Is the Galilean law of free fall an *a priori* truth?

Tomasz Bigaj*

Wydział Filozofii Uniwersytetu Warszawskiego

Abstrakt. W niniejszym artykule omawiam znany argument *a priori* Galileusza przeciwko Arystotelesowskiemu prawu spadku swobodnego, przedstawiając go w uwspółcześnionej matematycznej postaci. Pokazuję, że argument ten nie dowodzi, iż tempo spadku swobodnego jest stałe dla wszystkich ciał, chyba że założymy, że jest ono funkcją jednego addytywnego parametru. Jednakże jest całkowicie możliwe, że tempo spadku będzie dane w postaci stosunku dwóch addytywnych parametrów. W takim wypadku argument upada.
Słowa kluczowe: spadek swobodny, Arystoteles, Galileusz, suma wypukła, addytywność

Abstract. In this short note I discuss Galileo's well-known *a priori* argument against Aristotle's law of free fall, presented in a modern mathematical version. I show that the argument does not prove that the rate of fall should be constant for all bodies, unless we presuppose that the rate of fall is a function of one additive parameter. However, it is perfectly possible that the rate of fall will be given in the form of the ratio of two additive parameters, in which case the argument does not go through.
Keywords: free fall, Aristotle, Galileo, convex sum, additivity

Modern science owes a huge debt of gratitude to Galileo Galilei, the famous Italian polymath. His role in overturning the old, Aristotelian and Ptolemaic world view, while paving the way to the scientific revolution that later gave us Newton and Einstein, cannot be overestimated. Galileo was a keen observer and skillful experimentalist, but he also had an uncanny ability to invent simple and ingenious theoretical arguments in his numerous polemics with his scientific adversaries. For instance, in response to the Aristotelian explanation of the motion of projectiles in terms of the pressure from the displaced amount of air he astutely observed that if this explanation was correct, then an arrow shot perpendicularly to the direction of motion should fly faster than an arrow pointing in the direction of its motion. One of the most famous arguments that Galileo produced dealt with the phenomenon of free fall. As it turns out, even after four centuries the argument presents us with interesting challenges, particularly regarding the role of *a priori*, mathematical reasoning in discovering rules and principles governing the physical world.

The argument Galileo used against the then dominant Aristotelian theory of free fall is beautiful in its simplicity. Aristotle assumed, as seems natural, that the rate of fall for heavy objects (measured, for instance, by the amount of time required to reach the ground when dropped from a fixed height) is proportional to their weight. Here is a quote from *Physics* confirming this stance:

*ORCID: 0000-0002-8121-9789

We see that bodies which have a greater impulse either of weight or of lightness, if they are alike in other respects, move faster over an equal space, and in the ratio which their magnitudes bear to each other.

While the Aristotelian law of free fall in its above-given version does not withstand closer scrutiny, since it is easy to observe that a ten-kilogram body does not fall down ten times faster than a one-kilogram one, the final blow to the Peripatetic conception was dealt by argumentation *a priori*. Salviati, a character in the *Dialogue Concerning the Two Chief World Systems* who presents Galileo's own views, announces that: *even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one.*

The argument considers two bodies differing in weight only. Given Aristotle's law of free fall, the heavier body should reach the ground faster than the lighter one. But what will happen when we connect these bodies rigidly, so that they will have to move as one? On the one hand, the resulting body will be heavier than each of its components taken separately, therefore it should fall even faster. On the other hand, as Salviati observes: *it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter.*

Thus the rate of descent of the combined body should fall somewhere between the rates for individual components, which contradicts the earlier conclusion. Aristotle's law (as a matter of fact *any* law which implies that the rate of fall is an increasing function of weight) turns out to be inconsistent.

It may be remarked that Galileo's argument does not look entirely *a priori*, since the premise spelled out in the quotation above, no matter how intuitive, does not seem to be logically or conceptually necessary. The status of the assumption regarding the "hastening" and "retarding" phenomena is somewhat murky. It does not appear to be a straightforward generalization from experience, since it is based on the condition which does not occur in reality (in fact there are no "hasteners" or "delayers" in the Galilean sense – all objects fall at the same rate, no matter their weight).[1] Yet there is something very compelling about the counterfactual statement that if the considered objects had different "natural speeds" associated with their weights, then joining them together would result in an object whose natural speed would fall between the two values. We will simply assume this to be true without further justification, regardless of whether it is a conceptual truth or a basic fact about the world that is very unlikely to be refuted.

