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THERMAL STRESSES ASSOCIATED WITH A THERMOSENSITIVE MULTILAYERED DISC ANALYSED DUE TO POINT HEATING

V.B. Srinivas
Department of Mathematics, Anurag University, Venkatapur, Ghatkesar, Medchal-Malkajgiri District, Telangana, INDIA

V.R. Manthena*
Department of Mathematics, Priyadarshini J. L. College of Engineering, Nagpur, INDIA
E-mail: vkmanthena@gmail.com

S.D. Warbhe
Department of Mathematics, Laxminarayan Innovation Technological University, Nagpur, INDIA

G.D. Kedar
Department of Mathematics, RTM Nagpur University, Nagpur, INDIA

Navneet Kumar Lamba
Department of Mathematics, Shri Lemdeo Patil Mahavidyalaya, Mandhal, Nagpur, INDIA

In this paper, analytical solutions are presented for temperature and thermal behavior of a thermosensitive multilayered annular disc due to point heat source. Convective heating is applied to both the innermost and outermost layers. The nonlinearity of the thermal diffusivity Eq.is dealt using Kirchhoff's transformation technique. A finite integral transform in the form of Bessel's function is used to deal with the radial variable *r*. Fourier transform and angular eigen functions are also used to solve the thermal diffusivity equation. Deflection, resultant forces, shearing forces, resultant moments and thermal stresses are obtained. A mathematical representation is formulated for a 3-layered disc, with the inner, middle and outer layers composed of copper, zinc and aluminum respectively. The results are depicted graphically.

Key words: multilayered annular disc, thermosensitive, heat conduction, instantaneous point heat source, deflection, stresses.

1. Introduction

Multilayered configurations like composite and sandwich plates, circular discs, comprise multiple layers that offer superior structural and thermal characteristics. These structures find extensive applications in aviation, civil and offshore engineering. The careful consideration of mechanical performance under temperature variations is crucial in the production of multiple layered structures. Temperature changes cannot just cause significant inner pressures but may impact the characteristics of the materials within these structures. Due to increase in the use of mechanical structures in high temperature environment in the last three decades, the study of thermo-mechanical behavior of different materials with properties that vary with temperature has received attention.

Noda [1] discussed the influence of physical characteristics affected by temperature on thermal behavior of different solids. Olcer [2] presented a thorough examination of temperature profile by considering finite-length

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* To whom correspondence should be addressed

circular-cylinder. Gorman [3] investigated how temperature gradient in a radial parabolic manner affects lateral oscillations in orthotropic-circular-plates. Popovich *et al.* [4, 5] studied the heat conduction problems on various solids. Malzbender and Jülich [6], Kayhani *et al.* [7] derived thermal solutions for multilayered structures. Singh [8], Singh *et al.* [9], Kayhani *et al.* [10] obtained analytical solutions of heat transfer in different cylindrical and multiple layered structures. Norouzi [11] obtained explicit solution of thermoelastic profile for composite laminated structures with multiple spherical fiber-layers. Dalir and Nourazar [12] studied unsteady-state heat transfer of cylindrical structures having multiple concentric layers and obtained solutions for a 3-dimensional problem. Popovich and Kalynyak [13] developed an analytical framework and studied static thermal profile of multilayered thermally sensitive cylinder. Torabi and Zhang [14] found solutions for unsteady-state heat conduction with asymmetry, determined stresses of multiple layered solids. Manthena *et al.* [15] considered a thermally sensitive functionally graded rectangular plate and studied the effect of stress resultants on thermal stresses. Bhad *et al.* [16] studied the thermoelastic problem in multilayered elliptical composite plate with internal heat generation. Manthena *et al.* [17-22] studied the temperature and stress profile of various solids subjected to temperature dependent and independent material properties.

Jangid and Mukhopadhyay [23] presented an alternative solution for a initial-value and boundary-value problem. Etkin [24] elucidated the thermal-impulse's physical significance through the concept of entropy. Srinivas *et al.* [25] carried out thermoelastic analysis by taking rectangular-parallelepiped subjected to convection and temperature dependent characteristics. Razavi *et al.* [26], Balci and Akpinar [27], Bikram and Kedar [28], Etkin [29], Su *et al.* [30] solved various problems of steady state and un-steady state temperature profile and analyzed the corresponding thermal stresses. In order to ascertain the expression of temperature and stresses, Lamba [31] examined the behaviour of fractional time derivative in temperature-sensitive FG cylinders. Recently, in a thermo-diffusive medium, Yadav *et al.* [32] successfully established a significant memory response. Also the related work is reflected in [33, 34].

In this work the authors try to investigate the influence of point heating on temperature and thermal profile of a thermally sensitive multilayered annular circular disc. The k layered disc is defined over $r_{i-1} < r < r_i$, $0 < \theta < 2\pi$, $0 < z < h$. Heat conduction equation (HCE) is solved by integral-transform method and thermoelastic behavior is analyzed. Graphical analysis is carried out for a 3-layered disc.

2. Heat conduction equation and its solution

Figure 1 gives the depiction of the layered circular disc in a geometric form.

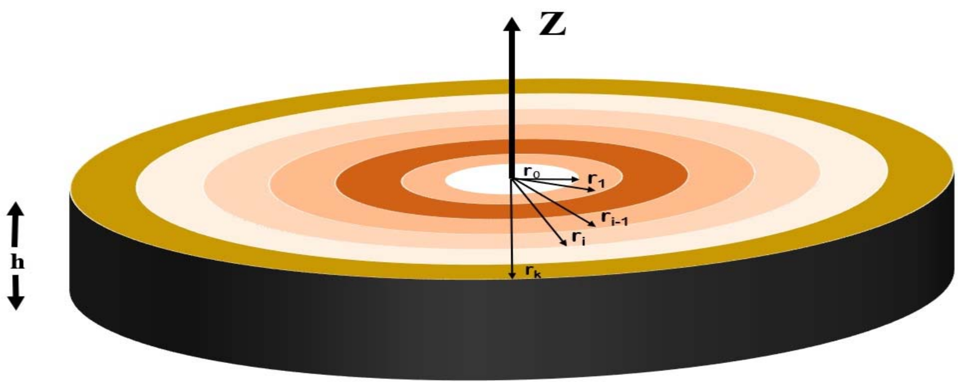

Fig.1. Multilayered disc.

The unsteady HCE with internal heat generation of a multilayered annular circular disc is [9]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda_i(T_i)\frac{\partial T_i}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\lambda_i(T_i)\frac{\partial T_i}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\lambda_i(T_i)\frac{\partial T_i}{\partial z}\right)+Q(r,\theta,z,t)=\rho_i\, C_i(T_i)\,\frac{\partial T_i}{\partial t} \tag{2.1}$$

where $\lambda_i(T_i)$, $C_i(T_i)$ are respectively, the temperature dependent thermal conductivity, specific heat capacity of the i^{th} layer, ρ_i is the density of the i^{th} layer, $Q(r,\theta,z,t)$ is the internal heat generation and $i=1,2,3,...,k$. Following [9], the initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.2) to (2.7).

$$T_i = 0, \quad \text{at} \quad t=0, \tag{2.2}$$

$$\lambda_1(T_1)\frac{\partial T_1}{\partial r} + h_0 T_1 = 0, \quad \text{at} \quad r=r_0, \tag{2.3}$$

$$\lambda_k(T_k)\frac{\partial T_k}{\partial r} + h_k T_k = f_k(z,\theta,t), \quad \text{at} \quad r=r_k, \tag{2.4}$$

$$\begin{gathered} T_i(r_{i-1},\theta,z,t) = T_{i-1}(r_{i-1},\theta,z,t), \\ \lambda_i(T_i)\left.\frac{\partial T_i}{\partial r}\right|_{r=r_{i-1}} = \lambda_{i-1}(T_{i-1})\left.\frac{\partial T_{i-1}}{\partial r}\right|_{r=r_{i-1}}, \end{gathered} \tag{2.5}$$

$$T_i\big|_{\theta=0} = T_i\big|_{\theta=2\pi}, \qquad \lambda_i(T_i)\left.\frac{\partial T_i}{\partial \theta}\right|_{\theta=0} = \lambda_i(T_i)\left.\frac{\partial T_i}{\partial \theta}\right|_{\theta=2\pi}, \tag{2.6}$$

$$T_i = 0, \quad \text{at} \quad z=0,\, h. \tag{2.7}$$

We use following dimensionless parameters.

