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RECURSIVE SET MEMBERSHIP ESTIMATION FOR OUTPUT–ERROR FRACTIONAL MODELS WITH UNKNOWN–BUT–BOUNDED ERRORS

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This paper presents a new formulation for set-membership parameter estimation of fractional systems. In such a context, the error between the measured data and the output model is supposed to be unknown but bounded with *a priori* known bounds. The bounded error is specified over measurement noise, rather than over an equation error, which is mainly motivated by experimental considerations. The proposed approach is based on the optimal bounding ellipsoid algorithm for linear output-error fractional models. A numerical example is presented to show effectiveness and discuss results.

Keywords: fractional calculus, set membership, estimation, unknown-but-bounded error.

1. Introduction

Over the past years, fractional differentiation has successfully been used for the description of memory and hereditary properties of some physical phenomena with non-exponential type decay like in fluid flow, rheology, diffusive transport, electrical networks and electromagnetic theory. The reader can refer to the work of Machado *et al.* (2011) and the references therein for more details. This great interest is motivated by the compact system representation obtained when modelling such phenomena using fractional models.

To deal with fractional modeling, parameter estimation theory using fractional models is often used to obtain a better system representation. A number of results are available for parameter estimation, mainly using different conventional approaches. For details, the reader can refer to the works of Narang *et al.* (2011), Victor *et al.* (2013) and Yakoub *et al.* (2015), and the references therein. The conventional approaches are relevant when *a priori* knowledge of statistical assumptions about measurement noises is available. However, this is not always the case in practice, and it is often more natural to assume that disturbances are unknown but bounded with *a priori* known bounds. In this context, the identification approach is called set-membership or bounded-error parameter estimation, and the result is not a parameter vector but a set of feasible parameter vectors. A parameter vector is said to be feasible if the error between the system output and the model output remains in the bounded-error interval.

The bounded-error approach to parameter estimation problems for linear and nonlinear rational systems has attracted the attention of many researchers. The reader can refer to the works of Milanese *et al.* (1996), Raissi *et al.* (2004) and Polyak *et al.* (2004), as well as the references therein, for more details. For fractional models, the development of bounded-error approaches has begun recently. For frequency-domain identification, there exist some approaches presented in the literature. One uses a set inversion algorithm (Malti *et al.*, 2010) and an interval-based global optimization algorithm (Amairi *et al.*, 2012). For time-domain parameter estimation, some methods formalizing the identification problem as an ellipsoidal or paralellotope set estimation problem have been used to characterize the set of all feasible parameters which are compatible with the model, the measured data and some prior error bounds (Amairi, 2015). In these methods the bounds of the error at each sampling time are specified over an equation error.

In practice, the bounded error is specified over a measurement noise rather than over an equation error, and then it is more suitable to use a set-membership approach based on the output error. A first attempt is made using an interval-based global optimization algorithm to estimate both the coefficients and the fractional orders of the system (Amairi *et al.*, 2010). The main disadvantage of this approach is the important computational burden which limits its use (the number of parameters to be estimated should be less than four even with reduced precision). In this paper, discussion will be concentrated on the use of set-membership parameter estimation using output-error fractional models. The proposed approach bounds the feasible set of parameters by an ellipsoid defined by its center and axis, and then a projection algorithm gives the interval parameter's uncertainties.

The paper is structured as follows. A brief review of fractional differentiation and linear fractional orders systems is presented in Section 2. Section 3 discusses the principle of set-membership parameter estimation of fractional models considering bounding the output error. A method based on the outer bounding ellipsoid algorithm is developed to identify a system using a fractional model. Section 4 presents a simulation example to show the effectiveness of the proposed method.

2. Preliminaries

Consider a SISO linear fractional order system governed by the following fractional differential equation:

$$y\left(t\right)+\sum_{i=1}^{n}a_i\mathbf{D}^{\upsilon_{ai}}y\left(t\right)=\sum_{j=0}^{m}b_j\mathbf{D}^{\upsilon_{bj}}u\left(t\right), \tag{1}$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{R}$ are the linear coefficients of the differential equation, $u(t)$ and $y(t)$ are respectively the input and the output signals, and $\mathbf{D}^{\upsilon}$ is the time domain differential operator.

The fractional orders υ_{ai} and υ_{bj} are allowed to be non-integer positive numbers and are ordered as follows for the identifiability purpose:

$$\begin{aligned}\upsilon_{a1} &< \upsilon_{a2} < \cdots < \upsilon_{an},\\ \upsilon_{b0} &< \upsilon_{b1} < \cdots < \upsilon_{bm},\end{aligned} \tag{2}$$

The υ-order fractional differentiation of a continuous-time function $g(t)$, relaxed at $t = 0$, i.e., $g(t) = 0$ for $t \le 0$, is numerically evaluated using the Grünwald definition (Samko *et al.*, 1993):

$$\mathbf{D}^{\upsilon}g\left(t\right)\triangleq\lim_{h\to 0}\frac{1}{h^{\upsilon}}\sum_{k=0}^{K}(-1)^k\binom{\upsilon}{k}g\left(t-kh\right),\quad \forall\upsilon\in\mathbb{R}, \tag{3}$$

where h is the sampling period, $t = Kh$ and

$$\binom{\upsilon}{k}=\begin{cases}1 & \text{if } k=0,\\ \dfrac{\upsilon\left(\upsilon-1\right)\ldots\left(\upsilon-k+1\right)}{k!} & \text{if } k>0\end{cases} \tag{4}$$

is Newton's binomial generalized to fractional orders.

The Laplace transform of the υ-order fractional differentiation of a function $g(t)$ with null initial conditions is defined by (Samko *et al.*, 1993)

$$\mathcal{L}\left\{D^{\upsilon}g(t)\right\}=s^{\upsilon}G(s), \tag{5}$$

where $G(s) = \mathcal{L}(g(t))$ and s is the Laplace operator.

