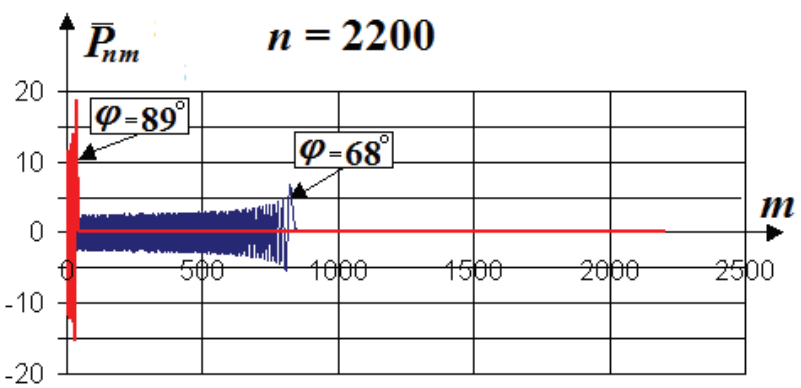

Fig. 8. The graphs of fnALFs change depending on m with n = 2200 for two geodetic latitudes: $\varphi = 68^0$ and $\varphi = 89^0$

Each graph of Figure 8 is based on 2200 points, the first graph – for latitude $\varphi = 68^0$ and second – for latitude $\varphi = 89^0$. As pointed out by (Jekeli et al., 2007) and (Fukushima, 2014a) the graph of the fnALFs can be divided into two parts. The first part oscillates and ends with maximum of fnALFs. The second part approaches to the axis m (Šprlák, 2011) asymptotically very quickly. As can be seen from the graphs on Figure 8, the first part of graph fnALFs is shortened and the second part

is lengthened with increasing latitude. This means that with increasing latitude, the number of extrema of the fnALFs decreases, but the absolute value of each remaining extrema increases. This property of functions $\overline{P}_{nm}$ is related to the fact that the sum of the squares of a set of variables $\overline{P}_{nm}$ with constant n and variable m does not depend on the latitude but only depends on n (see Eq. (58) below). At high latitudes this sum concentrates on a small number of variables $\overline{P}_{nm}$. At first the effect of "absolute zero" destroys the members of the set $\overline{P}_{nm}$ with a sufficiently large value m. Thus at high latitudes the members having values close to zero are forcibly nullified and almost have no effect to sum of $\overline{P}^2_{nm}$.

C. Jekeli et al. (2007) have used the above property of the Legendre functions which depends on latitude and reduced the amount of its computations for 36%.

The underflow problem is not dangerous by itself. Dangerous is the effect it generates, i.e. the effect of "absolute zero". Once appeared in the calculation of $\overline{P}_{mm}$ at a certain m, the "absolute zero" resets all $\overline{P}_{nm}$, no matter what values they take really. The effect of "absolute zero" is completely absent at the equator and increases with latitude in absolute value. However, due to the special property of the Legendre functions, the effect of "absolute zero" reaches its maximum not at the poles but at the latitudes $\pm 68^0$.

3. Methods of Computing the Legendre Functions

To solve the fnALFs calculations accuracy reduction problem caused by underflow, H.-G. Wenzel (1998) has suggested to use the scale factor 10^{200}. In the fnALFs calculations all of them were multiplied by this factor. The usage of this method became possible because the maximum absolute value of fnALFs does not exceed $(M + 1)$ (see Eq. (13)). Multiplying by the scale factor 10^{200} corresponds to increase of decimal logarithms of fnALFs by 200. Taking into account the fact that the decimal logarithm of the minimum subnormal number is equal -323,31, it can be concluded that when using a scale factor of 10^{200}, the calculations will involve the fnALFs, the decimal logarithms of which are at least -523,31.

Figure 9 shows a graph of $\log_{10}\left|\overline{P}_{mm}\right|$ depending of m for geodetic latitude 68^0. The value of m, at which $\log_{10}\left|\overline{P}_{mm}\right|$ is the minimum number, the decimal logarithm of which is greater than the minimum subnormal number, is equal to 764. This is the maximum degree of $\overline{P}_{mm}$ that theoretically should not be subjected to the effect of "absolute zero" (see Figs. 1-3). For the Wenzel method (Fig. 9) the maximum degree of the functions $\overline{P}_{mm}$ equals $M = 1236$. Since "absolute zero" destroys at the first place the fnALFs which values are close to zero, there is a lag effect in its appearance, due to which H.-G. Wenzel has increased the maximum degree to $M = 1900$, but only for $\left|\phi\right| < 70^0$.

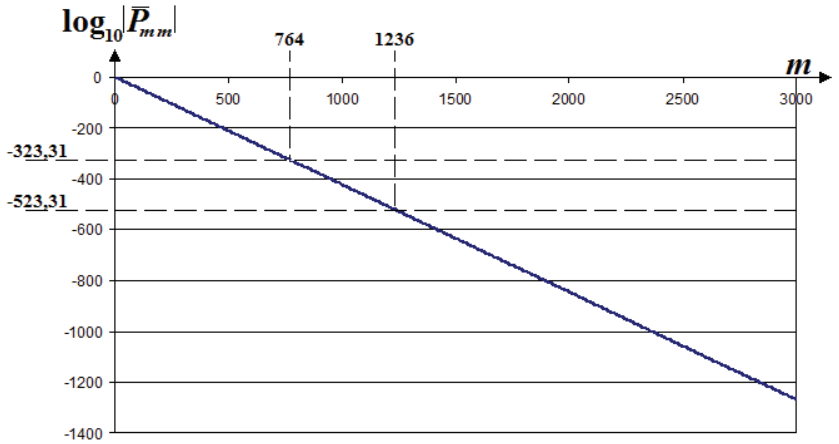

Fig. 9. Dependence of $\log_{10}\left|\overline{P}_{mm}\right|$ on m for $\varphi = 68^0$

S.A. Holmes and W.E. Featherstone (2002a; 2002b; 2002c) have given a more radical solution of the underflow problem. Their modified recursion method consists in computing not Legendre functions but functions related to Legendre functions with simple formulas.