Galileo's elegant thought experiment is sufficient to refute the theory of Aristotle, but can it establish the alternative law of free fall as proposed by the great Italian scientist? Galileo famously maintained that, ignoring air resistance, all bodies fall at the same rate regardless of their weight. This looks like a genuine empirical claim which must be experimentally corroborated to be accepted. And yet it is possible to give a straightforward mathematical argument apparently showing that the Galilean law of free fall must be true! Let us present this argument in some details. Let us first write the rate of fall of any heavy body (which nowadays we represent by its acceleration) as a function of its mass $f(m)$. Mass, in turn, is supposed to be additive with respect to the operation of physical summing. That is, for any non-overlapping bodies $a_1, \ldots a_n$, the mass of the physical sum of these bodies equals the sum of the individual masses: $m(a_1 \cup a_2 \cup \ldots a_n) = m(a_1) + m(a_2) + \cdots + m(a_n)$.

The most important premise of the argument is the condition that the function f representing the rate of fall must be such that the value of f for the sum of masses $m_1, \ldots m_n$ is a *convex sum* of the values of f for the individual masses, that is the value of $f(m_1 + \cdots + m_n)$ must fall somewhere between the minimal and the maximal values of $f(m_i)$. This is a straightforward generalization of Galileo's "hasteners" and "delayers" claim given above, and can be expressed mathematically as follows:

$$f\Big(\sum_i m_i\Big) = \sum_i p_i f(m_i), \tag{1}$$

where $\sum_i p_i = 1$ and $p_i \geq 0$ for all i. Now, it is easy to prove that any function f satisfying (1) must be identical on all rational numbers. Take any two rational numbers presented in the form of two fractions with common denominators: $\frac{m}{l}$ and $\frac{k}{l}$. From (1) it follows that

$$f\left(\frac{m}{l}\right) = f\Bigg(\underbrace{\frac{1}{l} + \cdots + \frac{1}{l}}_{m}\Bigg) = f\left(\frac{1}{l}\right) \tag{2}$$

and likewise for $\frac{k}{l}$

$$f\left(\frac{k}{l}\right) = f\Bigg(\underbrace{\frac{1}{l} + \cdots + \frac{1}{l}}_{k}\Bigg) = f\left(\frac{1}{l}\right) \tag{3}$$

from which we conclude that $f\left(\frac{m}{l}\right) = f\left(\frac{k}{l}\right)$. Assuming that f is continuous, we have proven that f is constant on all numbers.

This result is baffling. Galileo's law of free fall seems to be an important empirical claim about our world – a claim that could easily turn out to be incorrect. Its importance is highlighted by the well-known fact that the constancy of the rate of fall in a uniform gravitational field is directly connected with the identity between inertial mass and gravitational mass – the surprising and contingent fact that gave rise to Einstein's general theory of relativity. Surely, the fact that gravitational mass and inertial mass are equivalent should not be derivable *a priori* from some elementary mathematical considerations. But how can we explain the existence of a mathematical proof for the fact that the rate of fall for all bodies must be the same?

It is not difficult to observe that the culprit is the premise that the rate of fall is a function of mass *only*. To be accurate, the above argument will go through under the assumption that the function f depends on any additive parameter at all, not necessarily mass, as long as there is no functional dependence on some other parameters. But this assumption may very well be questioned. As an example let us consider a possible scenario in which the function f can be presented as a ratio of two functions g and h, each of which is an additive quantity characterizing a particular physical object a: $f(a) = \frac{g(a)}{h(a)}$.[2] Somewhat surprisingly, it can be shown that given this

1. James R. Brown insists that Galileo's reasoning regarding retarding and hastening is genuinely *a priori* (a thought experiment) and does not require any experimental verification. See J.R. Brown, "Why thought experiments transcend empiricism", in: C. Hitchcock (ed.), *Contemporary Debates in Philosophy of Science*, Blackwell 2004, pp. 23-43.