Applying the Laplace transform to the fractional differential equation (1) yields the following fractional transfer function:

$$F\left(s\right)=\frac{B(s)}{A(s)}=\frac{\sum\limits_{j=0}^{m}b_j s^{\upsilon_{bj}}}{1+\sum\limits_{i=1}^{n}a_i s^{\upsilon_{ai}}}. \tag{6}$$

Moreover, if $F(s)$ is ν-commensurate[1] with $\nu \in \mathbb{R}_+$, then it can be rewritten as follows:

$$F\left(s\right)=\frac{\sum\limits_{J=0}^{M}b_J s^{J\nu}}{1+\sum\limits_{I=1}^{N}a_I s^{I\nu}}, \tag{7}$$

where $N = \upsilon_{a_n}/\nu$ and $M = \upsilon_{b_m}/\nu$ are integers, and $\forall I \in \{1,\ldots N\}, \forall J \in \{0,\ldots,M\}$:

$$b_J=\begin{cases}b_j \text{ if } \exists j\in\{0,\ldots,m\} \text{ s.t.} \upsilon_{b_j}=J\nu,\\ 0 \text{ otherwise},\end{cases} \tag{8}$$

$$a_I=\begin{cases}a_i \text{ if } \exists i\in\{1,\ldots,n\} \text{ s.t. } \upsilon_{a_i}=I\nu,\\ 0 \text{ otherwise}.\end{cases} \tag{9}$$

A fractional system, represented by its transfer function (6), is bounded input bounded output (BIBO) stable if (Matignon, 1996) there exists R satisfying

$$\left|F\left(s\right)\right|\le R \tag{10}$$

and

$$\mathrm{Re}\left(p\right)\ge 0,$$

for any complex pole p.

In the case where $F(s)$ is irreducible, the stability property reads

$$\text{BIBO stability} \Leftrightarrow \; A\left(s\right)\neq 0 \quad\text{and}\; \mathrm{Re}\left(p\right)\ge 0, \forall p. \tag{11}$$

A commensurate-order system described by Eqn. (7) is BIBO stable iff (Matignon, 1996)

$$0<\nu<2 \tag{12}$$

and

$$\forall s_k^{\nu}\in\mathbb{C},\quad A\left(s_k^{\nu}\right)=0 \quad \text{such that} \quad \left|\arg\left(s_k^{\nu}\right)\right|>\nu\frac{\pi}{2}, \tag{13}$$

where s_k^{ν} is a pole of the commensurate transfer function.

[1] All differentiation orders are exactly divisible by the same number, an integral number of times.

3. Set-membership parameter estimation using the output-error fractional model

3.1. Problem statement. Consider a linear fractional model given by the following differential equation:

$$y^*(t) + \sum_{i=1}^{n} a_i \mathbf{D}^{\upsilon_{a_i}} y^*(t) = \sum_{j=0}^{m} b_j \mathbf{D}^{\upsilon_{b_j}} u^*(t), \qquad (14)$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{R}$ are the linear coefficients of the differential equation, and $y^*(t)$, $u^*(t)$ are respectively the noise-free output and input signals.

We assume that the system output is corrupted by additive noise η as shown in Fig. 1, i.e.,

$$y = y^* + \eta. \qquad (15)$$

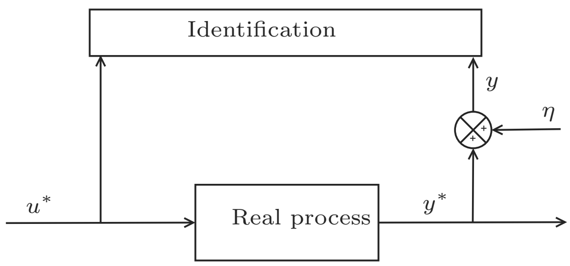

Fig. 1. Basic setup for the identification problem.

In order to derive a recursive algorithm for parameter estimation, an indirect approach with four steps is used here. It consists of discretization, linearization, estimation of pseudo-coefficients, and finally return to the continuous-time parameters.

Now, assuming that the discretization and linearization steps are performed, the discrete linear form of the differential equation (14) can be written as

$$y_K = -\sum_{i=0}^{n} \alpha_i \tilde{y}_{i,K} + \sum_{j=0}^{m} \beta_j \tilde{u}_{j,K} + e_K, \qquad (16)$$

where

$$\tilde{y}_{i,K} = \sum_{k=1}^{K} (-1)^k \binom{\upsilon_{a_i}}{k} y_{K-k}, \qquad (17)$$

$$\tilde{u}_{j,K} = \sum_{k=0}^{K} (-1)^k \binom{\upsilon_{b_j}}{k} u^*_{K-k}, \qquad (18)$$

$$\alpha_i = \frac{\dfrac{a_i}{h^{\upsilon_{a_i}}}}{1 + \displaystyle\sum_{l=1}^{n} \frac{a_l}{h^{\upsilon_{a_l}}}}, \quad \beta_j = \frac{\dfrac{b_j}{h^{\upsilon_{b_j}}}}{1 + \displaystyle\sum_{l=1}^{n} \frac{a_l}{h^{\nu_{a_l}}}}, \qquad (19)$$

$$e_K = \eta_K + \sum_{i=0}^{n} \alpha_i \tilde{\upsilon}_{i,K}, \quad \sum_{i=0}^{n} \alpha_i = 1, \qquad (20)$$

$$\tilde{\upsilon}_{i,K} = \sum_{k=1}^{K} (-1)^k \binom{\upsilon_{a_i}}{k} \eta_{K-k}, \qquad (21)$$

with h being the sampling period used in the discretization step. To ensure the stability of the discrete-time system, a stability test can be performed as mentioned by Ostalczyk (2012) or Busłowicz and Ruszewski (2015).

By replacing α_0 in the Eqn. (16) with $1 - \sum_{i=1}^{n} \alpha_i$ and taking into account that $\tilde{y}_{0,K} = 0, \forall K$, we obtain the following linear regression form:

$$y_K = -\tilde{y}_{0,K} + \tilde{\varphi}_K \tilde{\theta} + e_K, \qquad (22)$$

$$= \tilde{\varphi}_K \tilde{\theta} + e_K, \qquad (23)$$

where

$$\tilde{\theta} = [\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_m]^T, \qquad (24)$$

$$\tilde{\varphi}_K = \left[-\tilde{y}_{1,K}, \ldots, -\tilde{y}_{n,K}, \tilde{u}_{0,K}, \ldots, \tilde{u}_{m,K}\right]. \qquad (25)$$

