Opuscula Math. 42, no. 4 (2022), 549–560
https://doi.org/10.7494/OpMath.2022.42.4.549 Opuscula Mathematica

OSCILLATION OF EVEN ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MIXED DEVIATING ARGUMENTS

Blanka Baculikova

Communicated by Josef Diblík

Abstract. In the paper, we study oscillation and asymptotic properties for even order linear functional differential equations

$$y^{(n)}(t) = p(t)y(\tau(t))$$

with mixed deviating arguments, i.e. when both delayed and advanced parts of $\tau(t)$ are significant. The presented results essentially improve existing ones.

Keywords: higher order differential equations, mixed argument, monotonic properties, oscillation.

Mathematics Subject Classification: 34K11, 34C10.

1. INTRODUCTION

This paper is concerned with the oscillatory behavior of linear functional differential equations of the form

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

Throughout this paper it is assumed that

(H_1) $p \in C([t_0, \infty))$, $p(t) > 0$, n is even,
(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [T_y, \infty) \to R$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. An equation itself is said to be oscillatory if all its proper solutions are oscillatory. There are numerous papers devoted to oscillation of differential equation, see [1–15].

If $y(t)$ is a nonoscillatory solution of (E), then there exist an even integer $\ell \in \{0, 2, \dots, n\}$ and a $t_0 \geq T_y$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0, \infty) \text{ for } 0 \leq i \leq \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0, \infty) \text{ for } \ell \leq i \leq n. \end{aligned} \tag{1.1}$$

Such $y(t)$ is said to be a (nonoscillatory) solution of degree ℓ, and the set of all solutions of degree ℓ is denoted by $\mathcal{N}_\ell$. Consequently, the set $\mathcal{N}$ of all nonoscillatory solutions of (E) has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_n.$$

It is known that in the case where $\tau(t) \equiv t$ Eq.(E) always has solutions of degree 0 and n, that is, $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_n \neq \emptyset$ for ordinary differential equation

$$y^{(n)}(t) = p(t)y(t).$$

The situation for the case $\tau(t) \not\equiv t$ is different. In fact, it may happen that $\mathcal{N}_0 = \emptyset$ or $\mathcal{N}_n = \emptyset$ for (E) when the deviating argument is delayed ($\tau(t) < t$) or advanced ($\tau(t) > t$) and the deviation $|t - \tau(t)|$ is large enough, see e.g. pioneering works of Ladas *et al.* [15] and Koplatadze and Chanturija [11].

Later Kusano [13] studied (E) with $\tau(t)$ being of mixed type which means that delayed part

$$\mathcal{D}_\tau = \{t \in (t_0, \infty) : \tau(t) < t\}$$

and advanced part

$$\mathcal{A}_\tau = \{t \in (t_0, \infty) : \tau(t) > t\}$$

are both unbounded subset of (t_0, ∞). Kusano's main result is the following:

Let us denote

$$\tau_*(t) = \min\{\tau(t), t\}.$$

We employ two sequences $\{t_k\}$, $\{s_k\}$ such that

$$t_k \in \mathcal{D}_\tau, \quad t_k \to \infty \quad \text{as} \quad k \to \infty \tag{1.2}$$

and

$$s_k \in \mathcal{A}_\tau, \quad s_k \to \infty \quad \text{as} \quad k \to \infty. \tag{1.3}$$

Theorem 1.1. *Assume that there is a constant $\varepsilon > 0$ such that*

$$\int^{\infty} \big(\tau_*(t)\big)^{n-1-\varepsilon} p(t)\,\mathrm{d}t = \infty. \tag{1.4}$$