In the first modification these functions are:

$$p_{nm}^{I} = 10^{-280}\frac{\overline{P}_{nm}}{\cos^{m}\phi}, \tag{24}$$

in the second modification–

$$p_{nm}^{II} = 10^{-280}\frac{\overline{P}_{nm}}{\overline{P}_{mm}}. \tag{25}$$

Both p_{nm}^{I} and p_{nm}^{II} can be calculated according to the recursion equations (5–9) by substituting these functions in place of $\overline{P}_{nm}$. The method of modified functions p_{nm}^{I} and p_{nm}^{II} S.A. Holmes and W.E. Featherstone (2002a) have been combined with the method Clenshaw and Horner's scheme (Harris and Stocker, 1998), which allows not to return to the original Legendre functions in calculations of gravitational potential (1).

S.A. Holmes and W.E. Featherstone (2002a) recommend to use the functions p_{nm}^{I} and p_{nm}^{II} at $M \le 2700$, as at $n > 2814$ at points close to the poles the numerical values of the functions p_{nm}^{I} and p_{nm}^{II} become larger than 10^{308}, i.e. the overflow problem takes place.

The method of S.A. Holmes and W.E. Featherstone (2002a) is widely used to compute the disturbing potential (Peng and Xia, 2004), the Bouguer and isostatic anomalies (Balmino et al., 2012), the spherical harmonic analysis and synthesis

(Blais, 2008; Fantino and Casoto, 2009; Hirt, 2012), the gravitational potential of the topographic masses (Wang and Yang, 2013). GrafLab software (Bucha and Janák, 2013) for spherical harmonic synthesis contains three methods of fnALFs calculation, one of which is the method of S.A. Holmes and W.E. Featherstone (2002a).

Method similar to the method of (Holmes and Featherstone, 2002a) with special scale factors (Liu et al., 2011) was used to calculate the Legendre functions and its first derivatives to degree and order 3600.

D.W. Lozier and J.M. Smith (1981) proposed to calculate Legendre functions using the extended-range arithmetic, implemented with the special software which is currently part of the SLATEC Common Mathematical Library (http://performance.netlib.org/slatec). The analysis of accuracy and properties of this arithmetic was performed by (Wittwer et al., 2007). They showed that the extended-range arithmetic allows to increase the maximal degree of Legendre functions up to 100 000. However, the extended-range arithmetic procedures require special software connected to certain compilers (Fortran 77) and increase the Legendre functions computation time by 49 times.

C. Jekeli et al. (2007) in the quadruple precision environment calculated fnALFs using the first modification of S.A. Holmes and W.E. Featherstone (2002a) to the maximum degree and order $M = 23599$. A similar result is obtained by (Šprlák, 2011).

The calculation of first modified forward column recursion with Eq. (24) in the quadruple precision environment and subsequent determination of the Legendre functions and their first derivatives up to degree and order $M = 10800$ were accomplished by (Kwon and Lee, 2007).

According to T. Fukushima (2012a), displacement from the double-precision environment to quadruple precision environment does not solve the underflow problem either. It appears for classical forward column recursion, when $M = 21600$. Simultaneously the Legendre functions calculation time increases by 40–80 times.

Using a non-standard normalization and recursion formulas allowed (Yu et al., 2015) to calculate the Legendre functions up to degree and order $M = 20000$.

All of the aforementioned methods of Legendre functions calculation are aimed to solve the underflow problem. Therefore they have a limit of the maximum degree and order related to the overflow problem appearance. In 2012 three methods were proposed: the first two are based on the idea of Wenzel (1998) (Fukushima, 2012a; Balmino et al., 2012), and the third is a logarithmic method (Dmitrenko, 2012). These methods remove from calculations not the underflow problem but the effect of "absolute zero". Due to this property, they almost have no limit in the fnALFs maximum degree. Another property that is common to these methods is that full information about the fnALFs is stored in two functions: real and integer. Below there is a brief description of these methods.

3.1. Fukushima method

The exact theoretical substantiation of the Fukushima method can be found in (Fukushima, 2012a). Graphically the procedure for calculating the fnALFs by the Fukushima method is presented in Figure 10.

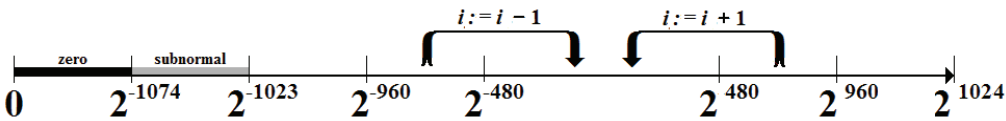

Fig. 10. Schematic representation of the numeric scale for double-precision numbers according to the standard IEEE754-2008, divided by intervals according to the Fukushima method

