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On the existence of a common solution to the Lyapunov equations

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Abstract. In this paper, a system of Lyapunov equations

$$A_i^* P + P A_i = -Q_i \quad (i = 1, \ldots, m), \tag{A}$$

is considered in which A_i are given $n \times n$ complex matrices, Q_i are unknown $n \times n$ Hermitian positive definite matrices and P, if any, is a common solution to the Lyapunov equations (A). Both sufficient and necessary and sufficient conditions are derived for the existence of such a matrix P. Examples are presented to illustrate the results.

Key words: common quadratic Lyapunov function, switched system, robust stability.

1. Introduction

The classical theorem of Lyapunov states that the equilibrium state of a linear time-invariant continuous-time system $\dot{x} = Ax$ is asymptotically stable if and only if there exists, for any positive definite Hermitian matrix Q, a positive definite Hermitian matrix P satisfying the following Lyapunov equation:

$$A^* P + PA = -Q.$$

If there exists, for a given set of matrices $\mathcal{A}$, a positive definite Hermitian matrix P such that:

$$\forall A \in \mathcal{A} \qquad A^* P + PA \quad \text{is negative definite,}$$

then the matrix P is said to be *a common solution to the Lyapunov equation* for the set of matrices $\mathcal{A}$; the function $V(x) = x^* P x$ is then *a common quadratic Lyapunov function* for $\mathcal{A}$.

The problem of the existence of a common solution to a given set of Lyapunov equations, which is closely related to the stability of an important and widely studied class of switched systems (for a brief survey of some recent results in this field see Klamka et al. [1]), has been extensively studied in the past three decades. For instance, results are known for pairwise commutating matrices (Narendra & Balakrishnan [2]), for matrices similar to a triangular matrix with a common similarity matrix (Mori et al. [3]), for Hermitian matrices (Cohen & Lewkowicz [4]), for matrices in a companion form (Shorten & Narendra [5]), or for 2×2 matrices (Shorten & Narendra [6], Cohen & Lewkowicz [7], Laffey & Šmigoc [8]). Unfortunately, there is a lack of analytical results applicable to the entire class of matrices.

The outline of the paper is as follows. After preliminary Sec. 2 we study in Sec. 3 the problem of the existence of a common solution to the Lyapunov equations. We provide both sufficient and necessary and sufficient conditions for a finite number of the Lyapunov equations to have such a solution. We also reconsider the problem of the existence of a common solution to the Lyapunov equations for two matrices A and A^*. In Sec. 4, we present illustrative examples and finally, in Sec. 5, we summarize the results.

2. Notations and preliminary results

Throughout this paper we use standard notations: $\mathbb{R}$ and $\mathbb{C}$ shall stand for the field of real and complex numbers, respectively; $\mathbb{R}^{n\times n}$ (resp. $\mathbb{C}^{n\times n}$) denotes the space of square matrices of dimension n with real (resp. complex) entries. For a matrix $A \in \mathbb{C}^{n\times n}$, A^T and A^* stand for its transpose and a conjugate transpose, respectively. $\sigma(\cdot)$ denotes the spectrum of a matrix (the set of all its eigenvalues). Let $P = P^* \in \mathbb{C}^{n\times n}$, we write $P > 0$ (resp. $P < 0$) to express that P is positive (resp. negative) definite. Recall also that a matrix is said to be (Hurwitz) *stable* if its spectrum is contained in the open left half of the complex plane.

For a matrix $A = [a_{\cdot 1}, \ldots, a_{\cdot n}] \in \mathbb{C}^{n\times n}$ where $a_{\cdot i}$ is its i-th column and for a vector $v = (v_1, \ldots, v_{n^2})^T \in \mathbb{C}^{n^2}$ we define

$$\mathrm{vec}(A) := \left(a_{\cdot 1}^T, a_{\cdot 2}^T, \ldots, a_{\cdot n}^T\right)^T \in \mathbb{C}^{n^2}$$

and

$$\mathrm{matr}(v) := \begin{pmatrix} v_1 & v_{n+1} & \cdots & v_{(n-1)n+1} \\ v_2 & v_{n+2} & \cdots & v_{(n-1)n+2} \\ \vdots & \vdots & & \vdots \\ v_n & v_{2n} & \cdots & v_{n^2} \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Obviously,

$$\mathrm{matr}\left(\mathrm{vec}(A)\right) = A, \qquad \mathrm{vec}\left(\mathrm{matr}(v)\right) = v.$$

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Recall that the Kronecker product of two $n \times n$ matrices A, B, denoted as $A \otimes B$, is an $n^2 \times n^2$ matrix of the form

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix},$$

where a_{ij} is the (i,j)-entry of A.