Supose moreover that there exist two sequences $\{t_k\}$, $\{s_k\}$ satisfying (1.2) *and* (1.3). *If*

$$\limsup_{k\to\infty} \int_{\tau(t_k)}^{t_k} \big(\tau(t_k) - \tau(s)\big)^{n-1} p(s)\,\mathrm{d}s > (n-1)! \tag{1.5}$$

and

$$\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} \big(\tau(t)-\tau(s_k)\big)^{n-1} p(t)\,\mathrm{d}t > (n-1)!, \tag{1.6}$$

then (E) *is oscillatory.*

Kusano employed the conditions (1.4), (1.5) and (1.6) to show that $\mathcal{N}_\ell = \emptyset$, $\ell = 2,4,\dots,n-2$, $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_n = \emptyset$, respectively. We observe that application of (1.4) is problematic for Euler type of equations ($p(t) \sim a/t^n$). The aim of this paper is to essentially improve all three conditions (1.4)–(1.6). Our significant progress will be demonstrated via equation

$$y^{(n)}(t) = p_0 y(t + \sin t)$$

for which Kusano obtained oscillation under condition

$$p_0 > \frac{(n-1)!}{(\sin 1 - 0.5)^{n-1}}. \tag{1.7}$$

Very recently the present author [4] contributed to the problem for the second order differential equations.

2. MAIN RESULTS

We introduce three results in which we shall show in successive steps that $\mathcal{N}_\ell = \emptyset$, $\ell = 2,4,\dots,n-2$, $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_n = \emptyset$.

Theorem 2.1. *Assume that*

$$\int^{\infty} \big(\tau_*(t)\big)^{n-1} p(t)\,\mathrm{d}t = \infty \tag{2.1}$$

and

$$\begin{aligned}\limsup_{t\to\infty} \Bigg\{ \int_{\tau_*(t)}^{t} \big(\tau_*(s)\big)^{n-1} p(s)\,\mathrm{d}s + \tau_*(t) \int_{t}^{\infty} \big(\tau_*(s)\big)^{n-2} p(s)\,\mathrm{d}s \\ + \frac{1}{\tau_*(t)} \int_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n} p(s)\,\mathrm{d}s \Bigg\} > 2(n-2)!.\end{aligned} \tag{2.2}$$

Then the classes $\mathcal{N}_\ell = \emptyset$ *for* (E) *for all* $\ell = 2,4,\dots,n-2$.

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t)$ of degree ℓ for some $\ell \in \{2,4,\dots,n-2\}$. Since $y^{(\ell-1)}(t)$ is positive and increasing, it is easy to see that

$$y^{(\ell-2)}(t) \ge \int_{t_1}^{t} y^{(\ell-1)}(s)\,\mathrm{d}s \ge y(t_1)(t-t_1).$$

Integrating $(\ell-2)$-times the last inequality, we are led to

$$y(t) \geq y(t_1)\frac{(t-t_1)^{\ell-1}}{(\ell-1)!} \geq ct^{\ell-1}, \qquad c = \frac{y(t_1)}{2(\ell-1)!}.$$

Taking into account (2.1), one gets

$$\int^{\infty} t^{n-\ell}y^{(n)}(t)\,\mathrm{d}t = \int^{\infty} t^{n-\ell}p(t)y(\tau(t))\,\mathrm{d}t \geq c\int^{\infty} t^{n-\ell}p(t)\tau^{\ell-1}(t)\,\mathrm{d}t$$
$$\geq \int^{\infty} \big(\tau_*(t)\big)^{n-1}p(t)\,\mathrm{d}t = \infty.$$

By Lemma 3.2 in [12], condition $\int^{\infty} t^{n-\ell}y^{(n)}(t)\,\mathrm{d}t = \infty$ guarantees that

$$\frac{y(t)}{t^{\ell}}\downarrow \qquad \frac{y(t)}{t^{\ell-1}}\uparrow \tag{2.3}$$

and

$$y(t) \geq \frac{t^{\ell}}{\ell!(n-\ell)!}\int_t^{\infty} s^{n-\ell-1}p(s)y(\tau(s))\,\mathrm{d}s + \frac{t^{\ell-1}}{\ell!(n-\ell)!}\int_{t_0}^{t} s^{n-\ell}p(s)y(\tau(s))\,\mathrm{d}s.$$