Information about values $\overline{P}_{nm}$ in the program, created by T. Fukushima (2012a), is stored in two variables: real z_{nm} and integer i_{nm}. The first values of the function $\overline{P}_{nm}$ with small n, m satisfy:

$$2^{-480} \le \left|\overline{P}_{nm}\right| < 2^{480}. \tag{26}$$

In this condition the functions z_{nm} and i_{nm} are equal:

$$z_{nm} = \overline{P}_{nm}, \quad i_{nm} = 0. \tag{27}$$

Recursive calculations with Eqs (5-9) in the program, created by T. Fukushima (2012a), are not for values $\overline{P}_{nm}$ but for values z_{nm}. Therefore, each value z_{nm} is tested with condition such as (26):

$$2^{-480} \le \left|z_{nm}\right| < 2^{480}. \tag{28}$$

If condition (28) stops running for any n, m, then there is a recalculation of the values z_{nm}, i_{nm} in the following formulas:

$$\text{If } \left|z_{nm}\right| < 2^{-480} \text{ then } z_{nm} := z_{nm} \cdot 2^{960},\ i_{nm} := i_{nm} - 1. \tag{29}$$

$$\text{If } \left|z_{nm}\right| \ge 2^{480} \text{ then } z_{nm} := z_{nm} \cdot 2^{-960},\ i_{nm} := i_{nm} + 1. \tag{30}$$

The procedure of recalculation can be considered as the process of value z_{nm} transferring to the interval satisfying the condition (28). This procedure is shown in Figure 10 with arrows. When $\left|z_{nm}\right| \ge 2^{480}$, z_{nm} moves from right to left. Likewise, when $\left|z_{nm}\right| < 2^{-480}$, z_{nm} moves from left to right.

As a result of these actions, all values z_{nm}, used in the recurrence formulas for the calculations are within the interval (28) that gives the possibility to calculate them as accurately as possible.

Eq. of link the fnALFs with values z_{nm}, i_{nm} is:

$$\overline{P}_{nm} = z_{nm} \cdot 2^{i_{nm} \cdot 960}. \tag{31}$$

According to T. Fukushima (2012a), this method allows to compute Legendre functions of extremely high degree up to 4 294 967 296 and increases Legendre computation time only by 10%.

This algorithm turned up to be so successful that T. Fukushima has used it for the finite difference of fnALFs (Fukushima, 2012b), for the first, second and third-order derivatives of fnALfs (Fukushima, 2012c; 2013), for definite integrals of fnALFs (Fukushima, 2014a), and also, with new method of computation the Second Kind Legendre functions, for external gravitational field of a general ring-like objects (Fukushima, 2014b, 2015, 2016).

3.2. Balmino method

G. Balmino and co-authors (2012) have proposed the algorithm for calculating fnALFs based on the first modification of (Holmes and Featherstone 2002a). Information about the fnALFs is stored in two functions: real H_{nm} and integer K_{nm}. The formulas of communication are as follows:

$$H_{nm} = \left(10^{-280}\right)^{K_{nm}+1} \frac{P_{nm}}{\cos^m \phi}. \tag{32}$$

Initially:

$$K_{nm} = 0. \tag{33}$$

So the initial calculations of functions H_{nm} up to n = 2815 completely coincide with calculations of modified functions p^I_{nm}. The method of modifications (Holmes and Featherstone, 2002a) has no underflow problem, but has the overflow problem. Therefore, the condition is testing:

$$\left|H_{nm}\right| < 10^{280}. \tag{34}$$

If the condition (34) is satisfied, the calculation process of the functions H_{nm} continues. If this condition is not satisfied, the value K_{nm} increases by one, i.e.

$$K_{nm} := K_{nm} + 1. \tag{35}$$

Simultaneously, such values as H_{nm}, $H_{n\text{-}1,m}$ are recalculating with Eqs.:

$$H_{nm} := H_{nm} \cdot 10^{-280},\ H_{n-1,m} := H_{n-1,m} \cdot 10^{-280}. \tag{36}$$

Further, the process of computing the functions H_{nm} with recursion formulas, similar to (5-9), continues.

For the inverse transform to the fnALFs the function Z is preliminary calculated:

$$Z = \log_{10}\left|\overline{P}_{nm}\right| = m\log_{10}(\cos\phi) + \log_{10}\left|H_{nm}\right| + 280\cdot(K_{nm}+1). \tag{37}$$

If $Z < 308$, then:

$$P_{nm} = 0, \tag{38}$$

else:

$$P_{nm} = 10^{Z}\cdot\mathrm{sgn}(H_{nm}). \tag{39}$$

3.3. Logarithmic method

Following (Dmitrenko, 2012), consider the functions:

$$p_{nm}^{\ln} = \ln\left|\overline{P}_{nm}\right|. \tag{40}$$

According to Eqs. (3-6), the functions $p_{nm}^{\ln}$ are determined for all latitudes except the poles and the equator, i.e. at:

$$90^0 < \phi < 0 \text{ and } 0 < \phi < 90^0. \tag{41}$$

When determining functions $p_{nm}^{\ln}$, the sign of functions $\overline{P}_{nm}$ is lost. Therefore, simultaneously with the introduction of functions $p_{nm}^{\ln}$ it is necessary to determine the integer functions k_{nm} which are equal to:

$$k_{nm} = \mathrm{sgn}(\overline{P}_{nm}). \tag{42}$$

Thus, the relation between functions $\overline{P}_{nm}$ and functions $p_{nm}^{\ln}$ can be written as:

$$\overline{P}_{nm} = k_{nm}\exp(p_{nm}^{\ln}). \tag{43}$$

Expressions for the functions $p_{mm}^{\ln}, p_{m,m-1}^{\ln}$ are resulted from taking a logarithm of Eqs. (7-9), i.e.

$$p_{0,0}^{\ln} = 0,\ p_{1,1}^{\ln} = 0.5\ln 3 + \ln(\cos\phi), \tag{44}$$

$$p_{mm}^{\ln} = 0.5\left(\ln 3 + \sum_{i=2}^{m}(\ln(2i+1) - \ln(2i))\right) + m\ln(\cos\phi), \tag{45}$$

$$p^{\ln}_{m,m-1} = 0.5\ln(2m+1) + \ln\left|\sin\phi\right| + p^{\ln}_{m-1,m-1}. \tag{46}$$