The return from the pseudo-coefficients (α_i and β_j) to the continuous-time parameters (a_i and b_j) can be performed by solving the linear system $Ax = B$, where

$$A_{i,j} = \begin{cases} \dfrac{(\alpha_i - 1)}{h^{\upsilon_{a_i}}} & \text{if } i = j, \\ \dfrac{\alpha_i}{h^{\upsilon_{a_j}}} & \text{if } i \neq j, \end{cases} \qquad (26)$$

$$B_i = -\alpha_i, \qquad (27)$$

$$x = (a_1, a_2, \ldots, a_n)^T, \qquad (28)$$

and the coefficients b_j are computed according to

$$b_j = \beta_j \sum_{i=0}^{n} a_i h^{\upsilon_{b_j} - \upsilon_{a_i}}. \qquad (29)$$

In the bounded-error context, the additive noise is assumed to be an unknown-but-bounded sequence, i.e., there exists a known γ such that

$$|\eta_K| \leq \gamma, \quad \forall K \geq 0. \qquad (30)$$

In this context, the problem statement is characterizing the feasible set of parameters Θ,

$$\Theta = \big\{\theta \in \mathbb{R}^{n+m+1} : |y_K - \hat{y}_K| \leq \gamma, \quad K = 1, \ldots, N_s\big\}. \qquad (31)$$

where N_s is the number of samples used and $\hat{y}_K$ is the estimated model output.

Using the return procedure explained above, the feasible set of parameters Θ can be obtained from the feasible set of pseudo-parameters $\tilde{\Theta}$,

$$\tilde{\Theta} = \Big\{\tilde{\theta} \in \mathbb{R}^{n+m+1}, \\ \big|y_K - \tilde{\varphi}_K^T \tilde{\theta}\big| \leq \gamma, \quad K = 1, \ldots, N_s\Big\}. \tag{32}$$

One way to solve this problem is by considering that the equation error is bounded. This was developed by Amairi (2015) who proposed the characterization using the outer bounding ellipsoid strategy adapted for estimation of fractional system parameters. To ensure good convergence, several procedures for the selection of the 'best' excitation signal and computation of the equation-error bounds were proposed and discussed. Also, when the system is commensurate, an iterative algorithm was proposed to deal with commensurate-order estimation.

In this paper, we propose the use of the bounded output-error context instead of the bounded equation-error one. This choice is motivated by the fact that it is more natural to specify the noise bounds than the equation-error bounds. In spite of the possibility to compute equation-error bounds from noise bounds as mentioned by Amairi (2015), the procedure remains inaccurate because they depend on the pseudo-parameter estimates α_i, the fractional orders υ_{a_i}, the noise bounds γ, and also the sampling period h.

3.2. Formulation in a bounded output-error context.

Equation (15) can be written as a linear regressive equation as

$$y_K = \tilde{\varphi}_K^* \tilde{\theta} + \eta_K, \tag{33}$$

where

$$\tilde{\varphi}_K^* = \left[-\tilde{y}_{1,K}^*, \ldots, \tilde{y}_{n,K}^*, \tilde{u}_{0,K}, \ldots, \tilde{u}_{m,K}\right]$$

is the free-noise regression vector and

$$\tilde{\theta} = \left[\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_m\right]^T$$

is the vector of pseudo-parameters.

The regression vector in Eqn. (33) is partially unknown because it contains the model output y_K^* and not the measured one y_K. One solution is to replace y_K^* by $y_K - \eta_K$, which leads to Eqn. (16), which can be rewritten in the regression form as in Eqn. (23).

The regression vector

$$\tilde{\varphi}_K = \left[-\tilde{y}_{1,K}, \ldots, -\tilde{y}_{n,K}, \tilde{u}_{0,K}, \ldots, \tilde{u}_{m,K}\right]$$

now contains the measured output, but it remains unknown because it depends on the pseudo-parameters to estimate. One way to overcome this dependence is to reformulate the linearization procedure using the bounded context of the additive noise.

Proposition 1. *Assume that $|\eta_K| \leq \gamma$. In the output-error bounding context, two conditions delimiting the feasible set of the pseudo-parameters should be satisfied:*

$$\tilde{\varphi}_K^- \tilde{\theta} \leq y_K + \gamma, \tag{34}$$

$$\tilde{\varphi}_K^+ \tilde{\theta} \geq y_K - \gamma, \tag{35}$$

where

$$\tilde{\varphi}_K^+ = \left[-\tilde{y}_{1,K} + \operatorname{sgn}(\alpha_1)\tilde{\gamma}_1, \ldots, -\tilde{y}_{n,K} + \operatorname{sgn}(\alpha_n)\tilde{\gamma}_n, \\ \tilde{u}_{0,K}, \ldots, \tilde{u}_{m,K}\right], \tag{36}$$

$$\tilde{\varphi}_K^- = \left[-\tilde{y}_{1,K} - \operatorname{sgn}(\alpha_1)\tilde{\gamma}_1, \ldots, -\tilde{y}_{n,K} - \operatorname{sgn}(\alpha_n)\tilde{\gamma}_n, \\ \tilde{u}_{0,K}, \ldots, \tilde{u}_{m,K}\right], \tag{37}$$

and

$$\tilde{\gamma}_i = (c(\upsilon_{a_i}) - 1)\gamma,$$

with

$$c(\upsilon_{a_i}) = \sum_{k=0}^{\lfloor \upsilon_{a_i} \rfloor} \left(1 + (-1)^{k + \lfloor \upsilon_{a_i} \rfloor}\right) \binom{\upsilon_{a_i}}{k}.$$

Here $\operatorname{sgn}(x)$ *denotes the function whose value is* $+1$ *if* $x \geq 0$ *and* -1 *if* $x < 0$.

Proof. The inequality

$$y_K - \gamma \leq y_K^* \leq y_K + \gamma, \quad \forall K \tag{38}$$

leads to

$$y_{K-k} - \gamma \leq y_{K-k}^* \leq y_{K-k} + \gamma, \\ \forall K, \quad k \in \{0, \ldots, K\}. \tag{39}$$

Taking into account the sign of $\binom{\upsilon_{a_i}}{k}$ and the parity of k, we obtain

$$(-1)^k \binom{\upsilon_{a_i}}{k} \left(y_{K-k} - \operatorname{sgn}\left((-1)^k \binom{\upsilon_{a_i}}{k}\right)\gamma\right) \\ \leq (-1)^k \binom{\upsilon_{a_i}}{k} y_{K-k}^* \\ \leq (-1)^k \binom{\upsilon_{a_i}}{k} \left(y_{K-k} + \operatorname{sgn}\left((-1)^k \binom{\upsilon_{a_i}}{k}\right)\gamma\right). \tag{40}$$