Since $y(t)$ is increasing, we get $y(\tau(s)) \geq y(\tau_*(s))$ which in view of the above inequality yields

$$y(\tau_*(t)) \geq \frac{1}{2(n-2)!}\Bigg\{\big(\tau_*(t)\big)^{\ell}\int_{\tau_*(t)}^{t} [\tau_*(s)]^{n-1}p(s)\frac{y(\tau_*(s))}{\big(\tau_*(t)\big)^{\ell}}\,\mathrm{d}s$$
$$+ \big(\tau_*(t)\big)^{\ell}\int_t^{\infty} \big(\tau_*(s)\big)^{n-2}p(s)\frac{y(\tau_*(s))}{\big(\tau_*(t)\big)^{\ell-1}}\,\mathrm{d}s$$
$$+ \big(\tau_*(t)\big)^{\ell-1}\int_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n}p(s)\frac{y(\tau_*(s))}{\big(\tau_*(t)\big)^{\ell}}\,\mathrm{d}s\Bigg\}.$$

Taking into account (2.3), we obtain

$$y(\tau_*(t)) \geq \frac{y(\tau_*(t))}{2(n-2)!}\Bigg\{\int_{\tau_*(t)}^{t} \big(\tau_*(s)\big)^{n-1}p(s)\,\mathrm{d}s + \tau_*(t)\int_t^{\infty} \big(\tau_*(s)\big)^{n-2}p(s)\,\mathrm{d}s$$
$$+ \frac{1}{\tau_*(t)}\int_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n}p(s)\,\mathrm{d}s\Bigg\}$$

which contradicts (2.2). □

Remark 2.2. The situation from Theorem 2.1 when $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$ is usually refereed as Property B of Eq. (E), thus this result can be reformulated in terms of Property B. It follows from Lemma 4.1 of [10] that condition (2.1) is necessary for Property B of (E). From this point of view, we have just one condition (2.2) for elimination of $n/2 - 1$ classes $\mathcal{N}_\ell$.

Now we turn our attention to elimination of the class $\mathcal{N}_0$.

Theorem 2.3. *Let there exist sequence* $\{t_k\}$ *satisfying* (1.2) *and* $\beta > 0$ *such that*

$$\frac{(t-\tau(t))^{n-1}}{(n-1)!}p(t) \geq \beta, \quad t \in [\tau(\tau(t_k)), \tau(t_k)], \quad k = 1, 2, \dots. \tag{2.4}$$

If

$$\limsup_{k\to\infty} \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} p(s) \int_{\tau(s)}^{\tau(t_k)} \mathrm{e}^{-\beta x} \big(x - \tau(s)\big)^{n-2} \,\mathrm{d}x\mathrm{d}s > (n-2)!, \tag{2.5}$$

then $\mathcal{N}_0 = \emptyset$ *for* (E).

Proof. Assume on the contrary that (E) has an eventually positive solution $y(t)$ of degree 0. Then for $u < v$ we have

$$y^{(n-2)}(u) \geq \int_u^v -y^{(n-1)}(x)\,\mathrm{d}x.$$

An integration in u over $[u, v]$ yields

$$-y^{(n-3)}(u) \geq \int_u^v \int_z^v -y^{(n-1)}(x)\,\mathrm{d}x\mathrm{d}z = \int_u^v -y^{(n-1)}(x)(x-u)\,\mathrm{d}x.$$

Repeating this procedure one gets

$$y(u) \geq \int_u^v -y^{(n-1)}(x)\frac{(x-u)^{n-2}}{(n-2)!}\,\mathrm{d}x \geq -y^{(n-1)}(v)\frac{(v-u)^{n-1}}{(n-1)!}. \tag{2.6}$$

We consider $t \in I = [\tau(\tau(t_k)), \tau(t_k)] \subset \mathcal{D}_\tau$, $k = 1, 2, \dots$. Setting $u = \tau(t)$ and $v = t$ into (2.6) one can see that

$$y(\tau(t)) \geq -y^{(n-1)}(t)\frac{(t-\tau(t))^{n-1}}{(n-1)!}$$

which in view of (E) and (2.4) yields

$$y^{(n)}(t) = p(t)y(\tau(t)) \geq -\beta\, y^{(n-1)}(t), \quad t \in I.$$

Now it is easy to verify that

$$e^{\beta t}\Big(-y^{(n-1)}(t)\Big)\downarrow \quad \text{on } I. \tag{2.7}$$

On the other hand, an integration of (E) yields

$$y^{(n-1)}(\tau(t_k)) \geq \int_{\tau(t_k))}^{t_k} p(s)y(\tau(s))\,\mathrm{d}s. \tag{2.8}$$