As the functions $\overline{P}_{mm}$ are always positive, and the sign of $\overline{P}_{m,m-1}$ is the same as of ϕ (see Eqs. (8-9)), the functions k_{mm} and $k_{m\text{-}1,m}$ are equal:

$$k_{mm} = 1, \quad k_{m,m-1} = \operatorname{sgn}(\phi). \tag{47}$$

To apply the logarithmic method it is necessary to rewrite Eq. (5) so that the value $p^{\ln}_{nm}$ can be obtained as a sum of several values. To do this, factor out the first member of the Eq. (5) to get (Dmitrenko, 2012):

$$\overline{P}_{nm} = a_{nm}\ \overline{P}_{n-1,m}\ \sin\phi\,(1 - D_{nm})\ , \tag{48}$$

where

$$D_{nm} = d_{nm}\frac{\overline{P}_{n-2,m}}{\overline{P}_{n-1,m}\sin\phi}, \tag{49}$$

or, considering Eq. (43),

$$D_{nm} = d_{nm}\frac{\exp(p^{\ln}_{n-2,m} - p^{\ln}_{n-1,m})}{\sin\phi}k_{n-2,m}\ k_{n-1,m}, \tag{50}$$

where:

$$d_{nm} = \sqrt{\frac{(n-m-1)(n+m-1)}{(2n-1)(2n-3)}}. \tag{51}$$

Next, the value D_{nm} should be analyzed. If the inequality

$$\left|D_{nm}\right| < 1, \tag{52}$$

is satisfied, the functions $p^{\ln}_{nm}$ can be calculated according to the following expression resulted from transformation of the Eq. (5),

$$p^{\ln}_{nm} = \ln\left|\sin\phi\right| + \ln a_{nm} + p^{\ln}_{n-1,m} + \ln(1 - D_{nm}). \tag{53}$$

In this case, the functions k_{nm} depend on the sign of the product $k_{n\text{-}1,m}\ \phi$, i.e.

$$k_{nm} = \operatorname{sgn}(k_{n-1,m}\ \phi)\ . \tag{54}$$

Similarly, if the inequality:

$$\left|D_{nm}\right| > 1, \tag{55}$$

is satisfied, in order to determine $p_{nm}^{\ln}$ the Eq. (5) is rewritten as follows:

$$p_{nm}^{\ln} = \ln b_{nm} + p_{n-2,m}^{\ln} + \ln\left(1 - \frac{1}{D_{nm}}\right). \tag{56}$$

In this case the sign of the functions k_{nm} is opposite to the sign of the functions $k_{n\text{-}2,m}$, i.e.

$$k_{nm} = -k_{n-2,m}. \tag{57}$$

If $D_{nm} = 1$ then there is the equality $\overline{P}_{nm} = 0$. For $D_{nm} = -1$ the functions $p_{nm}^{\ln}$ can be calculated either by formula (53), or by formula (56).

The Eqs. (40–57) completely solve the problem of determining fnALFs by the logarithmic method with forward column recursion.

Due to the fact that the software of Balmino method (Balmino et al., 2012) is not available from the public domain, further analysis is performed on (Fukushima, 2012) and logarithmic (Dmitrenko, 2012) methods.

4. Accuracy of Computing the Legendre Functions

To check the accuracy of computation fnALFs the Eq. (11) and the following condition (Holmes and Featherstone, 2002) can be used:

$$\sum_{m=0}^{n} \overline{P}_{nm}^{2} = 2n + 1, \tag{58}$$

To determine the Legendre functions calculations accuracy S.A. Holmes and W.E. Featherstone (2002a, 200b) have proposed to use the value named Numerical Accuracy that is calculated according to the formula (see Eq. (11)):

$$NA_{\phi} = \left| \frac{\sum_{n=0}^{M} \sum_{m=0}^{n} \overline{P}_{nm}^{2}(comp) - (M+1)^2}{(M+1)^2} \right|, \tag{59}$$

where $\sum_{n=0}^{M} \sum_{m=0}^{n} \overline{P}_{nm}^{2}(\text{comp})$ – the sum of $\overline{P}_{nm}^{2}$, calculated using one of the methods discussed above.

The logarithmic variation of formula (59) can be found in (Holmes and Featherstone, 2002c; Peng and Xia, 2004).

In the numerator part of the formula (59) there is a difference of two very close and very large values. The accuracy of this difference is much less than the accuracy of value $\sum_{n=0}^{M}\sum_{m=0}^{n}\overline{P}_{nm}^{2}(\text{comp})$. To increase the accuracy of the calculation NA_{ϕ} according to (58) the value δ_n, must be entered:

$$\delta_n = \sum_{m=0}^{n}\overline{P}_{nm}^{2}(comp) - (2n+1). \tag{60}$$

Then

$$\sum_{n=0}^{M}\sum_{m=0}^{n}\overline{P}_{nm}^{2}(comp) - (M+1)^2 = \sum_{n=0}^{M}\sum_{m=0}^{n}\overline{P}_{nm}^{2}(comp) - \sum_{n=0}^{M}(2n+1) = \sum_{n=0}^{M}\delta_n . \tag{61}$$

Substitution Eq. (61) into Eq. (59), gives:

$$NA_{\phi} = \left| \frac{\sum_{n=0}^{M}\delta_n}{(M+1)^2} \right| . \tag{62}$$

The value δ_n for the Fukushima and Balmino methods is determined from Eq. (60). For the logarithmic method, it is given by:

$$\delta_n = \sum_{m=0}^{n}\exp(2p_{nm}^{\ln}) - (2n+1). \tag{63}$$

where $p_{nm}^{\ln}$ is calculated using Eqs. (44-46, 53, 56).