Summing all these terms, we obtain

$$\sum_{k=1}^{K} (-1)^k \binom{\upsilon_{a_i}}{k} \left(y_{K-k} - \operatorname{sgn}\left((-1)^k \binom{\upsilon_{a_i}}{k}\right)\gamma\right) \\ \leq \tilde{y}_{i,K}^* \\ \leq \sum_{k=1}^{K} (-1)^k \binom{\upsilon_{a_i}}{k} \left(y_{K-k} + \operatorname{sgn}\left((-1)^k \binom{\upsilon_{a_i}}{k}\right)\gamma\right). \tag{41}$$

Furthermore, we have

$$\begin{aligned}\tilde{y}_{i,K} - \sum_{k=1}^{\infty} \left|(-1)^k \binom{\upsilon_{a_i}}{k}\right| \gamma &\le \tilde{y}_{i,K} - \sum_{k=1}^{K} \left|(-1)^k \binom{\upsilon_{a_i}}{k}\right| \gamma \\ &\le \tilde{y}^*_{i,K} \\ &\le \tilde{y}_{i,K} + \sum_{k=1}^{K} \left|(-1)^k \binom{\upsilon_{a_i}}{k}\right| \gamma \\ &\le \tilde{y}_{i,K} + \sum_{k=1}^{\infty} \left|(-1)^k \binom{\upsilon_{a_i}}{k}\right| \gamma.\end{aligned} \tag{42}$$

Consider

$$c(\upsilon_{a_i}) \triangleq \sum_{k=0}^{\infty} \left|\binom{\upsilon_{a_i}}{k}\right| < \infty,$$

which can be represented as a finite sum as (Samko *et al.*, 1993)

$$c(\upsilon_{a_i}) = \sum_{k=0}^{\lfloor \upsilon_{a_i} \rfloor} \left(1 + (-1)^{k+\lfloor \upsilon_{a_i} \rfloor}\right) \binom{\upsilon_{a_i}}{k}, \tag{43}$$

where $\lfloor \upsilon_{a_i} \rfloor$ means the largest integer not greater than υ_{a_i}. Then we obtain

$$\begin{aligned}\tilde{y}_{i,K} - (c(\upsilon_{a_i}) - 1)\gamma &\le \tilde{y}^*_{i,K} \\ &\le \tilde{y}_{i,K} + (c(\upsilon_{a_i}) - 1)\gamma.\end{aligned} \tag{44}$$

Taking into account the sign of α_i, we have

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \left(\tilde{y}_{i,K} - \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right) &\le \sum_{i=1}^{n} \alpha_i \tilde{y}^*_{i,K} \\ &\le \sum_{i=1}^{n} \alpha_i \left(\tilde{y}_{i,K} + \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right)\end{aligned} \tag{45}$$

Changing the sign, we get

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} - \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right) &\le -\sum_{i=1}^{n} \alpha_i \tilde{y}^*_{i,K} \\ &\le \sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} + \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right),\end{aligned} \tag{46}$$

which provides bounds for y_K,

$$\begin{aligned}&\sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} - \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right) + \sum_{j=1}^{m} \beta_j \tilde{u}_{j,K} \\ &\quad \le y^*_K \\ &\quad \le \sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} + \operatorname{sgn}(\alpha_i)(c(\upsilon_{a_i}) - 1)\gamma\right) \\ &\qquad + \sum_{j=0}^{m} \beta_j \tilde{u}_{j,K}.\end{aligned} \tag{47}$$

But y^*_K is also bounded by

$$y_K - \gamma \le y^*_K \le y_K + \gamma. \tag{48}$$

Accordingly, in order to make these two pairs of inequalities coherent, two conditions should be satisfied:

$$\sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} - \operatorname{sgn}(\alpha_i)\tilde{\gamma}_i\right) + \sum_{j=0}^{m} \beta_j \tilde{u}_{j,K} \le y_K + \gamma \tag{49}$$

and

$$\sum_{i=1}^{n} \alpha_i \left(-\tilde{y}_{i,K} + \operatorname{sgn}(\alpha_i)\tilde{\gamma}_i\right) + \sum_{j=0}^{m} \beta_j \tilde{u}_{j,K} \ge y_K - \gamma, \tag{50}$$

which lead respectively to (34) and (35). ■

Remark 1. With this reformulation, some additional *a priori* knowledge is needed, but it is limited to the values of the differentiation orders (υ_{a_i} and υ_{b_j}) and the signs of the autoregressive pseudo-parameters α_i determined by

$$\begin{aligned}\operatorname{sgn}(\alpha_i) &= \operatorname{sgn}(a_i)\operatorname{sgn}(\alpha_0), \\ \operatorname{sgn}(\alpha_0) &= \operatorname{sgn}\left(\sum_{l=0}^{n} \frac{a_l}{h^{\upsilon_{a_l}}}\right),\end{aligned} \tag{51}$$

where $a_0 = 1$, $\upsilon_{a_0} = 0$. Thereafter, using the new formulation, one assumption employed in the equation-error-based algorithm is relaxed. In fact, there is no necessity to know the values of α_i, and only their sign is required.

From a geometrical point of view, the inequalities (34) and (35) define a band $\Lambda_K = \left[H^-_K, H^+_K\right]$ of $\mathbb{R}^p$, where the band is contained between two non-parallel hyperplanes:

$$H^-_K = \left\{\tilde{\theta} \in \mathbb{R}^p : \tilde{\varphi}^+_K \tilde{\theta} = y_K - \gamma\right\}, \tag{52}$$

$$H^+_K = \left\{\tilde{\theta} \in \mathbb{R}^p : \tilde{\varphi}^-_K \tilde{\theta} = y_K + \gamma\right\}, \tag{53}$$

and $p = n + m + 1$ is the number of parameters to be estimated.

The problem considered in this paper is now reformulated as the characterization of the set of feasible pseudo-parameters $\tilde{\Theta}$,

$$\tilde{\Theta} = \Big\{ \tilde{\theta} \in \mathbb{R}^p : \quad y_K - \tilde{\varphi}_K^{+}\tilde{\theta} \leq \gamma, \\ y_K - \tilde{\varphi}_K^{-}\tilde{\theta} \geq -\gamma, \ K = 1, \dots, N \Big\}. \tag{54}$$

One way to characterize the set $\tilde{\Theta}_K$ is the intersection of the last K bands. This intersection operation yields a convex polyhedron with a complex geometric form:

$$\tilde{\Theta}_K = \bigcap_{k=1}^{K} \Lambda_k. \tag{55}$$

In the following, an ellipsoidal approach to fractional model estimation in the bounded output-error context is presented.