$$\begin{gathered} T_i{}^* = \frac{T_i}{T_0}, \quad r^* = \frac{r}{h}, \quad \theta^* = \frac{\theta}{2\pi}, \quad z^* = \frac{z}{h}, \quad t^* = \frac{\kappa_1 t}{h^2}, \quad \rho_i{}^* = \frac{\rho_i}{\rho_1}, \\ \left(r_i{}^*, h^*\right) = \frac{(r_i,h)}{h}, \quad E_i{}^* = \frac{E_i}{E_1}, \quad a^* = \frac{a h^2}{\kappa_1}, \quad \varpi_j^* = \frac{\varpi_j h^2}{\kappa_1}, \quad j=1,2, \end{gathered} \tag{2.8}$$

where T_0 is the ambient temperature, h is the thickness of the disc, $\kappa_1 = \lambda_1/\left(C_1\rho_1\right)$, is the thermal diffusivity of the inner layer, λ_1, C_1, ρ_1 are heat transfer properties, density, E_i is the Young's modulus, a, ϖ_j are the frequency.
The temperature dependent material properties $\lambda_i\left(T_i\right)$, $C_i\left(T\right)_i$, and heat flow $f_k(z,\theta,t)$ are taken as [4, 5, 13]

$$\begin{gathered} \lambda_i(T_i) = \lambda_1 \lambda_i{}^*(T_i{}^*), \quad C_i(T_i) = C_1 C_i{}^*(T_i{}^*), \\ f_k(z,\theta,t) = f_0 f_k{}^*(z^*,\theta^*,t^*), \quad Q(r,\theta,z,t) = Q_0 Q^*(r^*,\theta^*,z^*,t^*), \end{gathered} \tag{2.9}$$

where λ_1, C_1 have dimensions, f_0, Q_0 are the strength of the heat flow having relevant dimensions, and $\lambda_i{}^*\left(T_i{}^*\right)$, $C_i{}^*\left(T_i{}^*\right)$ are the dimensionless quantities, which are functions that describe the dependence of

these characteristics on dimensionless temperature, $f_k*(z*, \theta*, t*)$, $Q*(r*, \theta*, z*, t*)$ are the dimensionless functions which describe the space distribution of the heat flow.
Using Eqs (2.8-2.9), Eqs (2.1-2.7) reduces to the following dimensionless form (ignoring asterisks for convenience).

$$\begin{gathered}\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda_i(T_i)\frac{\partial T_i}{\partial r}\right)+\frac{1}{4\pi^2 r^2}\frac{\partial}{\partial\theta}\left(\lambda_i(T_i)\frac{\partial T_i}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\lambda_i(T_i)\frac{\partial T_i}{\partial z}\right)+P_0\,Q(r,\theta,z,t)= \\ =\rho_i\,C_i(T_i)\,\frac{\partial T_i}{\partial t}.\end{gathered} \tag{2.10}$$

The initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.11) to (2.16)

$$T_i=0, \quad \text{at} \quad t=0, \tag{2.11}$$

$$\lambda_1(T_1)\frac{\partial T_1}{\partial r}+Bi_1\,T_1=0, \quad \text{at} \quad r=\Phi_1, \tag{2.12}$$

$$\lambda_k(T_k)\frac{\partial T_k}{\partial r}+Bi_2\,T_k=Ki\,f_k(z,\theta,t), \quad \text{at} \quad r=\Phi_2, \tag{2.13}$$

$$\begin{gathered}T_i(r_{i-1},\theta,z,t)=T_{i-1}(r_{i-1},\theta,z,t), \\ \lambda_i(T_i)\left.\frac{\partial T_i}{\partial r}\right|_{r=r_{i-1}}=\lambda_{i-1}(T_{i-1})\left.\frac{\partial T_{i-1}}{\partial r}\right|_{r=r_{i-1}},\end{gathered} \tag{2.14}$$

$$\begin{gathered}T_i\big|_{\theta=0}=T_i\big|_{\theta=1}, \\ \lambda_i(T_i)\left.\frac{\partial T_i}{\partial\theta}\right|_{\theta=0}=\lambda_i(T_i)\left.\frac{\partial T_i}{\partial\theta}\right|_{\theta=1},\end{gathered} \tag{2.15}$$

$$T_i=0, \quad \text{at} \quad z=0,\,h. \tag{2.16}$$

Here $P_0=\dfrac{Q_0\,h^2}{\lambda_1 T_0}$ and $Ki=\dfrac{f_0\,h}{\lambda_1 T_0}$ are respectively the dimensionless Pomerantsev reference number and dimensionless Kirpichev reference number, $Bi_1=\dfrac{h_0 h}{\lambda_1}$, $Bi_2=\dfrac{h_k h}{\lambda_1}$ are the Biot criteria and $\Phi_1=(r_0/h)$, $\Phi_2=(r_k/h)$.
Introducing Kirchhoff's variable transformation [4, 5, 13]

$$\Theta_i(T_i)=\int_o^{T_i}\lambda_i(T_i)\,dT_i, \tag{2.17}$$

and considering the material with simple thermal nonlinearity (that is $[C_i(T_i)/\lambda_i(T_i)] \approx 1$), Eqs (2.10) to (2.16) become:

$$\frac{\partial^2\Theta_i}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta_i}{\partial r}+\frac{1}{4\pi^2 r^2}\frac{\partial^2\Theta_i}{\partial\theta^2}+\frac{\partial^2\Theta_i}{\partial z^2}+P_0\,Q(r,\theta,z,t)=\rho_i\,\frac{\partial\Theta_i}{\partial t}. \tag{2.18}$$

The initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.19)-(2.24).

$$\Theta_i=0, \quad \text{at} \quad t=0, \tag{2.19}$$

$$\frac{\partial\Theta_1}{\partial r}+Bi_1\,\Theta_1=0, \quad \text{at} \quad r=\Phi_1, \tag{2.20}$$

$$\frac{\partial\Theta_k}{\partial r}+Bi_2\,\Theta_k=Ki\,f_k(z,\theta,t), \qquad \text{at} \quad r=\Phi_2, \tag{2.21}$$

$$\begin{gathered}\Theta_i(r_{i-1},\theta,z,t)=\Theta_{i-1}(r_{i-1},\theta,z,t),\\ \left.\frac{\partial\Theta_i}{\partial r}\right|_{r=r_{i-1}}=\left.\frac{\partial\Theta_{i-1}}{\partial r}\right|_{r=r_{i-1}},\end{gathered} \tag{2.22}$$

$$\begin{gathered}\Theta_i\big|_{\theta=0}=\Theta_i\big|_{\theta=1},\\ \left.\frac{\partial\Theta_i}{\partial\theta}\right|_{\theta=0}=\left.\frac{\partial\Theta_i}{\partial\theta}\right|_{\theta=1},\end{gathered} \tag{2.23}$$

$$\Theta_i=0, \quad \text{at} \quad z=0,\,h. \tag{2.24}$$

To solve HCE (2.18), let:

$$f_k(\theta,z,t)=\delta(\theta-\theta_0)\,\delta(z-z_0)\,\exp(a\,t),$$

$$Q(r,\theta,z,t)=\delta(r-r_a)\,\delta(\theta-\theta_0)\,\delta(z-z_0)\,\delta(t).$$

Using Fourier transform on Eqs (2.18) and (2.24), we get:

$$\frac{\partial^2\bar\Theta_i}{\partial r^2}+\frac{1}{r}\frac{\partial\bar\Theta_i}{\partial r}+\frac{1}{4\pi^2 r^2}\frac{\partial^2\bar\Theta_i}{\partial\theta^2}-\psi_n^2\,\bar\Theta_i+\bar Q(r,\theta,\psi_n,t)=\rho_i\,\frac{\partial\bar\Theta_i}{\partial t}. \tag{2.25}$$

The conditions (2.19-2.23) become;

$$\bar\Theta_i=0, \quad \text{at} \quad t=0, \tag{2.26}$$

$$\frac{\partial\bar{\Theta}_1}{\partial r}+Bi_1\,\bar{\Theta}_1=0, \quad \text{at} \quad r=\Phi_1, \tag{2.27}$$

$$\frac{\partial\bar{\Theta}_k}{\partial r}+Bi_2\,\bar{\Theta}_k=\bar{f}_k(\psi_n,\theta,t), \quad \text{at} \quad r=\Phi_2, \tag{2.28}$$

$$\begin{gathered}\bar{\Theta}_i(r_{i-1},\theta,\psi_n,t)=\bar{\Theta}_{i-1}(r_{i-1},\theta,\psi_n,t),\\ \left.\frac{\partial\bar{\Theta}_i}{\partial r}\right|_{r=r_{i-1}}=\left.\frac{\partial\bar{\Theta}_{i-1}}{\partial r}\right|_{r=r_{i-1}},\end{gathered} \tag{2.29}$$

$$\begin{gathered}\bar{\Theta}_i\big|_{\theta=0}=\bar{\Theta}_i\big|_{\theta=1},\\ \left.\frac{\partial\bar{\Theta}_i}{\partial\theta}\right|_{\theta=0}=\left.\frac{\partial\bar{\Theta}_i}{\partial\theta}\right|_{\theta=1}\end{gathered} \tag{2.30}$$

where

$$\psi_n=n\pi/h,\ \bar{f}_k(\psi_n,\theta,t)=Ki\,z_0\sin(n\pi z_0/h)\delta(\theta-\theta_0)\exp(at),$$

$$\bar{Q}(r,\theta,z,t)=(z_0/P_0)\sin(n\pi z_0/h)\delta(r-r_a)\delta(\theta-\theta_0)\ \delta(t).$$

Considering periodic conditions, $\bar{\Theta}_i(r,\theta,t)$ is expanded as:

$$\bar{\Theta}_i(r,\theta,t)=\bar{\Theta}_{i0}(r,t)+\sum_{m=1}^{\infty}\bar{\Theta}_{imc}(r,t)\cos(m\theta)+\sum_{m=1}^{\infty}\bar{\Theta}_{ims}(r,t)\sin(m\theta). \tag{2.31}$$