The symbol of the absolute value in Eq. (54) ensures that the value NA_{ϕ} is always positive. However, the analysis of the variables δ_n shows that they have different signs. Figure 11 illustrates a graph of the change δ_n at the equator (ϕ = 0) for n varying from 0 to 8000. Because "absolute zero" effect is completely absent at the equator, the values δ_n were calculated for the fnALFs defined by Eqs. (5–9). The graph in Figure 11 shows the errors obtained directly from recursion formulas without taking into account any additional methods. Although most of the values δ_n on Figure 11 are negative, there are still values with a positive sign. The errors with different signs are mutually compensated and the total sum does not give the real situation of all errors of the method.

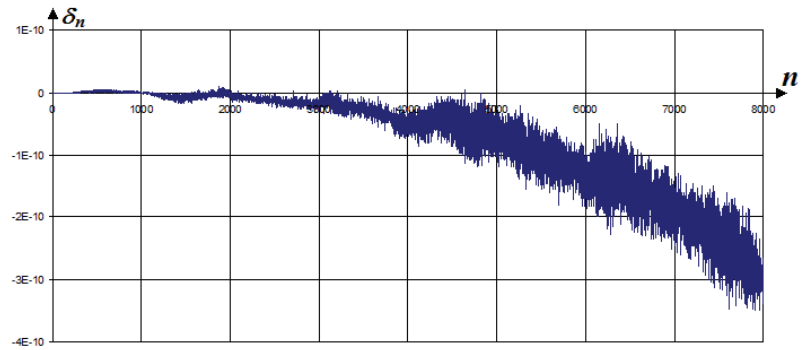

Fig. 11. Values δ_n dependence on n for latitude $\phi = 0$, determined by the classical method of forward columns recursion with Eqs. (5–9)

An even more clear result is shown in Figure 12, the graphs of change δ_n for the latitude $\phi = 60^0$ with Fukushima and logarithmic methods. As can be seen from the graphs, some of the quantities δ_n are positive and some of them are negative.

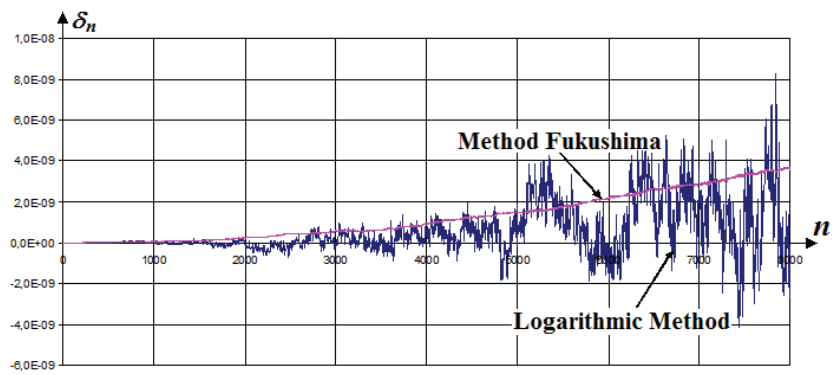

Fig. 12. Values δ_n depending on n for latitude $\phi = 60^0$, determined by the Fukushima and logarithmic methods

The value NA_ϕ for latitude $\phi = 60^0$ is equal to:

$$NA_{60} = 1{,}570\text{E-}13 \text{ (Fukushima method)},$$
$$NA_{60} = 7{,}253\text{E-}14 \text{ (Logarithmic method)}.$$

The change of sign of the function δ_n depending on the latitude for $n = 10800$ can be seen in Figure 3 (Kwon and Lee, 2007).

Eq. (62) gives a real result only if the values δ_n have the same sign for all values n. As signs of the values δ_n are different, more real result for the estimation the errors of calculation the Legendre functions gives Eq.:

$$NA'_{\phi} = \frac{\sum_{n=0}^{M} |\delta_n|}{(M+1)^2}. \tag{64}$$

The value NA'_{ϕ} for latitude $\phi = 60^0$ is equal to:

$$NA'_{60} = 1{,}570\text{E-}13 \text{ (Fukushima method)},$$
$$NA'_{60} = 1{,}03\text{E-}13 \text{ (Logarithmic method)}.$$

Numerical accuracy NA'_{ϕ} is determined for each latitude separately. Figure 13 shows graphs of the variation of NA'_{ϕ} for $M = 8000$ built by Fukushima and logarithmic methods. The graphics built for the geocentric latitude from -90^0 to 90^0 after 5 min. are as follows:

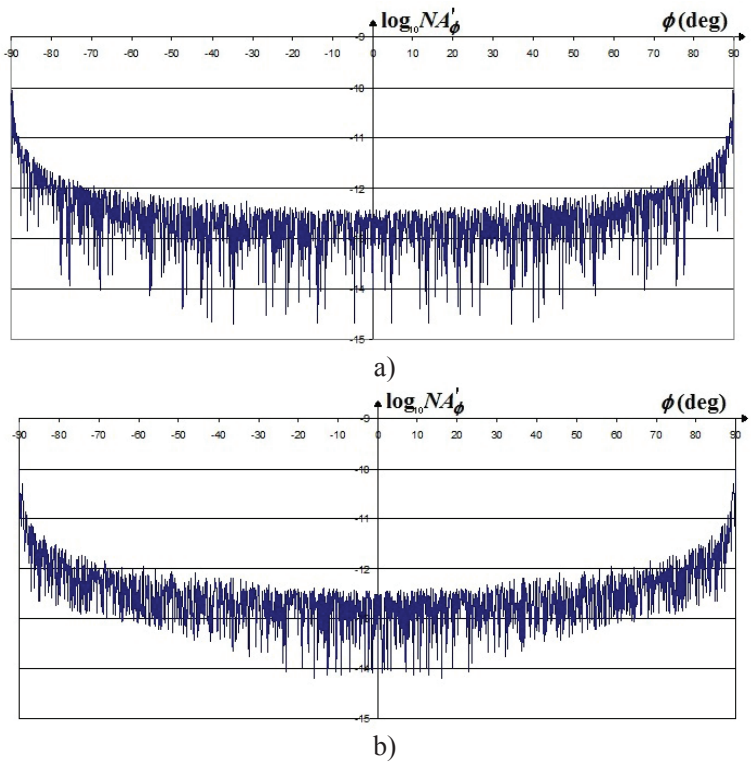

Fig. 13. Logarithm of numerical accuracy NA'_{ϕ} for $M = 8000$ for a) Fukushima method, b) Logarithmic method