2. Perhaps I should add here one clarificatory comment: the function f is not defined on the set of real numbers but on the set of physical objects, in contrast to the previous case, in which f could be defined as a function of mass, represented by real numbers. If we wanted to represent f purely mathematically, we would have to treat it as a function of two arguments: $f(g, h) = \frac{g}{h}$.

assumption, a condition which is a generalization of (1) is guaranteed to be satisfied:

$$f\Big(\bigcup_{i=1}^{n} a_i\Big) = \sum_{i=1}^{n} p_i f(a_i), \tag{4}$$

where p_i are defined as in formula (1), and $a_1, \dots a_n$ are some non-overlapping physical objects. Let a symbolize the physical union $\bigcup_{i=1}^{n} a_i$. Given that g and h are additive, we can derive the following:

$$\begin{aligned} f(a) &= \frac{g(a)}{h(a)} = \frac{\sum_{i=1}^{n} g(a_i)}{\sum_{i=1}^{n} h(a_i)} \\ &= \sum_{i=1}^{n} \frac{h(a_i)}{\sum_{j=1}^{n} h(a_j)} \frac{g(a_i)}{h(a_i)} \\ &= \sum_{i=1}^{n} \frac{h(a_i)}{\sum_{j=1}^{n} h(a_j)} f(a_i). \end{aligned} \tag{5}$$

We can clearly see that the numbers $\frac{h(a_i)}{\sum_{j=1}^{n} h(a_j)}$ sum up to 1, and thus (given the assumption of the positivity of the function $h(x)$) we conclude that $f(a)$ is a convex sum of the values $f(a_i)$, as stated in (4). Thus the condition imposed by Galileo on the function representing the rate of fall is automatically satisfied. But this is the case regardless of the fact that the value of the function f may vary from body to body. Hence it is theoretically possible that the rate of fall will be different for different bodies.

As a matter of fact, this is not a mere theoretical possibility. Let us consider an electrically charged particle in a uniform electrostatic field (such as produced by a capacitor consisting of two infinite charged plates) as an example. In that case the value of the acceleration of such a particle is given as $\frac{q}{m}E$, where q is the charge of the particle, m – it's mass and E the (constant) value of the electric field. Since both the charge and mass are additive quantities, the Galilean condition (4) is satisfied, as proven above. Combining two particles whose rates of fall are given respectively by $\frac{q_1}{m_1}E$ and $\frac{q_2}{m_2}E$, where $\frac{q_1}{m_1} < \frac{q_2}{m_2}$, we obtain a particle with a rate of fall equal to $\frac{q_1+q_2}{m_1+m_2}E$, and thus greater than that of particle 1 but smaller than the rate of particle 2. Consequently, the faster particle will hasten the slower one, while the slower will retard the faster, exactly as demanded by Galileo. And all this is perfectly compatible with the fact that different particles can have different rates of descent, depending on their charge-to-mass ratio.

An analogous situation would occur if gravitational mass were different from inertial mass. In that hypothetical scenario the acceleration of a free-falling body would be proportional to the ratio of its gravitational mass m_g to its inertial mass m_i like so: $a = \frac{m_g}{m_i} g$. This by itself would be sufficient to make the condition (4) true, but if this ratio varied among different bodies, the Galilean law of free fall would be violated. Thus it is conceptually possible that Galileo's condition of "hasteners" and "delayers" may be satisfied, and yet objects fall at different rates in the gravitational field of the Earth.

In his generalization of the *a priori* argument against Aristotle's law of free fall, Galileo made a logical error, which nevertheless turned out to be a happy one. He assumed, following Aristotle, that the rate of fall is represented by a function of one additive parameter (i.e. mass), and from this assumption he derived that this function must be constant. However, he overlooked the fact that the crucial premise of his derivation – the assumption that the rate of fall of a composition of two bodies must be somewhere between the rates of fall for these individual bodies taken separately – can be made true by representing the rate of fall of an object as a ratio of two independent additive parameters. But Galileo was proved right – whether by pure luck or thanks to the intuition of a genius. As a matter of fact, the ratio of the gravitational mass to the inertial mass for any object is constant, and therefore the Galilean law of free fall is correct. However, this fact had to be verified by experiment, quite independently from any *a priori* reasoning no matter how compelling.