Similarly, the expression for heat supply is taken as:

$$\bar{f}_k(\theta,t)=\bar{f}_{k0}(\theta,t)+\sum_{m=1}^{\infty}\bar{f}_{kmc}(\theta,t)\cos(m\theta)+\sum_{m=1}^{\infty}\bar{f}_{kms}(\theta,t)\sin(m\theta). \tag{2.32}$$

Using the orthogonality conditions along axial direction, the coefficients in Eq.(2.32) are obtained as:

$$\begin{gathered}\bar{f}_{i0}(r,t)=\int_0^1\bar{f}_i(r,\theta,t)\,d\theta,\\ \bar{f}_{imc}(r,\theta,t)=\int_0^1\bar{f}_i(r,\theta,t)\cos(m\theta)\,d\theta,\\ \bar{f}_{ims}(r,\theta,t)=\int_0^1\bar{f}_i(r,\theta,t)\sin(m\theta)\,d\theta.\end{gathered} \tag{2.33}$$

Using Eqs (2.31) and (2.32) in Eqs (2.25) and (2.30), we get;

$$\left(\frac{\partial^2\bar{\bar{\Theta}}_{im}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial r}-\frac{m^2}{4\pi^2 r^2}\bar{\bar{\Theta}}_{im}-\psi_n^2\bar{\bar{\Theta}}_{im}\right)+\bar{\bar{Q}}_m(r,t)=\rho_i\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial t}. \tag{2.34}$$

The conditions (2.26-2.29) become

$$\bar{\bar{\Theta}}_{im}=0, \quad \text{at} \quad t=0, \tag{2.35}$$

$$\frac{\partial\bar{\bar{\Theta}}_{1m}}{\partial r}+Bi_1\,\bar{\bar{\Theta}}_{1m}=0, \quad \text{at} \quad r=\Phi_1, \tag{2.36}$$

$$\frac{\partial\bar{\bar{\Theta}}_{km}}{\partial r}+Bi_2\,\bar{\bar{\Theta}}_{km}=\bar{\bar{f}}_{km}(t), \quad \text{at} \quad r=\Phi_2, \tag{2.37}$$

$$\begin{gathered}\bar{\bar{\Theta}}_{im}(r_{i-1},t)=\bar{\bar{\Theta}}_{i-1,m}(r_{i-1},t),\\ \left.\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial r}\right|_{r=r_{i-1}}=\left.\frac{\partial\bar{\bar{\Theta}}_{i-1,m}}{\partial r}\right|_{r=r_{i-1}}\end{gathered} \tag{2.38}$$

where

$$\bar{\bar{f}}_{km}(t)=A_1\exp(at),\ A_1=Ki\theta_0 z_0\sin(m\theta_0/2)\sin(n\pi z_0/h),$$

$$\bar{\bar{Q}}(r,\psi_n,t)=A_2\delta(r-r_a)\delta(t),\ A_2=(z_0\theta_0/P_0)\sin(n\pi z_0/h)\sin(m\theta_0/2).$$

Using integration and operating eqn. (2.34) by $\int_{r_{i-1}}^{r_i} r\,S_{im}(r)\,dr$, we get [9]

$$\begin{gathered}\int_{r_{i-1}}^{r_i}\left(\frac{\partial^2 S_{im}(r)}{\partial r^2}+\frac{1}{r}\frac{\partial S_{im}(r)}{\partial r}-\frac{m^2}{4\pi^2 r^2}S_{im}(r)-\psi_n^2 S_{im}(r)\right)r\,\bar{\bar{\Theta}}_{im}\,dr+\\ +\left[r\,S_{im}(r)\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial r}-r\,\bar{\bar{\Theta}}_{im}\frac{\partial S_{im}(r)}{\partial r}\right]_{r_{i-1}}^{r_i}+\int_{r_{i-1}}^{r_i}(\bar{\bar{Q}}_m(r,t))\,r\,S_{im}(r)\,dr=\int_{r_{i-1}}^{r_i}\left(\rho_i\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial t}\right)r\,S_{im}(r)\,dr.\end{gathered} \tag{2.39}$$

$S_{im}(r)$ in Eq.(2.39) is chosen so that it satisfies the following differential equation

$$\frac{\partial^2 S_{im}(r)}{\partial r^2}+\frac{1}{r}\frac{\partial S_{im}(r)}{\partial r}+\left(-\frac{m^2}{4\pi^2 r^2}-\psi_n^2+\alpha_{im}^2\right)S_{im}(r)=0. \tag{2.40}$$

Subject to inner and outer surface, interface conditions

$$\frac{dS_{1m}}{dr} + Bi_1 S_{1m} = 0, \tag{2.41}$$

$$\frac{dS_{km}}{dr} + Bi_2 S_{km} = 0, \tag{2.42}$$

$$\begin{gathered} S_{im}(r_{i-1}) = S_{i-1,m}(r_{i-1}), \\ \left.\frac{dS_{im}}{dr}\right|_{r=r_{i-1}} = \left.\frac{dS_{i-1,m}}{dr}\right|_{r=r_{i-1}}. \end{gathered} \tag{2.43}$$

Solution of these eqns. in terms of Bessel's function is expressed as:

$$S_{imp}(\alpha_{imp} r) = a_{imp} J_0(\alpha_{imp} r) + b_{imp} Y_0(\alpha_{imp} r).$$

Orthogonality condition

$$\sum_{i=1}^{k} \int_{r_{i-1}}^{r_i} r\, S_{imp}(\alpha_{imp} r) S_{imp}(\alpha_{imq} r)\, dr = \left| \begin{array}{ll} 0\,; & p \neq q, \\ S_{imp}(\alpha_{imp})\,; & p = q, \end{array} \right. \tag{2.44}$$

$$\kappa_i \alpha_{imp}^2 = \kappa_1 \alpha_{1mp}^2 \tag{2.45}$$

Using Eq.(2.40), Eq.(2.39) becomes:

$$\begin{gathered} \left[r\, S_{imp}(r) \frac{\partial \bar{\bar{\Theta}}_{im}}{\partial r} - r\, \bar{\bar{\Theta}}_{im} \frac{\partial S_{imp}(r)}{\partial r} \right]_{r_{i-1}}^{r_i} + \int_{r_{i-1}}^{r_i} (\bar{\bar{Q}}_m(r,t))\, r\, S_{imp}(r)\, dr = \\ = \int_{r_{i-1}}^{r_i} \left(\rho_i \frac{\partial \bar{\bar{\Theta}}_{im}}{\partial t} + \alpha_{imp}^2 \bar{\bar{\Theta}}_{im} \right) r\, S_{imp}(r)\, dr. \end{gathered} \tag{2.46}$$

Using Eq.(2.45), Eq.(2.46) becomes:

$$\begin{gathered} \left[r\, S_{imp}(r) \frac{\partial \bar{\bar{\Theta}}_{im}}{\partial r} - r\, \bar{\bar{\Theta}}_{im} \frac{\partial S_{imp}(r)}{\partial r} \right]_{r_{i-1}}^{r_i} + \int_{r_{i-1}}^{r_i} (\bar{\bar{Q}}_m(r,t))\, r\, S_{imp}(r)\, dr = \\ = \int_{r_{i-1}}^{r_i} \left(\rho_i \frac{\partial \bar{\bar{\Theta}}_{im}}{\partial t} + (\kappa_1 / \kappa_i)\, \alpha_{1mp}^2 \bar{\bar{\Theta}}_{im} \right) r\, S_{imp}(r)\, dr. \end{gathered} \tag{2.47}$$

After simplification, we get:

$$\sum_{i=1}^{k}\lambda_i\left[r\,S_{imp}(r)\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial r}-r\,\bar{\bar{\Theta}}_{im}\frac{\partial S_{imp}(r)}{\partial r}\right]_{r_{i-1}}^{r_i}+\sum_{i=1}^{k}\int_{r_{i-1}}^{r_i}\lambda_i\bar{\bar{Q}}_m(r,t)\,r\,S_{imp}(r)\,dr=$$
$$=\sum_{i=1}^{k}\int_{r_{i-1}}^{r_i}\lambda_i\left(\rho_i\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial t}+(\kappa_1/\kappa_i)\alpha_{1mp}^2\bar{\bar{\Theta}}_{im}\right)r\,S_{imp}(r)\,dr. \tag{2.48}$$

We define

$$\bar{\bar{\Theta}}_{mp}^{\varphi}=\sum_{i=1}^{k}\lambda_i\int_{r_{i-1}}^{r_i}r\,S_{imp}(r)\,\bar{\bar{\Theta}}_{im}\,dr,$$
$$\bar{\bar{Q}}_{mp}^{\varphi}=\sum_{i=1}^{k}\int_{r_{i-1}}^{r_i}\lambda_i\,\bar{\bar{Q}}_{im}(r,t)\,r\,R_{imp}(r)\,dr. \tag{2.49}$$

Hence Eq.(2.48) becomes:

$$\rho_i\frac{d\,\bar{\bar{\Theta}}_{mp}^{\varphi}}{d\,t}+(\kappa_1/\kappa_i)\alpha_{1mp}^2\bar{\bar{\Theta}}_{mp}^{\varphi}=\sum_{i=1}^{k}\lambda_i\left[r\,S_{imp}(r)\frac{\partial\bar{\bar{\Theta}}_{im}}{\partial r}-r\,\bar{\bar{\Theta}}_{im}\frac{\partial S_{imp}(r)}{\partial r}\right]_{r_{i-1}}^{r_i}+\bar{\bar{Q}}_{mp}^{\varphi}. \tag{2.50}$$