There is no possibility to conclude on the basis of Figure 13, which of the two methods under consideration is more accurate. With the exception of the specifics, Figure 13 a) and 13 b) are almost identical. Therefore, for the analysis of the numerical accuracy

calculated for all latitudes from $\phi_{\min}$ to $\phi_{\max}$ with a step $\Delta\phi$, the following value can be proposed:

$$NA' = \frac{\Delta\phi}{(\phi_{\max} - \phi_{\min})} \sum_{1}^{N\phi} NA'_{\phi}, \tag{65}$$

where N_ϕ – the number of values NA'_ϕ in the computation of NA'.

Figure 14 shows the graphs NA' depending on the maximum degree M from 0 to 8000 for both methods. A value NA' calculated according to the formula (65) with $\phi_{\min} = -90^0$, $\phi_{\max} = 90^0$, $\Delta\phi = 5'$ allows to trace the trend of numerical accuracy depending on the maximum degree of fnALFs.

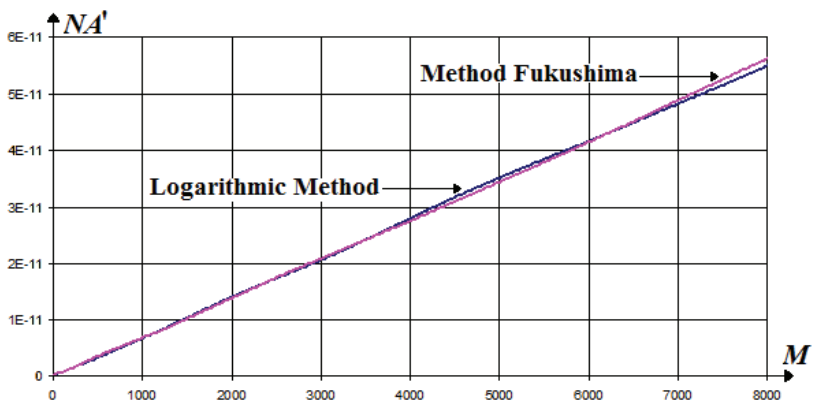

Fig. 14. Dependence of the numerical accuracy NA' of the maximum degree M of fnALFs for Fukushima and logarithmic methods

The maximum values NA' for $M = 8000$ for the Fukushima and logarithmic methods are equal, respectively:

$NA' = 5.6\text{E-}11$ (Fukushima method),
$NA' = 5.5\text{E-}11$ (Logarithmic method).

Therefore, both methods have the same accuracy of the calculation fnALFs.

Since the method (Balmino et al., 2012) is based on the same recursion formulas (5–9) as method (Fukushima, 2012), there is a reason to believe that their accuracy is the same.

5. CPU Time of Computing the Legendre Functions

The comparison of the formulas for calculation of the functions z_{nm} (Fukushima method) with formulas for calculation $p_{nm}^{\ln}$ (logarithmic method) allows to conclude that the logarithmic method should require more computation time than the Fukushima method does.

The functions z_{nm}, underlying the Fukushima method, are calculated by using the same recursion Eq. (5) that fnALFs does, but with taking into consideration the fact whether they satisfy or do not the condition (28). With the increase in the maximum degree of Legendre functions the amount of computation by the formula (5) will also increase. This formula contains two multiplication operations and one subtraction. Additional calculations for the function z_{nm} transmission in the interval (28), and final computation of fnALFs by the Eqs. (31) increase the time of its calculation, as indicated in (Fukushima, 2012a), not more than 10%.

Functions $p_{nm}^{\ln}$, underlying the logarithmic method, calculated by Eqs. (50, 53, 56) that contain the operations of addition, subtraction, multiplication, division, and logarithmic and exponential functions. Therefore, the total computation time of functions $p_{nm}^{\ln}$ of the logarithmic method is more than functions z_{nm} calculation time of the Fukushima method. For maximum degree $M = 8000$ calculation time of fnALFs by logarithmic method is 6 times more than calculation time of these functions by the Fukushima method. This result is obtained by using the original programs available in (Fukushima, 2012; Dmitrenko, 2012), without any additional techniques to minimize the CPU time.

The comparison of the formulas of communication fnALFs with the original functions for methods (Balmino et al., 2012) and (Fukushima, 2012) allows to make an assumption about the CPU time of both methods.

When calculating functions $\overline{P}_{nm}$ using the Eq. (31) for Fukushima method, the fact that the value $i_{nm} < 2$ (see Sect. 2.1) is taken into consideration. Therefore, in program presented (Fukushima, 2012), the function $2^{i_{nm} \cdot 960}$ is replaced with an array containing such values:

0 for $i_{nm} < -1$ ($\overline{P}_{nm} \cong 0$);

2^{-960} for $i_{nm} = -1$;

$2^{0} = 1$ for $i_{nm} = 0$;

2^{960} for $i_{nm} = 1$.