Employing (2.6) with $u=\tau(s)$ and $v=\tau(t_k)$ with $s\in[\tau(t_k),t_k]$ we get

$$y(\tau(s)) \geq \int_{\tau(s)}^{\tau(t_k)} -y^{(n-1)}(x)\frac{(x-\tau(s))^{n-2}}{(n-2)!}\,\mathrm{d}x. \tag{2.9}$$

Combining (2.9) together with (2.8) we obtain

$$\begin{aligned} y^{(n-1)}(\tau(t_k)) &\geq \int_{\tau(t_k))}^{t_k} p(s)\int_{\tau(s)}^{\tau(t_k)} -y^{(n-1)}(x)\frac{(x-\tau(s))^{n-2}}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}s \\ &\geq -y^{(n-1)}(\tau(t_k))e^{\beta\tau(t_k)}\int_{\tau(t_k))}^{t_k} p(s)\int_{\tau(s)}^{\tau(t_k)} e^{-\beta x}\frac{(x-\tau(s))^{n-2}}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

where we have used (2.7). This contradicts (2.5) and the proof is complete. □

Remark 2.4. It is easy to see that (2.5) for $\beta=0$ reduces to (1.5) and we conclude that our criterion improves that of Kusano.

The following considerations are intended to simplify condition (2.5), namely to avoid computation of double integral with exponential function.

Corollary 2.5. *Let there exist sequence* $\{t_k\}$ *satisfying* (1.2) *and* $\beta>0$ *such that* (2.4) *holds. If*

$$\begin{aligned} \limsup_{k\to\infty}\int_{\tau(t_k)}^{t_k} p(s)\Bigg\{&\frac{\big(\tau(t_k)-\tau(s)\big)^{n-1}}{(n-1)!}+\beta\frac{\big(\tau(t_k)-\tau(s)\big)^{n}}{n!} \\ &+\beta^2\frac{\big(\tau(t_k)-\tau(s)\big)^{n+1}}{(n+1)!}\Bigg\}\,\mathrm{d}s>1, \end{aligned} \tag{2.10}$$

then $\mathcal{N}_0=\emptyset$ *for* (E).

Proof. We employ the identity

$$\int_{\tau(s)}^{\tau(t_k)} e^{-\beta x} \frac{\big(x-\tau(s)\big)^{n-2}}{(n-2)!}\,dx$$
$$= e^{-\beta\tau(t_k)} \left\{ \frac{\big(\tau(t_k)-\tau(s)\big)^{n-1}}{(n-1)!} + \beta\frac{\big(\tau(t_k)-\tau(s)\big)^{n}}{n!} + \beta^2\frac{\big(\tau(t_k)-\tau(s)\big)^{n+1}}{(n+1)!} \right\}$$
$$+ \beta^3 \int_{\tau(s)}^{\tau(t_k)} e^{-\beta x} \frac{\big(x-\tau(s)\big)^{n+1}}{(n+1)!}\,dx.$$

Then

$$\int_{\tau(s)}^{\tau(t_k)} e^{-\beta x} \frac{\big(x-\tau(s)\big)^{n-2}}{(n-2)!}\,dx \geq e^{-\beta\tau(t_k)} \left\{ \frac{\big(\tau(t_k)-\tau(s)\big)^{n-1}}{(n-1)!} + \beta\frac{\big(\tau(t_k)-\tau(s)\big)^{n}}{n!} + \beta^2\frac{\big(\tau(t_k)-\tau(s)\big)^{n+1}}{(n+1)!} \right\}.$$

It follows from the last inequality that (2.10) implies (2.5) and the proof is complete. □

Remark 2.6. We can include more corresponding terms into (2.10) but this yields only very little progress and so exactly three terms are enough.