Applying the interface conditions (2.38) and (2.43), yields

$$\frac{d\,\bar{\bar{\Theta}}_{mp}^{\varphi}}{d\,t}+A_3\bar{\bar{\Theta}}_{mp}^{\varphi}=A_4\exp(at)+A_5\,\delta(t) \tag{2.51}$$

where

$$A_3=(\kappa_1/\rho_i\,\kappa_i)\alpha_{1mp}^2,\quad A_4=A_1(\lambda_k\,r_k/\rho_k),$$

$$A_5=A_2\;F(r_i,r_{i-1}),\quad F(r_i,r_{i-1})=\int_{r_{i-1}}^{r_i}[\delta(r-r_a)\,r\,S_i(r)\,dr]$$

and

$$\bar{\bar{\Theta}}_{mp}^{\varphi}=0,\quad \text{at}\quad t=0. \tag{2.52}$$

Using Laplace transform (LT), LT inversion on Eqs (2.51), (2.52), we get:

$$\bar{\bar{\Theta}}_{mp}^{\varphi}=E_1[\exp(at)-\exp(-A_3t)]+A_5\exp(-A_3t). \tag{2.53}$$

Here $E_1=(A_4/A_3+a)$.

The generalized Fourier series expansion of $\bar{\bar{\Theta}}_{im}(r,t)$ is

$$\bar{\bar{\Theta}}_{im}(r,t)=\sum_{p=1}^{\infty}c_{mp}(t)S_{imp}(r), \tag{2.54}$$

where $\quad c_{mp}(t)=[\sum_{i=1}^{k}\lambda_i\int_{r_{i-1}}^{r_i} r\,S_{imp}(r)\bar{\bar{\Theta}}_{im}\,dr]/[S_{imp}(\alpha_{imp})].$

Now using Eq.(2.49), we get

$$c_{mp}(t)=[\bar{\bar{\Theta}}^{\varphi}_{mp}]/[S_{imp}(\alpha_{imp})].$$

Using Eq.(2.54) in Eq.(2.31), we get

$$\bar{\Theta}_i(r,\theta,t)=\sum_{p=1}^{\infty}\Psi_2 S_{i0p}(r)+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_3 S_{imp}(r)\cos(m\theta)+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_4 S_{imp}(r)\sin(m\theta)\,, \tag{2.55}$$

where

$$\Psi_2=[\bar{\bar{\Theta}}^{\varphi}_{0p}]/[S_{i0p}(\alpha_{i0p})],\quad \Psi_3=[\bar{\bar{\Theta}}^{\varphi}_{mpc}]/[S_{imp}(\alpha_{imp})],\quad \Psi_4=[\bar{\bar{\Theta}}^{\varphi}_{mps}]/[S_{imp}(\alpha_{imp})]\,.$$

Applying inverse Fourier Sine transform on the above Eq.(2.55), yields

$$\begin{aligned}\Theta_i(r,z,\theta,t)=\sum_{n=1}^{\infty}\Bigg\{&\sum_{p=1}^{\infty}\Psi_2 S_{i0p}(r)+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_3 S_{imp}(r)\cos(m\theta)+\\ &+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_4 S_{imp}(r)\sin(m\theta)\Bigg\}\sin(n\pi z/h).\end{aligned} \tag{2.56}$$

Heat conductivity is taken as [1]:

$$\lambda_i(T_i)=\lambda_{i0}\exp(\varpi_1 T_i),\quad \varpi_1\le 0\,. \tag{2.57}$$

Here λ_{i0} is the dimensionless reference value of thermal conductivity defined by,

$$\lambda_{i0}{*}=\frac{\lambda_{i0}}{\lambda_{(i-1)0}}\,.$$

Substituting Eq.(2.57) in Eq.(2.17), yields

$$\Theta_i=(\lambda_{i0}/\varpi_1)[\exp(\varpi_1 T_i)-1]. \tag{2.58}$$

Using Eq.(2.58) in Eq.(2.56), yields

$$T_i(r,z,\theta,t)=(1/\varpi_1)\log_e[g(r,z,\theta,t)+1] \tag{2.59}$$

where

$$g(r,z,\theta,t)=\sum_{n=1}^{\infty}(\varpi_1/\lambda_{i0})\left\{\sum_{p=1}^{\infty}\Psi_2 S_{i0p}(r)+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_3 S_{imp}(r)\cos(m\theta)+\right.$$
$$\left.+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_4 S_{imp}(r)\sin(m\theta)\right\}\sin(n\pi z/h).$$

We use the following logarithmic expansion

$$\log_e[g(r,z,\theta,t)+1]=[g(r,z,\theta,t)]+(1/2)[g(r,z,\theta,t)]^2+(1/3)[g(r,z,\theta,t)]^3+\dots \tag{2.60}$$

Ignoring terms with order greater than one, we get:

$$\log_e\left(g(r,z,\theta,t)+1\right)]=g(r,z,\theta,t).$$

Hence Eq.(2.59) becomes

$$T_i(r,z,\theta,t)=\sum_{n=1}^{\infty}(1/\lambda_{i0})\left\{\sum_{p=1}^{\infty}\Psi_2 S_{i0p}(r)+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_3 S_{imp}(r)\cos(m\theta)+\right.$$
$$\left.+\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\Psi_4 S_{imp}(r)\sin(m\theta)\right\}\sin(n\pi z/h). \tag{2.61}$$

3. Thermoelastic analysis

In the cylindrical coordinate system, the boundary conditions for thin disc with support at both ends are [35]:

$$\nabla^2\nabla^2 w^{(i)}=\frac{-1}{(1-\nu_i)D_i}\nabla^2 M_T^{(i)} \tag{3.1}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},\qquad D_i=\frac{E_i\,h^3}{12\left(1-\nu_i^2\right)}. \tag{3.2}$$

Subject to conditions

$$w^{(i)}=\frac{\partial w^{(i)}}{\partial t}=0,\quad \text{at}\quad t=0. \tag{3.3}$$

The problem is restricted under thermal load by an elastic reaction along the boundaries $r=r_0$, $r=r_k$, [16, 22]

$$\frac{\partial w^{(i)}}{\partial r}+w^{(i)}=0, \quad \text{at} \quad r=r_0,$$

$$\frac{\partial w^{(k)}}{\partial r}+w^{(k)}=0, \quad \text{at} \quad r=r_k,$$

$$w^{(i)}=w^{(i-1)}, \quad \text{at} \quad r=r_{i-1}, \tag{3.4}$$

$$\frac{\partial w^{(i)}}{\partial r}=\frac{\partial w^{(i-1)}}{\partial r} \quad \text{at} \quad r=r_{i-1}.$$

It is assumed that the constants of proportionality specified as per Hooke's law are one.
Resultant and shearing forces, moments are [35]:

$$N_{rr}^{(i)}=N_{\theta\theta}^{(i)}=N_{r\theta}^{(i)}=0, \tag{3.5}$$

$$Q_{rr}^{(i)}=-D_i\frac{\partial}{\partial r}\left(\nabla^2 w^{(i)}\right)-\frac{1}{1-\nu_i}\frac{\partial M_T^{(i)}}{\partial r},$$

$$Q_{\theta\theta}^{(i)}=-D_i\frac{1}{r}\frac{\partial}{\partial\theta}\left(\nabla^2 w^{(i)}\right)-\frac{1}{1-\nu_i}\frac{1}{r}\frac{\partial M_T^{(i)}}{\partial\theta}, \tag{3.6}$$

$$M_{rr}^{(i)}=-D_i\left[\frac{\partial^2 w^{(i)}}{\partial r^2}+\frac{\nu_i}{r}\frac{\partial w^{(i)}}{\partial r}\right]-\frac{1}{1-\nu_i}M_T^{(i)},$$

$$M_{\theta\theta}^{(i)}=-D_i\left[\nu_i\frac{\partial^2 w^{(i)}}{\partial r^2}+\frac{1}{r}\frac{\partial w^{(i)}}{\partial r}\right]-\frac{1}{1-\nu_i}M_T^{(i)}. \tag{3.7}$$

Here $M_{rr}^{(i)}$ satisfies the condition

$$\left.M_{rr}^{(i)}\right|_{r=r_0}=0, \qquad 0<\theta<1. \tag{3.8}$$

Stress components are

$$\sigma_{rr}^{(i)}=\frac{1}{h}N_{rr}^{(i)}+\frac{12z}{h^3}M_{rr}^{(i)}+\frac{1}{1-\nu_i}\left(\frac{1}{h}N_T^{(i)}+\frac{12z}{h^3}M_T^{(i)}-\alpha_i(T_i)\,E_i\,T_i\right),$$

$$\sigma_{\theta\theta}^{(i)}=\frac{1}{h}N_{\theta\theta}^{(i)}+\frac{12z}{h^3}M_{\theta\theta}^{(i)}+\frac{1}{1-\nu_i}\left(\frac{1}{h}N_T^{(i)}+\frac{12z}{h^3}M_T^{(i)}-\alpha_i(T_i)\,E_i\,T_i\right), \tag{3.9}$$

$$M_T^{(i)} = E_i \int_0^h \alpha_i(T_i)\, T_i\, z\, dz \ , \qquad N_T^{(i)} = E_i \int_0^h \alpha_i(T_i)\, T_i\, dz. \tag{3.10}$$