So, the calculation procedure for functions $\overline{P}_{nm}$ in Fukushima method, contains one multiplication operation and one call of array with four of the above numbers.

Eq. (37) of the method (Balmino et al., 2012) for calculation of the value Z includes a logarithmic function, two operations of summation, a call of integer-indexed array ($280 \cdot (K_{nm} + 1)$) and array of values $m \cdot \log_{10}(\cos\phi)$. Final Eq. (39) for calculation $\overline{P}_{nm}$ contains a power function, operation of multiplication and one call of integer-indexed array. Therefore, it can be assumed that the CPU time of the fnALFs computation with method of (Balmino et al., 2012) is more than the time needed with the method (Fukushima 2012).

Thus, the Fukushima method is the best of all available methods of calculation the Legendre functions of arbitrary degree and order. Its main advantage is the calculation speed.

6. Conclusions

The fnALFs calculation problems analysis shows that the underflow problem is not dangerous by itself. The main problem that generates the gross errors in its calculations is the problem named as the effect of "absolute zero".

Due to the fact that forward column recursion (Eq. (5)) calculates $\overline{P}_{nm}$ with one value m, and with $m \le n \le M$, once appeared in the calculation $\overline{P}_{nm}$, the "absolute zero" spreads to all $\overline{P}_{nm}$ from a set with one value m, regardless of whether it is equal to zero or not real. The effect of "absolute zero" is completely absent at the equator and increases with latitude in absolute value. However, due to the special property of the Legendre functions, the effect of absolute zero reaches a maximum not at the poles but at latitudes $\pm 68^0$.

The use of subnormal numbers does not save the situation for the fnALFs, but only masks the problem for a time. As a result, the accuracy of the fnALFs with the appearance of subnormal numbers in their calculations is continuously decreasing.

Computing methods of the fnALFs with the scale factor usage and modified recursion method proposed by S.A Holmes and W.E. Featherstone (2002a) remove the underflow problem from calculations. As a result, these methods have limitations in the maximum degree M because of the appearance of the overflow problem. Three methods of the fnALFs calculating: (Balmino et al., 2012; Fukushima, 2012; Dmitrenko, 2012) removed the effect of "absolute zero" from the calculations. The methods are interesting also because they have practically no limit for the maximum degree M of the fnALFs.

It is shown that the formula for numerical accuracy NA calculation represented in (Holmes and Featherstone, 2002a, 2002b, 2002c) can give an erroneous result because of errors with different signs in the calculation $\overline{P}_{nm}$ in the set with one value n, and with $0 \le m \le n$. A more accurate formula for NA calculation is given here.

It is shown that the numerical accuracy of the Fukushima method and the logarithmical method is the same. But the CPU time of the fnALFs calculation with logarithmic method was 6 times more (M = 8000) than the time functions calculation by the Fukushima method.

There is an assumption about the CPU time of the fnALFs calculation with method Balmino et al. (2012), according to which its calculation time must be greater than the CPU time of the calculation the fnALFs with method of Fukushima. Therefore, the Fukushima method is the best. Its main advantage is the computational speed, which is an important factor in the calculation of such great Legendre functions number as 2 401 336 for EGM2008. The time difference between this method and other will rise with increasing of M.

Acknowledgements

The authors thank the anonymous referee for valuable advices that improve the quality of the present article.

References

Abramowitz, W. and Stegan, I.A. (1972). *Handbook of mathematical functions*, 9th edn. Dover Publications, New York (470 pp).

Balmino, G., Vales, N., Bonvalot, S. and Briais, A. (2012). Spherical Harmonic modeling to ultra-high degree of Bouguer and isostatic anomalies. *J. geod*., 86: 499–520. DOI: 10.1007/s00190-011-0533-4.

Blais, J.A.R. (2008). Discrete spherical harmonic transforms: Numerical preconditioning and optimization. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), *5102 LNCS (PART 2)*, 638–645.

Bucha, B. and Janák, J. (2013). A MATLAB-based graphical user interface program for computing functionals of the geopotential up to ultra-high degrees and orders. *Comp. Geosci.*, 56, 186–196.

Dmitrenko, A. (2012). *Modern transformation of determination the geoid* [in Russian]. Krivoy Rog: Mineral ISBN 978-966-72-4 (218 pp).

Fantino, E. and Casoto, S. (2005). Comparison among spherical harmonic synthesis methods for functionals of the gravity field. *Newton's Bulletin 3*, 32–49.

Fantino, E. and Casoto, S. (2009). Methods of harmonic synthesis for global geopotential models and their first-, second- and third-order gradients. *J. Geod.*, 83, 595–619. DOI: 10.1007/s00190-008-0275-0.

Fukushima, T. (2012a). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating points numbers. *J. Geod*., 86, 271–285. DOI: 10.1007/s00190-011-0519-2.

Fukushima, T. (2012b). Recursive computation of finite difference of associate Legendre functions. *J. Geod.*, 86, 745–754. DOI: 10.1007/s00190-012-0553-8.

Fukushima, T. (2012c). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers: II first-, second-, and third-order derivatives. *J. Geod.*, 86, 1019–1028. DOI: 10.1007/s00190-012-0561-8.

Fukushima, T. (2013). Recursive computation of oblate spheroidal harmonics of the second kind and their first-, second-, and third-order derivatives. *J. Geod.*, 87, 303–309. DOI: 10.1007/s00190-012-0599-7.

Fukushima, T. (2014a). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers: III integral. *Comp. Geosci.*, 63, 17–21. DOI: 10.1016/j.cageo.2013.10.010.

Fukushima, T. (2014b). Prolate spheroidal harmonic expansion of gravitational field. *Astron. J.*, 147, 152–160.