3.3. Set-membership parameter estimation through the outer bounding ellipsoid (OBE).

The proposed approach consists in enclosing $\tilde{\Theta}_K$ with an ellipsoid

$$\tilde{E}_K = \Big\{ \tilde{\theta} \in \mathbb{R}^p : (\tilde{\theta} - \tilde{c}_{K-1})^T \tilde{M}_{K-1}^{-1} (\tilde{\theta} - \tilde{c}_{K-1}) \leq \tilde{\sigma}_K^2 \Big\}, \tag{56}$$

where $\tilde{c}_{K-1}$ is the center of the ellipsoid and $\tilde{M}_{K-1}$ is a positive definite matrix. The axes of the ellipsoid are determined from the eigenvectors of the matrix $\tilde{\sigma}_K^2 \tilde{M}_{K-1}$.

The ellipsoid $\tilde{E}_K$ contains the intersection of the band Λ_K and the ellipsoid $\tilde{E}_{K-1}$, taking into account the new observations (y_K and u_K^*)

$$\tilde{E}_K \supset \tilde{E}_{K-1} \cap \Lambda_K. \tag{57}$$

The construction of $\tilde{E}_K$ needs knowledge of the band position Λ_K with respect to the current ellipsoid $\tilde{E}_{K-1}$. In fact, there are five possible cases:

1. If

$$y_K - \gamma \geq \tilde{\varphi}_K^{+}\tilde{c}_{K-1} + \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T}$$

or

$$y_K + \gamma \leq \tilde{\varphi}_K^{-}\tilde{c}_{K-1} - \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T},$$

then $\tilde{E}_{K-1} \cap \Lambda_K = \emptyset$. In this case, the new observation is treated as erroneous and $\tilde{E}_K = \tilde{E}_{K-1}$.

2. If

$$y_K - \gamma \leq \tilde{\varphi}_K^{+}\tilde{c}_{K-1} - \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T}$$

and

$$y_K + \gamma \geq \tilde{\varphi}_K^{-}\tilde{c}_{K-1} + \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T},$$

then $\tilde{E}_{K-1} \subset \Lambda_K$. The new observation is considered redundant and $\tilde{E}_K = \tilde{E}_{K-1}$.

3. If

$$y_K - \gamma > \tilde{\varphi}_K^{+}\tilde{c}_{K-1} - \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T}$$

and

$$y_K + \gamma \geq \tilde{\varphi}_K^{-}\tilde{c}_{K-1} + \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T},$$

then only H_K^{-} intersects $\tilde{E}_{K-1}$ and Λ_K is replaced by $\left[H_K^{-}, H_K^{++}\right]$ with

$$H_K^{++} = \Big\{ \tilde{\theta} \in \mathbb{R}^p, \tilde{\varphi}_K^{+}\tilde{\theta} = \tilde{\varphi}_K^{+}\tilde{c}_{K-1} \\ + \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T} \Big\}. \tag{58}$$

The hyperplane H_K^{++} is tangent to $\tilde{E}_{K-1}$ and parallel to H_K^{+}. An optimal bounding ellipsoid (OBE) algorithm can be directly applied to obtain $\tilde{E}_K$.

4. If

$$y_K - \gamma < \tilde{\varphi}_K^{+}\tilde{c}_{K-1} - \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T}$$

and

$$y_K + \gamma < \tilde{\varphi}_K^{-}\tilde{c}_{K-1} + \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T},$$

then only H_K^{+} intersects $\tilde{E}_{K-1}$ and Λ_K is replaced by $\Lambda_K' = \left[H_K^{--}, H_K^{-}\right]$ with

$$H_K^{--} = \Big\{ \tilde{\theta} \in \mathbb{R}^p, \tilde{\varphi}_K^{-}\tilde{\theta} = \tilde{\varphi}_K^{-}\tilde{c}_{K-1} \\ + \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T} \Big\}. \tag{59}$$

The hyperplane H_K^{--} is tangent to $\tilde{E}_{K-1}$ and parallel to H_K^{-}, and then an OBE algorithm can be directly applied to obtain $\tilde{E}_K$.

5. If

$$y_K - \gamma > \tilde{\varphi}_K^{+}\tilde{c}_{K-1} - \sqrt{\tilde{\varphi}_K^{+}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{+}\right)^T}$$

and

$$y_K + \gamma < \tilde{\varphi}_K^{-}\tilde{c}_{K-1} + \sqrt{\tilde{\varphi}_K^{-}\tilde{M}_{K-1}\left(\tilde{\varphi}_K^{-}\right)^T},$$

then the two hyperplanes H_K^{-} and H_K^{+} are strictly inside $\tilde{E}_{K-1}$ but they are not parallel. Thus, the band Λ_K cannot be defined with a quadratic form and then it is not possible to apply directly an OBE algorithm to obtain $\tilde{E}_K$.

For the last case, two algorithms have been proposed in the literature to construct the ellipsoid $\tilde{E}_K$. The first algorithm (CLE) is due to Clement and Gentil (1988), and the construction consists of two steps. The second one (FER) has been proposed by Ferreres and M'Saad (1997) and needs only one step, which consists in the construction of a hyperplane parallel to H_K^- or H_K^+.