Here $\alpha_i(T_i)$ is the temperature dependent coefficient of linear thermal expansion assumed as:

$$\alpha_i(T_i) = \alpha_{i0} \exp(\varpi_2 T_i), \quad \varpi_2 \geq 0 \ . \tag{3.11}$$

Here α_{i0} is the dimensionless reference value of coefficient of linear thermal expansion defined by,

$$\alpha_{i0}{}^* = \frac{\alpha_{i0}}{\alpha_{(i-1)0}} \ .$$

Using Eqs (2.61) and (3.11), in Eq.(3.10), we get $M_T^{(i)}$ and $N_T^{(i)}$ as:

$$\begin{aligned} M_T^{(i)} = \left(\alpha_i E_i / \lambda_{i0}\right) \sum_{n=1}^{\infty} \Psi_5 \Bigg\{ & \sum_{p=1}^{\infty} \Psi_2 S_{i0p}(r) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_3 S_{imp}(r) \cos(m\theta) + \\ & + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_4 S_{imp}(r) \sin(m\theta) \Bigg\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} N_T^{(i)} = \left(\alpha_i E_i / \lambda_{i0}\right) \sum_{n=1}^{\infty} \Psi_6 \Bigg\{ & \sum_{p=1}^{\infty} \Psi_2 S_{i0p}(r) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_3 S_{imp}(r) \cos(m\theta) + \\ & + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_4 S_{imp}(r) \sin(m\theta) \Bigg\} \end{aligned} \tag{3.13}$$

where

$$\Psi_5 = [(h^2 / 24n^2\pi^2)][(-3n\pi(8+3\varpi_2^2)\cos(n\pi) + \varpi_2(3+6n^2\pi^2 - 3\cos(2n\pi) + n\pi\varpi_2\cos(3n\pi))],$$

$$\Psi_6 = [(h / 24n\pi)][24 + 12n\pi\varpi_2 + 8\varpi_2^2 - 3(8+3\varpi_2^2)\cos(n\pi) + \varpi_2^2\cos(3n\pi)].$$

Using Eq.(3.12), the deflection $w^{(i)}$ of the i^{th} layer from Eq.(3.1) is obtained as:

$$\begin{aligned} w^{(i)} = & \left(12(1+\nu)\alpha_i \Psi_5 / \lambda_{i0} h^3\right) \times \Big[(\Phi_1 + \Phi_2 + \Phi_3) / \Big(\Phi_1 + \left((m^2 + r^2)/r^2\right)\Phi_2 + \\ & + \left((m^2+r^2)/r^2\right)\Phi_3 \Big) \Big] \times \sum_{n=1}^{\infty} \left\{ \Phi_1 S_{i0p}(r) + \Phi_2 S_{imp}(r) + \Phi_3 S_{imp}(r) \right\} \end{aligned} \tag{3.14}$$

where $\quad \Phi_1 = \sum_{p=1}^{\infty} \Psi_2, \quad \Phi_2 = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_3 \cos(m\theta), \quad \Phi_3 = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \Psi_4 \sin(m\theta).$

Using Eqs (3.12) and (3.14), Eqs (3.7), (3.9) are solved with the help of Mathematica software.

4. Numerical analysis

For numerical analysis, a mathematical model is formulated for a 3-layered disc, with the inner, middle and outer layers composed of copper, zinc and aluminum, respectively [22].

Let ambient temperature $T_0 = 20$, $r_0 = 1, r_1 = 2, r_2 = 3, r_3 = 4$, inner layer $r_0 < r < r_1$, middle layer $r_1 < r < r_2$, outer layer $r_2 < r < r_3$.

Figures 2(a), 2(b), 2(c) shows temperature along θ, z, r respectively. Here temperature distribution follows a sinusoidal nature. Along axial direction, its magnitude suddenly increases and reaches to zero towards the end. Along radial direction, the temperature slowly increases and attains peak and reduces towards the inner-layer.

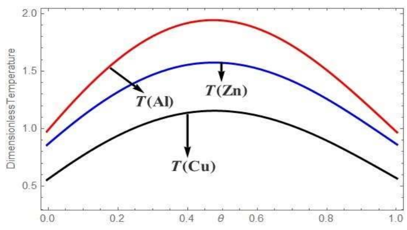

Fig.2a. Plot of temperature along θ

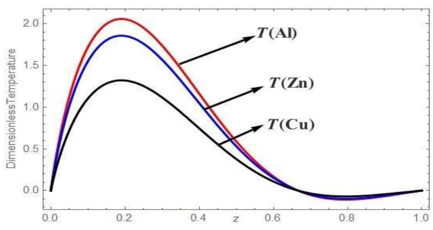

Fig.2b. Plot of temperature along z

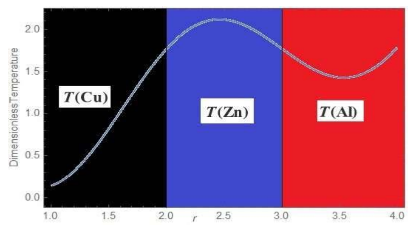

Fig.2c. Plot of temperature along r

The following Figs (3-7) on left represent homogeneous case, while on right represent nonhomogeneous case.

Figures 3(a), 3(b) show the dimensionless deflection along θ, r respectively. Due to the application of heat, the deflection is more in outer-layer as compared to other layers. Its magnitude is more in the temperature dependent case as compared to the temperature independent case.

Figures 4(a), 4(b), 5(a), 5(b) show the dimensionless resultant moments along θ, r. The moment $M_{\theta\theta}$. is tensile in nature, while M_{rr} is seen to be tensile and compressive in different regions.

Figures 6(a), 6(b), 6(c) show $\sigma_{\theta\theta}$, while Figs 7(a), 7(b), 7(c) show σ_{rr} along θ, z, r. $\sigma_{\theta\theta}$ is tensile along θ, z, while radially its nature changes in both cases. Magnitude of σ_{rr} gradually increases with increase in θ, z, while along r direction it becomes compressive and attains tensile nature towards the inner-layer.

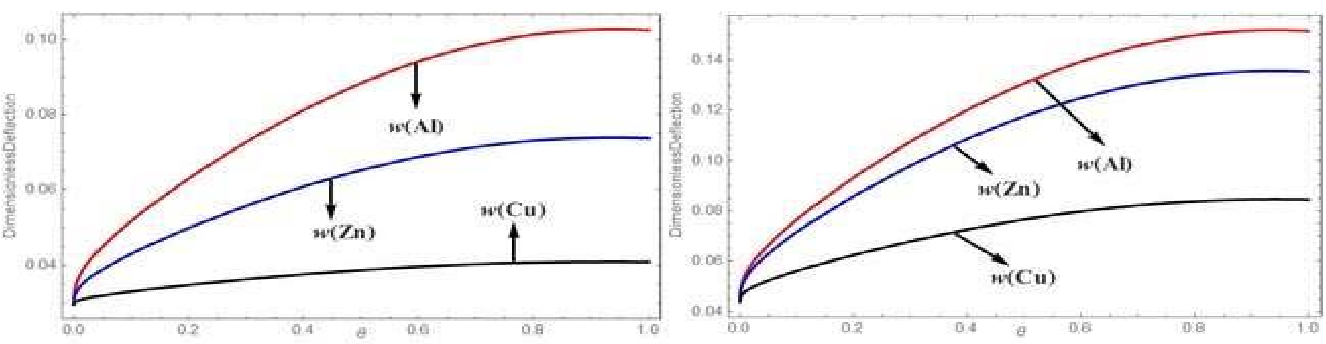

Fig.3(a). Plot of deflection along θ.

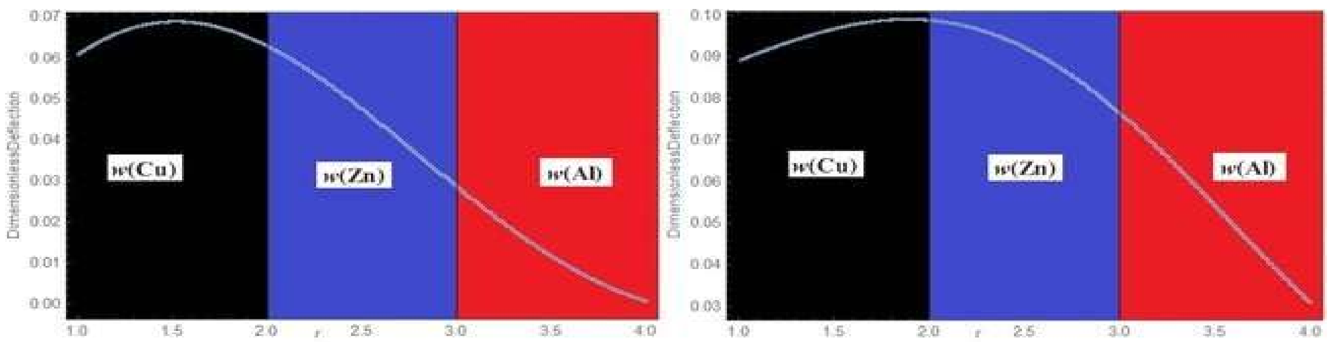

Fig.3(b). Plot of deflection along *r*.