Now we present criterion for elimination of the class $\mathcal{N}_n$.

Theorem 2.7. *Let there exist sequence* $\{s_k\}$ *satisfying* (1.3) *and* $\gamma > 0$ *such that*

$$\frac{(\tau(t)-t)^{n-1}}{(n-1)!}p(t) \geq \gamma, \quad t \in [\tau(s_k), \tau(\tau(s_k))], \quad k = 1, 2, \dots \tag{2.11}$$

If

$$\limsup_{k\to\infty} e^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} p(t) \int_{\tau(s_k)}^{\tau(t)} e^{\gamma x}\big(\tau(t)-x\big)^{n-2}\,dx dt > (n-2)!, \tag{2.12}$$

then $\mathcal{N}_n = \emptyset$ *for* (E).

Proof. Assume on the contrary that (E) has an eventually positive solution $y(t)$ of degree n. Then for $u < v$ we have

$$y^{(n-2)}(v) \geq \int_u^v y^{(n-1)}(x)\,dx.$$

An integration in v over $[u,v]$ yields

$$y^{(n-3)}(v) \geq \int_u^v \int_u^z y^{(n-1)}(x)\,\mathrm{d}x\mathrm{d}z = \int_u^v y^{(n-1)}(x)(v-x)\,\mathrm{d}x.$$

Repeating this procedure we have

$$y(v) \geq \int_u^v y^{(n-1)}(x)\frac{(v-x)^{n-2}}{(n-2)!}\,\mathrm{d}x \geq y^{(n-1)}(u)\frac{(v-u)^{n-1}}{(n-1)!}. \tag{2.13}$$

We consider $t \in J = [\tau(s_k), \tau(\tau(s_k))] \subset \mathcal{A}_\tau$, $k = 1, 2, \ldots$ Setting $u = t$ and $v = \tau(t)$ into (2.13) one can see that

$$y(\tau(t)) \geq y^{(n-1)}(t)\frac{(\tau(t)-t)^{n-1}}{(n-1)!},$$

which in view of (E) and (2.11) yields

$$y^{(n)}(t) = p(t)y(\tau(t)) \geq \gamma\, y^{(n-1)}(t) \quad t \in I,$$

which implies that

$$\mathrm{e}^{-\gamma\, t} y^{(n-1)}(t) \uparrow \quad \text{on } J. \tag{2.14}$$

On the other hand, an integration of (E) yields

$$y^{(n-1)}(\tau(s_k)) \geq \int_{s_k}^{\tau(s_k))} p(t)y(\tau(t))\,\mathrm{d}t. \tag{2.15}$$

Employing (2.13) with $u = \tau(s_k)$ and $v = \tau(t)$ with $t \in [\tau(t_k), t_k]$ we get

$$y(\tau(t)) \geq \int_{\tau(s_k)}^{\tau(t)} y^{(n-1)}(x)\frac{(\tau(t)-x)^{n-2}}{(n-2)!}\,\mathrm{d}x. \tag{2.16}$$

Combining (2.16) together with (2.15) we obtain

$$\begin{aligned} y^{(n-1)}(\tau(s_k)) &\geq \int_{s_k}^{\tau(s_k)} p(t) \int_{\tau(s_k)}^{\tau(t)} y^{(n-1)}(x)\frac{(\tau(t)-x)^{n-2}}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}t \\ &\geq y^{(n-1)}(\tau(s_k))\mathrm{e}^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} p(t) \int_{\tau(s_k)}^{\tau(t)} \mathrm{e}^{\gamma\, x}\frac{(\tau(t)-x)^{n-2}}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where we have used (2.14). This contradicts (2.12) and the proof is complete. □

Corollary 2.8. *Let there exist sequence* $\{s_k\}$ *satisfying* (1.3) *and* $\gamma > 0$ *such that* (2.11) *holds. If*

$$\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} p(t)\left\{ \frac{\big(\tau(t)-\tau(s_k)\big)^{n-1}}{(n-1)!} + \gamma \frac{\big(\tau(t)-\tau(s_k)\big)^{n}}{n!} + \gamma^2 \frac{\big(\tau(t)-\tau(s_k)\big)^{n+1}}{(n+1)!} \right\} \mathrm{d}t > 1, \tag{2.17}$$

then $\mathcal{N}_n = \emptyset$ *for* (E).