Theorem 1. (Ferreres and M'Saad, 1997) *If H_K^- and H_K^+ are strictly inside $\tilde{E}_{K-1}$, then two parallel hyperplanes can be defined as follows:*

- *H_K^{-*} is parallel to H_K^-:*

$$H_K^{-*} = \left\{\tilde{\theta} \in \mathbb{R}^p, \tilde{\varphi}_K^+ \tilde{\theta} = f_1^*\right\}, \tag{60}$$

- *H_K^{+*} is parallel to H_K^+:*

$$H_K^{+*} = \left\{\tilde{\theta} \in \mathbb{R}^p, \tilde{\varphi}_K^- \tilde{\theta} = f_2^*\right\}, \tag{61}$$

where

$$f_1^* = \tilde{\varphi}_K^+ \tilde{c}_{K-1} + \frac{\lambda_2 m_{12} + \sqrt{(m_{22} - \lambda_2^2)(m_{11}m_{22} - m_{12}^2)}}{m_{22}}, \tag{62}$$

$$f_2^* = \tilde{\varphi}_K^- \tilde{c}_{K-1} + \frac{\lambda_1 m_{12} - \sqrt{(m_{11} - \lambda_1^2)(m_{11}p_{22} - m_{12}^2)}}{m_{11}}, \tag{63}$$

with

$$\begin{aligned} \lambda_1 &= y - \gamma - \tilde{\varphi}_K^+ \tilde{c}_{K-1}, \\ \lambda_2 &= y + \gamma - \tilde{\varphi}_K^- \tilde{c}_{K-1}, \\ m_{11} &= \sqrt{\tilde{\varphi}_K^- \tilde{M}_{K-1} \left(\tilde{\varphi}_K^-\right)^T}, \\ m_{12} &= \sqrt{\tilde{\varphi}_K^- \tilde{M}_{K-1} \left(\tilde{\varphi}_K^+\right)^T}, \\ m_{22} &= \sqrt{\tilde{\varphi}_K^+ \tilde{M}_{K-1} \left(\tilde{\varphi}_K^+\right)^T}. \end{aligned} \tag{64}$$

After the construction of the parallel hyperplane, the use of an OBE algorithm is possible. As mentioned in the literature, the FER algorithm produces an ellipsoid with less pessimism than the CLE algorithm (Ferreres and M'Saad, 1997). To optimize the result of the FER algorithm, it is possible to construct two ellipsoids ($\tilde{E}_K^{'}$,$\tilde{E}_K^{''}$) using (H_K^{-*}, H_K^{+*}), and then calculate their intersection.

The obtained ellipsoid should take into account the old information represented by the ellipsoid $\tilde{E}_{K-1}$ and the new observations defined by hyperplanes using the weights ρ_K and δ_K:

$$\begin{aligned} \tilde{E}_K = \Big\{\tilde{\theta} \in \mathbb{R}^{n+m+1} | \\ \rho_K (\tilde{\theta} - \tilde{c}_{K-1})^T \tilde{M}_{K-1}^{-1} (\tilde{\theta} - \tilde{c}_{K-1}) \\ + \delta_K (y_K - \tilde{\varphi}_K^T \tilde{\theta})^2 \le \rho_K \tilde{\sigma}_{K-1}^2 + \delta_K \gamma \Big\}. \end{aligned} \tag{65}$$

The new values of $\tilde{c}_K$, $\tilde{M}_K$ and $\tilde{\sigma}_K$ are recursively obtained from their previous values using the following expressions (Fogel and Huang, 1982):

$$\tilde{e}_K = y_K - \tilde{\varphi}_K^T \tilde{c}_{K-1}, \tag{66}$$

$$\tilde{g}_K = \tilde{\varphi}_K^T \tilde{M}_{K-1} \tilde{\varphi}_K, \tag{67}$$

$$\tilde{M}_K = \frac{1}{\rho_K} \left[\tilde{M}_{K-1} - \frac{\delta_K \tilde{M}_{K-1} \tilde{\varphi}_K \tilde{\varphi}_K^T \tilde{M}_{K-1}}{\rho_K + \delta_K \tilde{g}_K} \right], \tag{68}$$

$$\tilde{c}_K = \tilde{c}_{K-1} + \delta_K \tilde{M}_K \tilde{\varphi}_K \tilde{e}_K, \tag{69}$$

$$\tilde{\sigma}_K^2 = \rho_K \tilde{\sigma}_{K-1}^2 + \delta_K \gamma_K^2 - \frac{\rho_K \delta_K \tilde{e}_K^2}{\rho_K + \delta_K \tilde{g}_K}, \tag{70}$$

where

$$\rho_K = \frac{1}{\tilde{\sigma}_{K-1}^2}, \quad \delta_K = \frac{q_K}{\gamma^2}.$$

The variable q_K is obtained from the minimization of measure μ_V proportional to the ellipsoid volume (Fogel and Huang, 1982):

$$\mu_V(K) = \det\left(\tilde{\sigma}_K^2 \tilde{M}_K\right). \tag{71}$$

Here q_K is the positive solution of

$$d_2 q_K^2 + d_1 q_K + d_0 = 0, \tag{72}$$

with

$$\begin{aligned} d_2 &= (p-1)\tilde{\sigma}_{K-1}^4 \tilde{g}_K^2, \\ d_1 &= ((2p-1)\gamma^2 - \tilde{\sigma}_{K-1}^2 \tilde{g}_K + \tilde{e}_K)\tilde{\sigma}_{K-1}^2 \tilde{g}_K, \\ d_0 &= (p(\gamma^2 - \tilde{e}_K^2) - \tilde{\sigma}_{K-1}^2 \tilde{g}_K)\gamma^2. \end{aligned}$$

3.4. Return to continuous parameters.

Using the general OBE algorithm described by Eqns. (66)–(70), the interval vector containing the coefficients α_i and β_j in guaranteed way can be determined as

$$[\tilde{\theta}] = [\tilde{\theta}^-, \tilde{\theta}^+], \tag{73}$$

where

$$\tilde{\theta}^- = \tilde{c}_{N_s} - \tilde{\Delta}, \tag{74}$$

$$\tilde{\theta}^+ = \tilde{c}_{N_s} + \tilde{\Delta}, \tag{75}$$

$$\tilde{\Delta} = \sqrt{\mathrm{diag}\left(\tilde{\sigma}_{N_s}^2 \tilde{M}_{N_s}\right)}. \tag{76}$$

The return to the interval continuous-time parameters vector $[\theta]$ from the interval pseudo-coefficients vector $[\tilde{\theta}]$ can be performed by solving the interval linear system of equations $[A]\,[x] = [B]$, where

$$[A]_{i,j} = \begin{cases} \dfrac{([\alpha_i] - 1)}{h^{v_{a_i}}} & \text{if } i = j, \\ \dfrac{[\alpha_i]}{h^{v_{a_j}}} & \text{if } i \neq j, \end{cases} \tag{77}$$

$$[B]_i = -[\alpha_i], \tag{78}$$

$$[x] = ([a_1], [a_2], \dots, [a_n])^T . \tag{79}$$

The coefficients $[b_j]$ are determined directly by

$$[b_j] = [\beta_j] \sum_{i=0}^{n} [a_i] h^{v_{b_j} - v_{a_i}}. \tag{80}$$

Remark 2. Owing to the use of the relations (74) and (75), the ellipsoid obtained using the previous procedure is conservative. Moreover, solving the interval linear system of equations may lead to completely erroneous results.