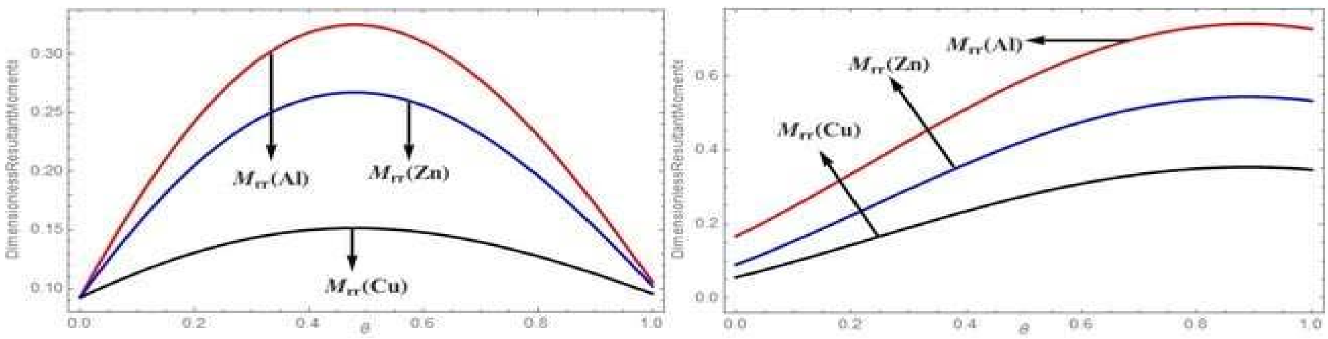

Fig.4(a). Plot of $M_{\theta\theta}$ along θ.

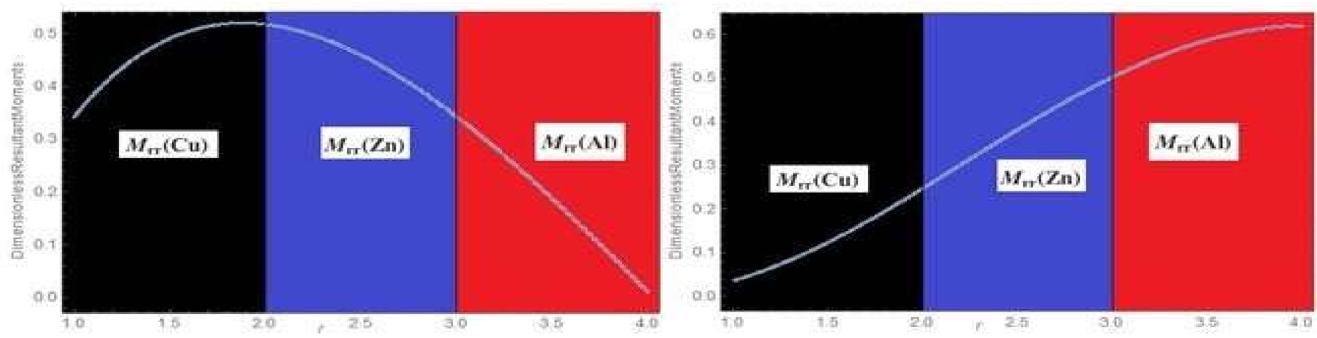

Fig.4(b). Plot of $M_{\theta\theta}$ along *r*.

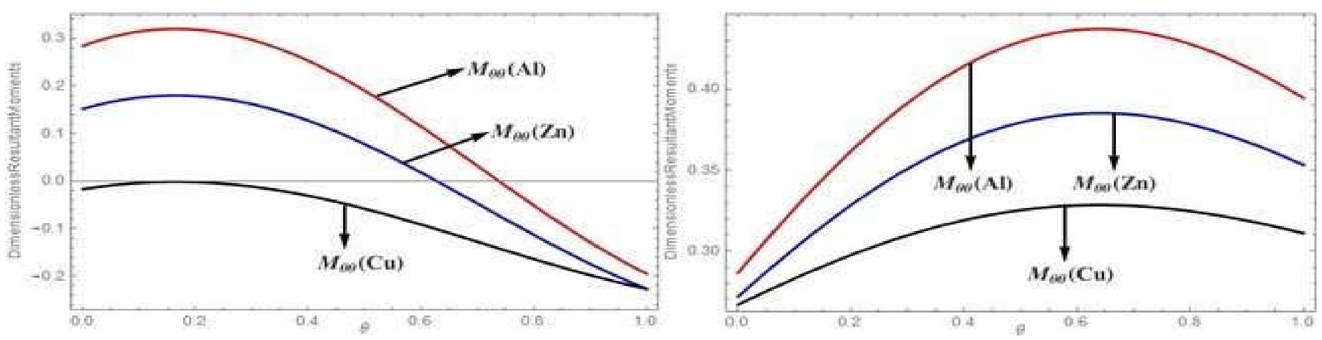

Fig.5(a). Plot of M_{rr} along θ.

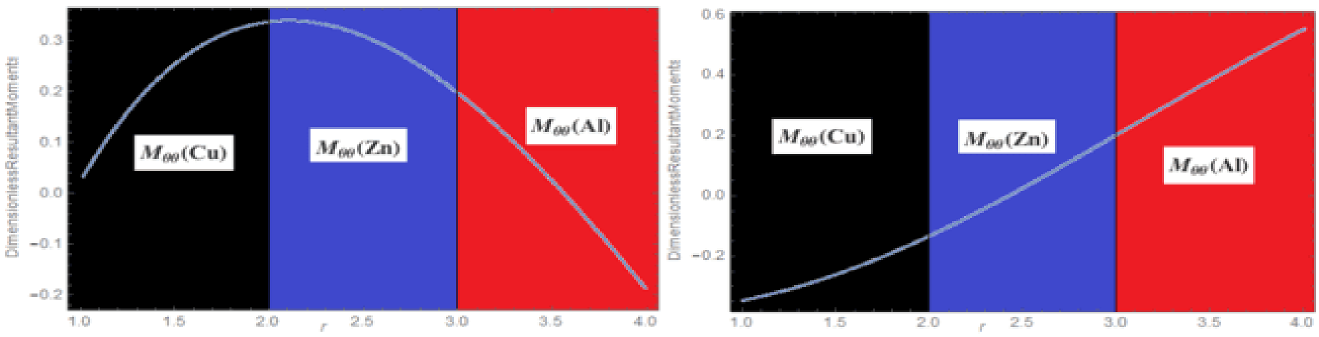

Fig.5(b). Plot of M_{rr} along *r*.

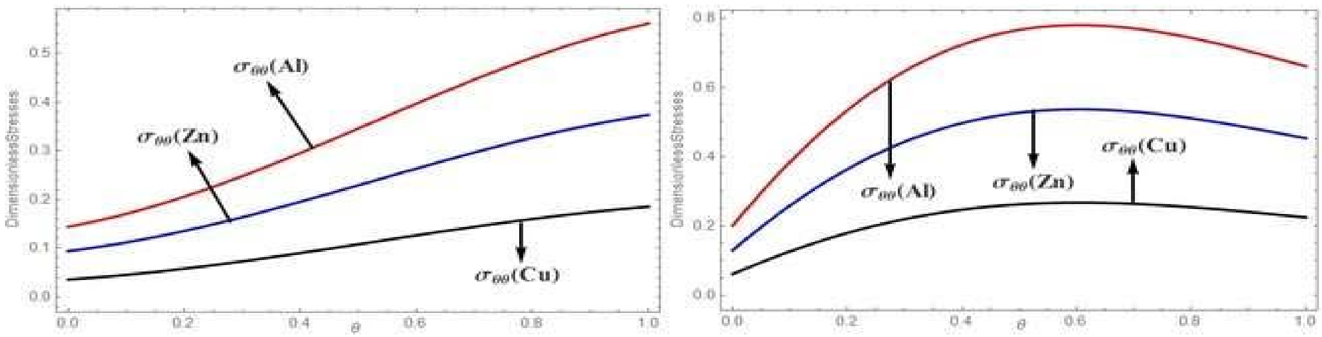

Fig.6(a). Plot of $\sigma_{\theta\theta}$ along θ.

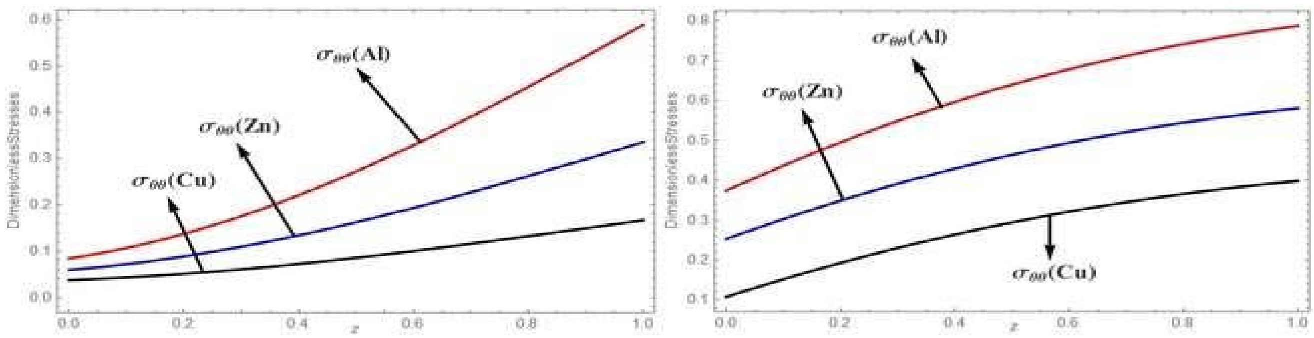

Fig.6(b). Plot of $\sigma_{\theta\theta}$ along *z*.