Fukushima, T. (2015). Numerical Computation of Point Values, Derivatives, and Integrals of Associated Legendre Function of the First Kind and Point Values and Derivatives of Oblate Spheroidal Harmonics of the Second Kind of High Degree and Order. International Association of Geodesy Symposia 145, 1–5.

Fukushima, T. (2016). Zonal toroidal harmonic expansions of external gravitational fields for ring-like objects, *Astron. J*., 152, 35 (31pp).

Harris, J.W. and Stocker, H. (1998). *Handbook of mathematics and computational science*. Springer, Berlin Heidelberg New York (1028 pp).

Heiskanen, W.A. and Moritz, H. (1967). *Physical geodesy*. Freeman and Co, San Francisco (364 pp).

Hirt, C. (2012). Efficient and accurate high-degree spherical harmonic synthesis of gravity field functionals at the Earth's surface using the gradient approach. *J. Geod*., 86, 729–744. DOI:10.1007/s00190-012-0550-y.

Holmes, S.A. and Featherstone, W.E. (2002a). A unified approach to the Clenshaw summation and the recursive computation of very high degree and order normalized associated Legendre functions. *J. Geod*., 76, 279–299. DOI: 10.1007/s00190-002-0216-2.

Holmes, S.A., Featherstone, W.E. (2002b). SHORT NOTE: Extending simplified high-degree synthesis methods to second latitudinal derivatives of geopotential. *J. Geod*., 76, 447–450. DOI: 10.1007/s00190-002-0268-3.

Holmes, S.A. and Featherstone, W.E. (2002c). A simple and stable approach to high degree and order spherical harmonic synthesis. Vistas for Geodesy in the New Millennium. Of the series International Association of Geodesy Symposia. 125, 259–264.

IEEE Computing Society (2008). 754–2008 – *IEEE Standard for Floating-Point Arithmetic*. doi:10.1109/IEEESTD.2008.4610935 (70 pp).

Jekeli, C. (2006). *Geometric Reference Systems in geodesy*: Division of Geodesy and Geospatial Science School of Earth Science, Ohio State University (202 pp). http://www.uacg.bg/filebank/att_1855.pdf.

Jekeli, C., Lee, K.J. and Kwon, J.H. (2007). On the computation and approximation of ultra-high-degree spherical harmonic series. *J. Geod.*, 81, 603–615. DOI: 10.1007/s00190-006-0123-z.

Kahan, W. (1996). IEEE Standard 754 for Binary Floating-Point Arithmetic: Lecture notes on the status of IEEE. https://www.cs.berkeley.edu/~ wkahan/ieee754status/ieee754.ps.

Kwon, J.H. and Lee, J.-K. (2007). Numerical computation of ultra-high-degree legendre function. *Journal of the Korean Society of Surveying Geodesy Photogrammetry and Cartography*, 25 (1), 63–68.

Liu, Z., Liu, S. and Huang O. (2011). Scale factors in recursion of ultra-high degree and order Legendre functions and Horner's scheme of summation. *Acta Geodaetica et Cartographica Sinica*, 40 (4), 454–458.

Lozier, D.W. and Smith, J.M. (1981). Algorithm 567 Extended-range arithmetic and normalized Legendre polynomials. *ACM Trans. Math. Softw*., 7, 141–146.

Moritz, H. (1980). Geodetic reference system 1980. *Bull. Geod.*, 54(4), 395–405.

NIMA. (2000). *Department of Defense World Geodetic System 1984: its definition and relationships with local geodetic systems* (3rd ed.). US National Imagery and Mapping Agency Technical Report 8350.2 (175 pp).

Olver, F.W.J., Lozier, D.W., Boisvert, R.E. and Clark, C.W. (eds) (2010). *NIST Handbook of Mathematical Functions*. Cambridge Univ Press, Cambridge. http://dlmf.nist.gov/.

Pavlis, N.K., Holmes, S.A., Kenyon, S.C. and Factor, J.K. (2012). The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). *J. Geophys. Res*., 117, B04406, 1–38.

Peng, F.-Q. and Xia , Z.-R. (2004). Algorithms for computing ultra-high-degree disturbing geopotential elements. *Acta Geophysica Sinica*, 47 (6), 1023–1028.

Šprlák, M. (2011). On the numerical problems of spherical harmonics: Numerical and algebraic methods avoiding instabilities of the associated legendre's functions. *ZFV – Zeitschrift fur Geodasie, Geoinformation und Landmanagement*, 136 (5), 310–320.

Tschering, C.C., Rapp, R.H. and Goal C. (1983). A comparison of methods for computing gravimetric quantities from high degree spherical harmonic expansions. *Manuscr. Geod*. 8, 249–272.

Wang, Y.M. and Yang, X. (2013). On the spherical and spheroidal harmonic expansion of the gravitational potential of the topographic masses. *J. Geod.*, 87, 909–921. DOI: 10.1007/s00190-013-0654-z.

Wenzel, H.-G. (1998). Ultra-high degree geopotential models GPM98A, B, and C to degree 1800. Submit. Proceedings of the Joint Meeting of the International Gravity Commission and International Geoid Commission, 7–12 September, Trieste, Italy.

Wittwer, T., Klees, R., Seitz, K. and Heck, B. (2008). Ultra-high degree spherical harmonic analysis and synthesis using extended-range arithmetic. *J. Geod.*, 82, 223–229. DOI: 10.1007/s00190-007-0172-y.

Yu, J.-H., Zeng, Y.-Y., Zhu, Y.-C. and Meng, X.-C. (2015). A recursion arithmetic formula for Legendre functions of ultra-high degree and order on every other degree. *Chinese J. Geophysics*, 58(3), 748–755. DOI: 10.6038/cjg20150305.