To solve this problem, the following Algorithm 1 proposed by Amairi (2015) is used.

Algorithm 1. Return to continuous parameters.

Require: $\tilde{\Delta}, \tilde{c}_{N_s}$

- Apply the relations (74) and (75) to obtain $\tilde{\theta}^-$ and $\tilde{\theta}^+$.
- Construct a grid Ω^d of $n+m+1$ vectors obtained from $\left[\tilde{\theta}^-, \tilde{\theta}^+\right]$ using a suitable stepsize.
- Filter the grid by removing all elements that do not belong to the ellipsoid $\tilde{E}_{N_s}$. The obtained filtered grid is denoted by Ω_f^d.
- Return to the continuous-time parameter from each element belonging to Ω_f using Eqns. (26)–(29). The obtained set is denoted by Ω_f^c.

$\{$Returns $\Omega_f^c\}$

Knowing that the return procedure is linear, the obtained set Ω_f^c will be an ellipsoid that contains all feasible continuous parameters.

Remark 3. If the signs of the pseudo-parameters α_i are not assumed to be known *a priori*, then a pre-estimation step can be performed using a classic identification algorithm. The algorithm should produce accurate estimates by filtering the data and using an iterative technique such as the algorithm *srivcf* proposed by Victor *et al.* (2013). Finally, Eqn. (51) is applied to determine the sign of α_i.

Remark 4. As long as the system is commensurate, an iterative algorithm can be developed to estimate the fractional commensurate order. The algorithm can be similar to the one proposed by Amairi (2015), or to other algorithms that minimize the output error.

In the following, the proposed method is accessed through a numerical example. A comparative study between the equation-error OBE algorithm and the output-error OBE algorithm is presented to show the efficiency of the proposed method.

4. Numerical example

Consider the BIBO stable commensurate-order fractional system described by the following transfer function:

$$\begin{aligned} G(s,\theta^*) &= \frac{k}{\left(\dfrac{s}{\omega_0}\right)^{2\nu} + 2\xi\left(\dfrac{s}{\omega_0}\right)^{\nu} + 1} \\ &= \frac{b_0}{a_2 s^{2\nu} + a_1 s^{\nu} + 1}, \end{aligned} \tag{81}$$

where $(k, \xi, \omega_0)^T = (1, -0.5, 2)^T$ is the real parameter vector and $\nu = 0.5$ is the real commensurate-order. The real parameter vector can be also written as $\theta^* = (a_2, a_1, b_0)^T = (0.5, -0.707, 1)^T$.

System output y with constant sampling period $h = 0.05$ s is generated using a pseudo random binary sequence (PRBS) input uniformly distributed in $[-1, 1]$, with its power spectral density (PSD) shown in Fig. 2.

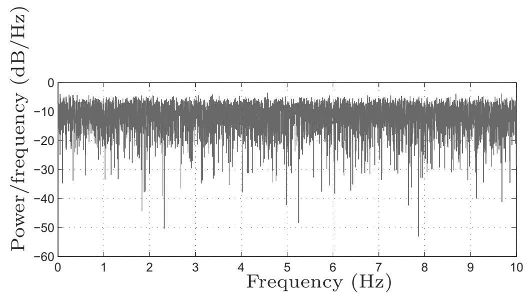

Fig. 2. Power spectral density of the input signal.

An unknown-but-bounded noise η is added to the system output, where $|\eta_K| \leq \gamma,\ \forall K \in \{1, \dots, 10000\}$.

4.1. Sign determination. Before applying the proposed algorithm, the sign of each pseudo-parameter $\alpha_i,\ i = 0, 1, 2$ should be determined. In this case, the simplified refined instrumental variable for the continuous-time fractional models (srivcf) algorithm is used because it is known to produce accurate estimates (Victor *et al.*, 2013). Using Eqn. (51) leads to the sign of α_i as mentioned in Table 1.

Table 1. Signs of the pseudo-parameters $\alpha_i, i = 0, 1, 2$.

Pseudo-parameter	sign
α_0	$+1$
α_1	-1
α_2	$+1$

4.2. Comparative study. In order to compare the results obtained with the output-error approach with those presented by Amairi (2015), a noise realization with $\gamma = 0.005$ is used as presented in Fig. 3.

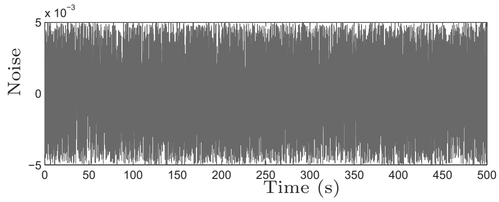

Fig. 3. Additive bounded noise with $\gamma = 0.005$.

By applying the proposed method using the Fogel–Huang (FH) algorithm (Fogel and Huang, 1982; Milanese *et al.*, 1996), an ellipsoid enclosing the set of feasible pseudo-parameters $\tilde{\Theta}$ is obtained. Figure 4 presents the evolution of the ellipsoid volume.

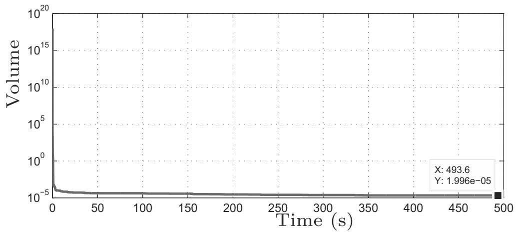

Fig. 4. Evolution of the ellipsoid volume using the output-error approach.

Now, the return to continuous parameters is used to obtain the values relative to the center of the ellipsoid. Figure 5 and Table 2 show respectively the time evolution of these estimated parameters and the final values for the equation-error and output-error approaches.

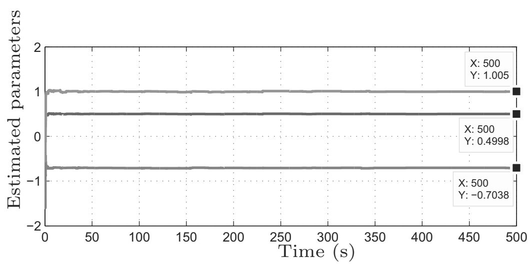

Fig. 5. Evolution of the estimated parameters relative to the ellipsoid center.