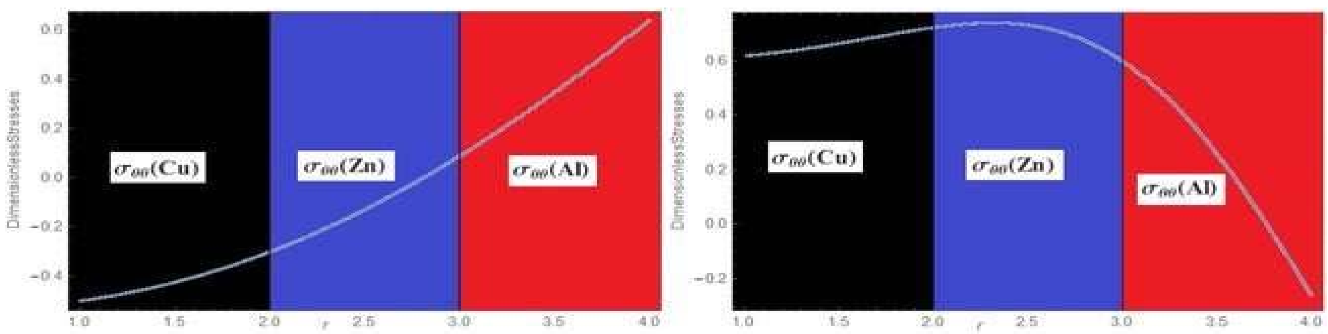

Fig.6(c). Plot of $\sigma_{\theta\theta}$ along *r*.

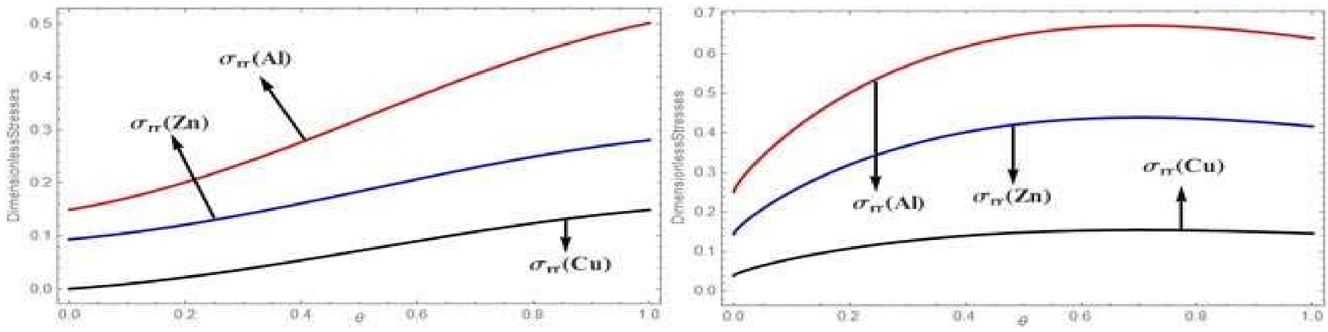

Fig.7(a). Plot of σ_{rr} along θ.

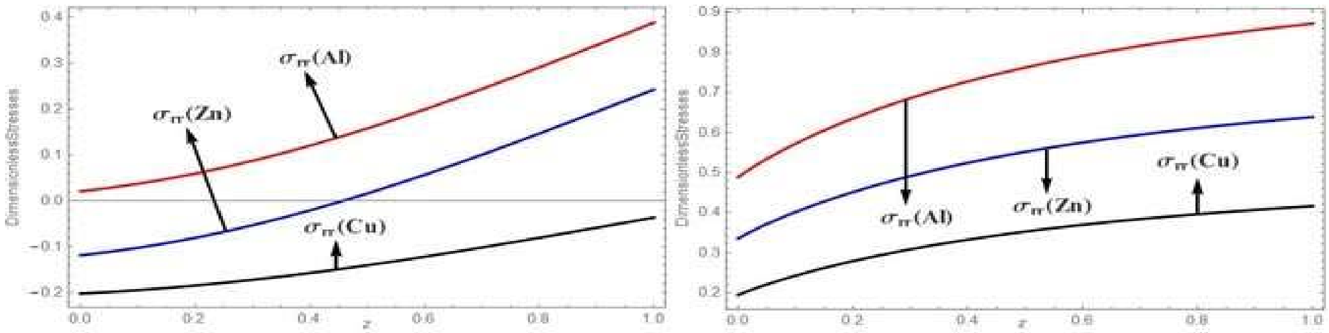

Fig.7(b). Plot of σ_{rr} along *z*.

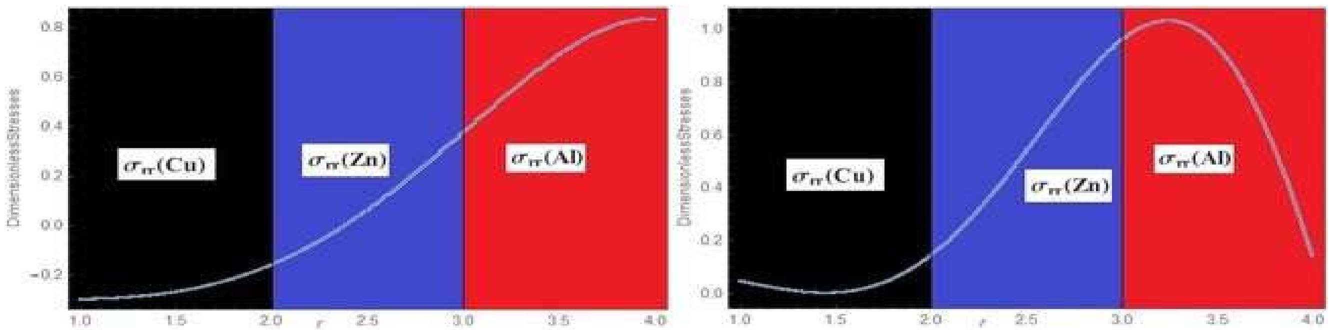

Fig.7(c). Plot of σ_{rr} along *r*.

5. Validation of the results

This paper presents a mathematical model for a multilayered thin annular circular disc. The asymmetric heat conduction problem with time-dependent boundary conditions and heat source is solved using the finite integral transform approach. The temperature distribution, along with its corresponding deflection, resultant moments, and thermal stress distributions, has been derived. As a limiting case, if we consider homogeneous material properties, the results agree with [22].

6. Conclusion

Thermal behavior of a multiple layered annular circular disc due to instantaneous point heating is investigated in this paper by taking thermally sensitive material properties. The HCE is solved using

Kirchhoff's method and finite integral transform method. In the temperature independent case, the radial stress suddenly changes to compressive nature as the heat passes from middle layer to inner layer, whereas it is tensile in nature for all regions in the temperature dependent case. The point heat source generates heat in the annular disc in the middle layer. This heat source causes the temperature rise in the middle layer, while heat propagates accordingly in the inner and outer layers. The annular disc experiences notable variations in thermoelastic properties due to the introduction of point heat source.

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Nomenclature

$C_i(T_i)$ – specific heat capacity of the i^{th} layer
D_i – bending rigidity of the i^{th} layer
E_i – Young's modulus of the ith layer
h – thickness of the disc
h_0, h_K – surface coefficients
$M_{rr}^{(i)}, M_{\theta\theta}^{(i)}$ – resultant moments
$M_T^{(i)}, N_T^{(i)}$ – thermally induced resultant moments of the i^{th} layer
$N_{rr}^{(i)}, N_{\theta\theta}^{(i)}, N_{r\theta}^{(i)}$ – resultant forces
$Q(r,\theta,z,t)$ – internal heat generation
$Q_{rr}^{(i)}, Q_{\theta\theta}^{(i)}$ – shearing forces
T_i – temperature of the i^{th} layer
T_0 – ambient temperature
$w^{(i)}$ – deflection of the i^{th} layer
$\alpha_i(T_i)$ – temperature dependent coefficient of linear thermal expansion
$\lambda_i(T_i)$ – thermal conductivity of the i^{th} layer
ν_i – Poisson's ratio of the i^{th} layer
ρ_i – density of the i^{th} layer
$\sigma_{rr}, \sigma_{\theta\theta}$ – components of stress functions

References

[1] Noda N. (1986): *Thermal stresses in materials with temperature dependent properties*.– Applied Mechanics Reviews, vol.44, pp.383-397, https://api.semanticscholar.org/CorpusID:120719142
[2] Olcer N.Y. (1968): *A general unsteady heat flow problem in a finite composite hollow circular cylinder under boundary conditions of second kind*.– Nuclear Engineering and Design, vol.7, No.2, pp.97-112. https://doi.org/10.1016/0029-5493(68)90052-6
[3] Gorman D.G. (1985): *Thermal gradient effects upon the vibrations of certain composite circular plates Part I*: *Plane Orthotropic*.– Journal of Sound and Vibration, vol.101, No.3, pp.337-345, https://doi.org/10.1016/S0022-460X(85)80134-6
[4] Popovich V.S. and Fedai B.N. (1997): *The axisymmetric problem of thermoelasticity of a multilayer thermosensitive tube*.– Journal of Mathematical Sciences, vol.86, pp.2605-2610, https://doi.org/10.1007/BF02356105
[5] Popovych V.S. and Makhorkin I.M. (1998): *On the solution of heat conduction problems for thermosensitive bodies*.– Journal of Mathematical Sciences, vol.88, No.3, pp.352-359, https://doi.org/10.1007/BF02365251
[6] Malzbender J. and Jülich F. (2004): *Mechanical and thermal stresses in multilayered materials*.– Journal of Applied Physics, vol.95, No.4, pp.1780-1782. DOI: 10.1063/1.1642289

[7] Kayhani M., Nourouzi M. and Delooei A.A. (2010): *An exact solution of axi-symmetric conductive heat transfer in cylindrical composite laminate under the general boundary condition*.– International Journal of Mechanical, Aerospace, Industrial, Mechatronic and Manufacturing Engineering, vol.4, pp.776-783.