If we pick up the previous results we immediately obtain criterion for oscillation of (E).

Theorem 2.9. *Assume that all conditions of Theorem 2.1, Corollary 2.5 and Corollary 2.8 hold true. Then* (E) *is oscillatory.*

We support novelty of the paper with the following illustrative example.

Example 2.10. Consider the equation

$$y^{(n)}(t) = p_0 y(t+\sin t), \tag{E_x}$$

where $p_0 > 0$ is a constant. Clearly , the deviating argument $\tau(t) = t + \sin t$ is of mixed type and $\tau_*(t) \ge t-1$. Consequently, it is easy to see that (2.1) and (2.2) hold true and by Theorem 2.1 the classes $\mathcal{N}_\ell = \emptyset$ for all $\ell = 2, 4, \dots, n-2$.

We choose the positive constant a such that $a = \cos a$ (≈ 0.739085) and if we set $t_k = (-\pi/2) + 2k\pi + a$, $k = 1, 2, \dots$, then $t_k \in \mathcal{D}_\tau$ and moreover $\tau(t_k) = (-\pi/2) + 2k\pi$ and $\tau(\tau(t_k)) = (-\pi/2) + 2k\pi - 1$.

A simple computation shows that for $t \in [\tau(\tau(t_k)), \tau(t_k)]$

$$\begin{aligned} \frac{(t-\tau(t))^{n-1}}{(n-1)!}\, p(t) &= \frac{p_0}{(n-1)!}(-\sin t)^{n-1} \\ &\ge \frac{p_0}{(n-1)!}(-\sin(\tau(\tau(t_k))))^{n-1} \\ &= \frac{p_0}{(n-1)!}(\cos 1)^{n-1} = \beta. \end{aligned}$$

On the other hand, by the Jensen inequality

$$\begin{aligned} \int_{\tau(t_k)}^{t_k} (\tau(t_k)-\tau(s))^{n-1}\,\mathrm{d}s &\ge (t_k-\tau(t_k)) \left(\frac{1}{t_k-\tau(t_k)} \int_{\tau(t_k)}^{t_k} (\tau(t_k)-\tau(s))\,\mathrm{d}s \right)^{n-1} \\ &= \frac{1}{a^{n-2}} \left(\sin a - \frac{a^2}{2} \right)^{n-1} = a \left(\frac{\sqrt{1-a^2}}{a} - \frac{a}{2} \right)^{n-1}. \end{aligned}$$

Therefore criterion (2.10) reduces to a simple condition

$$\begin{aligned} ap_0\Bigg\{ &\frac{1}{(n-1)!}\left(\frac{\sqrt{1-a^2}}{a}-\frac{a}{2}\right)^{n-1}+\frac{\beta}{n!}\left(\frac{\sqrt{1-a^2}}{a}-\frac{a}{2}\right)^{n} \\ &+\frac{\beta^2}{(n+1)!}\left(\frac{\sqrt{1-a^2}}{a}-\frac{a}{2}\right)^{n+1}\Bigg\}>1 \end{aligned} \tag{2.18}$$

and by Corollary 2.5 the class $\mathcal{N}_0=\emptyset$.

Now, if we set $s_k=(\pi/2)+2k\pi-a$, $k=1,2,\dots$, then $s_k\in\mathcal{A}_\tau$ and moreover $\tau(s_k)=(\pi/2)+2k\pi$ and $\tau(\tau(s_k))=(\pi/2)+2k\pi+1$.

It is easy to verify that for $t\in[\tau(s_k),\tau(\tau(s_k))]$

$$\begin{aligned} \frac{(\tau(t)-t)^{n-1}}{(n-1)!}\,p(t) &= \frac{p_0}{(n-1)!}(\sin t)^{n-1} \\ &\ge \frac{p_0}{(n-1)!}(\sin(\tau(\tau(t_k))))^{n-1} \\ &= \frac{p_0}{(n-1)!}(\cos 1)^{n-1}=\gamma=\beta. \end{aligned}$$

Proceeding exactly as above, we can verify that criterion (2.17) reduces again to (2.18) and by Corollary 2.8 the class $\mathcal{N}_n=\emptyset$. Finally, we conclude that (2.18) guarantees oscillation of (E_x).