Table 2. Ellipsoid center.

Approach	Center
equation-error OBE	$(0.497, -0.705, 0.997)$
output-error OBE	$(0.499, -0.703, 1.004)$

From the previous results it is clear that the proposed method leads to an ellipsoid with a reduced volume and has good convergence properties. But the OBE algorithm naturally leads to conservatism in the pseudo-parameter space and more in the continuous-time parameter space. This conservatism can be reduced using the appropriate projection algorithm (Algorithm 1). Figure 6 presents the ellipsoid enclosing the set of all the feasible continuous-time parameters with 12 data recirculations. Data recirculation is a technique used to optimize the results (reduction of the ellipsoid volume and an improvement of the convergence properties) (Milanese *et al.*, 1996).

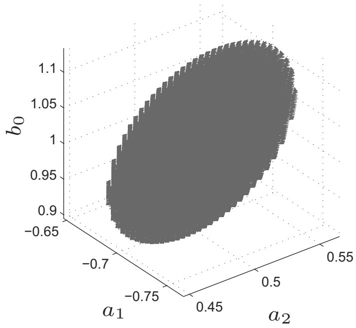

Fig. 6. Ellipsoid enclosing the set of all feasible continuous-time parameters with 12 data recirculations.

To provide more evidence on the effectiveness of the proposed method, the obtained ellipsoid is projected in the continuous-parameter space after obtaining the set Ω_f by Algorithm 1. As shown in Table 3, the proposed method leads to minimum-width intervals for any number of recirculations.

4.3. Monte Carlo simulations. In several situations, the noise may not be a white process or may not even be stationary. It may have sinusoidal components, or it may be impulsive in other cases. Therefore, in this section the performances of the proposed method are shown when the noise is, e.g., an additive sum of white noise and a sinusoid.

Consider as input a PRBS signal uniformly distributed in $[-1, 1]$. The noise is generated by

$$\eta_K = A \sin(\omega K h) + \frac{\gamma}{2} w, \tag{82}$$

Table 3. Interval continuous-time parameters based on the number of recirculations (projection of Ω_f^c onto the parameter space).

Algorithm		Number of recirculations		
		1	5	12
Equation error	$[a_2]$	$[0.4432, 0.5786]$	$[0.4504, 0.5703]$	$[0.4509, 0.5690]$
	$[a_1]$	$[-0.7836, -0.6264]$	$[-0.7809, -0.6303]$	$[-0.7753, -0.6347]$
	$[b_0]$	$[0.8523, 1.2156]$	$[0.8600, 1.2031]$	$[0.8576, 1.2012]$
Output error	$[a_2]$	$[0.4458, 0.5713]$	$[0.4575, 0.5555]$	$[0.4593, 0.5512]$
	$[a_1]$	$[-0.7708, -0.6423]$	$[-0.7599, -0.6538]$	$[-0.7564, -0.6574]$
	$[b_0]$	$[0.8795, 1.1669]$	$[0.8929, 1.1405]$	$[0.8948, 1.1340]$

where w is a white process uniformly distributed in $[-1, 1]$, A satisfies $0 \leq A \leq \gamma/2$, ω is a frequency taken randomly between $\pi/20$ and $\pi/5$, and $h = 0.05$ s.

The noise η_K remains bounded with a known bound $\gamma = 0.005$, and 500 Monte Carlo runs of the proposed OBE algorithm with $N_s = 10000$ are performed. Figure 7 shows two arbitrary noise sequences taken from the 500 generated ones. Table 4 contains the continuous-time parameters relative to the center of the ellipsoid, the volume, and the FIT, which is defined as follows:

$$\text{FIT} = 100 \times \left(1 - \sqrt{\frac{\sum_{k=1}^{N_s} (y_k - \hat{y}_k)^2}{\sum_{k=1}^{N_s} (y_k - \bar{y})^2}} \right), \tag{83}$$

where $\bar{y}$ is the mean value of the output signal and $\hat{y}$ is the estimated output signal.

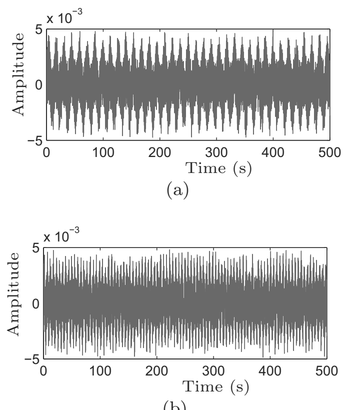

Fig. 7. Different bounded noise signals with sinusoidal components.

Table 4. Mean values of the continuous parameters relative to the ellipsoid center.

$\overline{a}_2$	$\overline{a}_1$	$\overline{b}_0$	$\overline{\text{Volume}}(\times 10^{-5})$	FIT
0.5022	−0.7035	1.007	1.618	98.55

The obtained results confirm the consistency of the proposed algorithm even if the noise is not white. This is confirmed by the interval parameters obtained after projection of Ω_f as shown in Table 5.

Table 5. Interval continuous-time parameters obtained by the output-error OBE algorithm when the noise is not white (projection of Ω_f^c onto the parameter space).

$[a_2]$	$[0.4375, 0.5938]$
$[a_1]$	$[-0.8113, -0.6183]$
$[b_0]$	$[0.8149, 1.2597]$

5. Conclusion

This paper focused on set-membership parameter estimation of a fractional system in a bounded-error context using outer bounding ellipsoid algorithms. The main idea consisted in reformulating the characterization of the ellipsoid enclosing the feasible set of discrete parameters (pseudo-parameters) using the bounded output-error context instead of the equation-error one. First, a simple method to determine the sign of the pseudo-parameters was presented and tested. This information was then used in the proposed set-membership method to enclose the feasible set of pseudo-parameters with an ellipsoid of a reduced volume. Once obtained, the ellipsoid was transformed into another ellipsoid that encloses the feasible set of continuous parameters using an algorithm that offers minimum conservatism.

The performances of the proposed method were evaluated through a second-order fractional system. Monte Carlo simulations showed that the method provides proper interval parameters even if the noise is not white. Therefore, the method offers a framework to estimate uncertainties of a fractional system, especially for an adaptive control scheme. Hence, it will be interesting, in future developments, to extend this method to deal with colored output noise and adapt it to the errors-in-variables context.

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