[8] Singh S., Jain P.K. and Rizwan-uddin (2010): *Analytical solution of time-dependent multilayer heat conduction problems for nuclear applications*.– Proceedings of the 1st International Nuclear and Renewable Energy Conference (INREC10), Amman, Jordan, March 21-24, https://doi.org/10.1109/INREC.2010.5462601

[9] Singh S., Jain P.K. and Rizwan-uddin (2011): *Finite integral transform method to solve asymmetric heat conduction in a multilayer annulus with time-dependent boundary conditions*.– Nuclear Engineering and Design, vol.241, No.1, pp.144-154. https://doi.org/10.1016/j.nucengdes.2010.10.010

[10] Kayhani M., Nourouzi M. and Delooei A.A. (2012): *A general analytical solution for heat conduction in cylindrical multilayer composite laminates*.– International Journal of Thermal Sciences, vol.52, No.1, pp.73-82, 10.1016/j.ijthermalsci.2011.09.002

[11] Norouzi M., Delouei A.A. and Seilsepour M. (2013): *A general exact solution for heat conduction in multilayer spherical composite laminates*.– Composite Structures, vol.106, pp.288-295, http://dx.doi.org/10.1016/j.compstruct.2013.06.005

[12] Dalir N. and Nourazar S.S. (2014): *Analytical solution of the problem on the three-dimensional transient heat conduction in a multilayer cylinder*.– Journal of Engineering Physics and Thermophysics, vol.87, No.1, pp.89-97, DOI: 10.1007/s10891-014-0988-2

[13] Popovych V.S. and Kalynyak B.M. (2016): *Mathematical modelling and methods for the determination of the static thermoelastic state of multilayer thermally sensitive cylinders*.– Journal of Mathematical Sciences, vol.215, pp.218-242, https://doi.org/10.1007/s10958-016-2833-y

[14] Torabi M. and Zhang K. (2016): *Analytical solution for transient temperature and thermal stresses within convective multilayer discs with time-dependent internal heat generation, Part I: Methodology*.– Journal of Thermal Stresses, vol.39, No.4, pp.398-413, DOI: 10.1080/01495739.2016.1152132

[15] Manthena V.R., Lamba N.K., Kedar G.D. and Deshmukh K.C. (2016): *Effects of stress resultants on thermal stresses in a functionally graded rectangular plate due to temperature dependent material properties*.– International Journal of Thermodynamics, vol.19, No.4, pp.235-242, DOI: 10.5541/ijot.5000201356

[16] Bhad P., Varghese V. and Khalsa L. (2017): *Heat production in a simply supported multilayer elliptic annulus composite plate and its associated thermal stresses*.– Journal of Stress Analysis, vol.2, issue 2, DOI: 10.22084/jrstan.2018.15081.1034

[17] Manthena V.R., Lamba N.K. and Kedar G.D. (2018): *Estimation of thermoelastic state of a thermally sensitive functionally graded thick hollow cylinder: a mathematical model*.– Journal of Solid Mechanics, vol.10, No.4, pp.766-778, doi: 20.1001.1.20083505.2018.10.4.6.4

[18] Manthena V.R. and Kedar G.D. (2018): *Mathematical modeling of thermoelastic state of a functionally graded thermally sensitive thick hollow cylinder with internal heat generation*.– International Journal of Thermodynamics, vol.21, No.4, pp.202-212, https://doi.org/10.5541/ijot.434180

[19] Manthena V.R. and Kedar G.D. (2018): *Transient thermal stress analysis of a functionally graded thick hollow cylinder with temperature dependent material properties*.– Journal of Thermal Stresses, vol.41, No.5, pp.568-582, DOI: 10.1080/01495739.2017.1402669

[20] Manthena V.R. and Kedar G.D. (2019): *On thermoelastic problem of a thermosensitive functionally graded rectangular plate with instantaneous point heat source*.– Journal of Thermal Stresses, vol.42, No.7, pp.849-862, https://doi.org/10.1080/01495739.2019.1587327

[21] Manthena V.R. (2019): *Uncoupled thermoelastic problem of a functionally graded thermosensitive rectangular plate with convective heating*.– Archive of Applied Mechanics, vol.89, No.8, pp.1627-1639, DOI: 10.1007/s00419-019-01532-1

[22] Manthena V.R., Srinivas V.B. and Kedar G.D. (2020): *Analytical solution of heat conduction of a multilayered annular disc and associated thermal deflection and thermal stresses*.– Journal of Thermal Stresses, vol.43, No.5, pp.563-578, https://doi.org/10.1080/01495739.2020.1735975

[23] Jangid K. and Mukhopadhyay S. (2020): *Variational and reciprocal principles on the temperature-rate dependent two-temperature thermoelasticity theory*.– Journal of Thermal Stresses, vol.43, issue 7, pp.816-828, DOI: 10.1080/01495739.2020.1753607

[24] Etkin V. (2021): *Solving the problem of thermodynamic inequalities*.– International Journal of Thermodynamics, vol.24, No.1, pp.54-60, https://doi.org/10.5541/ijot.874737

[25] Srinivas V.B., Manthena V.R., Bikram J. and Kedar G.D. (2021): *Fractional order heat conduction and thermoelastic response of a thermally sensitive rectangular parallelepiped*.– International Journal of Thermodynamics, vol.24, issue 1, pp.62-73, https://doi.org/10.5541/ijot.849663

[26] Razavi S.E., Adibi T. and Hassanpour H. (2021): *Thermo-flow analysis of cylinder with crossed splitter plates with a characteristics-based scheme*.– International Journal of Thermodynamics, vol.24, No.2, pp.83-91. https://doi.org/10.5541/ijot.783614

[27] Balci E. and Akpinar S. (2021): *Quaternary element incorporation effects on thermal properties and crystal-micro structure of Cu-Al-Fe high temperature shape memory alloys*.– International Journal of Thermodynamics, vol.24, No.2, pp.119-126, https://doi.org/10.5541/ijot.805275

[28] Bikram J. and Kedar G.D. (2021): *Study of thermoelastic behaviour, efficiency and effectiveness of rectangular fin with fractional order energy balance equation*.– International Journal of Thermodynamics, vol.24, No.3, pp.216-225, https://doi.org/10.5541/ijot.879603

[29] Etkin V. (2021): *Synthesis of equilibrium and non-equilibrium thermodynamics*.– International Journal of Thermodynamics, vol.24, No.4, pp.91-101, https://doi.org/10.5541/ijot.912365

[30] Su C.H., Hwu C. and Nguyen V.T. (2021): *Transient thermal stress analysis of temperature-dependent anisotropic viscoelastic solids*.– Journal of Thermal Stresses, vol.44, No.12, pp.1495-1513. https://doi.org/10.1080/01495739.2021.2000344

[31] Lamba N.K. (2022): *Thermosensitive response of a functionally graded cylinder with fractional order derivative*.– International Journal of Applied Mechanics and Engineering, vol.27, No.1, pp.107-124. https://doi.org/10.2478/ijame-2022-0008

[32] Yadav A.K., Singh A. and Jurczak P. (2023): *Memory dependent triple-phase-lag thermo-elasticity in thermo-diffusive medium*.– International Journal of Applied Mechanics and Engineering, vol.28, No.4, pp.137-162, DOI: https://doi.org/10.59441/ijame/172631

[33] Roy S. and Lahiri A. (2020): *A study on fractional order thermoelastic half space*.– International Journal of Applied Mechanics and Engineering, vol.25, No.4, pp.191-202, doi:10.2478/ijame-2020-0058.

[34] Lamba N. (2023): *Impact of memory-dependent response of a thermoelastic thick solid cylinder*.– Journal of Applied and Computational Mechanics, vol.9, No.4, pp.1135-1143, doi: 10.22055/jacm.2023.43952.4149

[35] Noda N., Hetnarski R.B. and Tanigawa Y. (2003): *Thermal Stresses*, second edition.– Taylor & Francis, New York.

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