Remark 2.11. To show the progress and novelty of our results, we compare our criterion (2.18) obtained for (E_x) together with Kusano's (1.7). If we set $n=4$, then (1.7) guarantees oscillation of (E_x) for $p_0>150.7$ while (2.18) requires only $p_0>43.5$. The progress is manifest.

3. SUMMARY

In this paper we tried to fulfill the certain gap in the oscillation theory concerning differential equations with mixed arguments. Our criteria are of high generality, easily verifiable and improve the known ones. Moreover our results are applicable also for n-odd when

$$\mathcal{N}=\mathcal{N}_1\cup\mathcal{N}_3\cup\ldots\cup\mathcal{N}_n.$$

In this case the situation is simpler due to absence of class $\mathcal{N}_0$.

REFERENCES

[1] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations*, Kluver Academic Publishers, Dotrecht, 2002.

[2] B. Baculikova, *Oscillation of second-order nonlinear noncanonical differential equations with deviating argument*, Appl. Math. Lett. **91** (2019), 68–75.

[3] B. Baculikova, *Oscillatory behavior of the second order noncanonical differential equation*, Electron. J. Qual. Theory Differ. Equ. **2019**, no. 89, 1–17.

[4] B. Baculikova, *Oscillation and asymptotic properties of second order half-linear differential equations with mixed deviating arguments*, Mathematics **2021**, 9(20), 2552.

[5] G.E. Chatzarakis, B. Dorociakova, R. Olah, *An oscillation criterion of linear delay differential equations*, Adv. Difference Equ. (2021), Article no. 85.

[6] O. Došly, P. Řehák, *Half-linear Differential Equations*, vol. 202, North-Holland Mathematics Studies, 2005.

[7] J. Dzurina, I. Jadlovska, *Oscillation of n-th order strongly noncanonical delay differential equations*, Appl. Math. Lett. **115** (2021), Article no. 106940.

[8] S.R. Grace, I. Jadlovska, A. Zafer, *Oscillatory behavior of n-th order nonlinear delay differential equations with a nonpositive neutral term*, Hacet. J. Math. Stat. **49** (2021), no. 2, 766–776.

[9] I.T. Kiguradze, T.A. Chaturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Acad. Publ., Dordrecht, 1993.

[10] R. Koplatadze, *On differential equations with a delayed argument having properties A and B*, Differ. Uravn. (Minsk) **25** (1989), 1897–1909.

[11] R. Koplatadze, T.A. Chanturia, *On oscillatory properties of differential equations with deviating arguments*, Tbilisi Univ. Press, Tbilisi, 1977.

[12] R. Koplatadze, G. Kvinkadze, I.P. Stavroulakis, *Properties A and B of n-th order linear differential equations with deviating argument*, Georgian Math. J. **6** (1999), no. 6, 553–566.

[13] T. Kusano, *On even order functional differential equations with advanced and retarded arguments*, J. Differential Equations **45** (1982), no. 1, 75–84.

[14] T. Kusano, *Oscillation of even order linear functional differential equations with deviating arguments of mixed type*, J. Math. Anal. Appl. **98** (1984), 341–347.

[15] G. Ladas, V. Lakshmikantham, J.S. Papadakis, B.G. Zhang, *Oscillation of higher-order retarded differential equations generated by retarded argument*, Delay and Functional Differential Equations and Their Applications, pp. 219–231, Academic Press, New York, 1972.

Blanka Baculikova
blanka.baculikova@tuke.sk

Technical University of Košice
Faculty of Electrical Engineering and Informatics
Department of Mathematics
Letná 9, 042 00 Košice, Slovakia

Received: March 11, 2022.
Revised: April 27, 2022.
Accepted: May 12, 2022.