Let $L(A) := \left(A^T \otimes I_n\right) + (I_n \otimes A^*)$, where I_n is the identity matrix of dimension n. It is known that the matrix $L(A)$ is invertible if and only if $\sigma(-A) \cap \sigma(A^*) = \varnothing$ (e.g. Horn & Johnson [10]). In particular, the matrix $L(A)$ is invertible if A is stable.

The following theorem (Horn & Johnson [10]) plays a key role in our further considerations.

Theorem 1. Let A and Q be two $n \times n$ complex matrices and let A be such that the matrix $L(A)$ is invertible. Then there exists an $n \times n$ matrix P being the unique solution to the following Lyapunov equation

$$A^*P + PA = -Q. \tag{1}$$

Moreover, if Q is Hermitian then so is P.

Note that Eq. (1) can be rewritten in the equivalent form (see e.g. Horn & Johnson [10])

$$L(A)\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q),$$

and hence, its only solution can be expressed as

$$P = -\,\mathrm{matr}\left(L^{-1}(A)\,\mathrm{vec}(Q)\right), \tag{2}$$

where $L^{-1}(A)$ denotes the inverse of $L(A)$.

Given matrices A, B, $Q \in \mathbb{C}^{n\times n}$, where $Q = [q_{ij}]$ is a Hermitian matrix and suppose the matrix $L(A)$ is nonsingular. Let $\beta_{ij} \in \mathbb{C}^{n^2}$ (for $i = 1,\ldots,n$, $j = 1,\ldots,n$) be $i + n\cdot(j-1)$-th column of $L(B)L^{-1}(A)$, i.e.

$$L(B)L^{-1}(A)$$
$$= [\beta_{11}, \beta_{21}, \ldots, \beta_{n1}, \beta_{12}, \beta_{22}, \ldots, \beta_{n2}, \ldots, \beta_{1n}, \ldots, \beta_{nn}],$$

and put

$$H_{ij}(A,B) := \mathrm{matr}(\beta_{ij}), \tag{3}$$

for $i = 1,\ldots,n$, $j = 1,\ldots,n$. Also, let

$$\mathrm{vec}(Q) = (q_{11}, q_{21}, \ldots, q_{n1}, q_{12}, \ldots, q_{n2}, \ldots, q_{nn})^T \in \mathbb{C}^{n^2}.$$

Under these notations, we have

$$L(B)L^{-1}(A)\,\mathrm{vec}(Q) = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} q_{ij},$$

and thus

$$\mathrm{matr}\left(L(B)L^{-1}(A)\,\mathrm{vec}(Q)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}(A,B) q_{ij}. \tag{4}$$

The next proposition, following straightforward from the above considerations, is given without a proof.

Proposition 2. Let A and B be two $n \times n$ complex matrices and suppose the matrix $L(A)$ is nonsingular. Moreover, let Q be an $n \times n$ complex matrix and assume that an $n \times n$ complex matrix P is a solution to the following Lyapunov equation

$$A^*P + PA = -Q. \tag{5}$$

Then

$$B^*P + PB = -\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}\left(A,B\right) q_{ij}. \tag{6}$$

Additionally, if Q is Hermitian then so is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}(A,B) q_{ij}.$$

Now we formulate and justify some important properties of the matrices $H_{ij}(A,B)$. These matrices play a key role in our further considerations. Matrices $Q_k = [q_{lm}^{(k)}]$ $(k = 1, ..., n)$ occurring in the theorem are defined as follows:

$$q_{lm}^{(k)} = \begin{cases} 1 & \text{for} \quad l = m = k, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. Let A and B be two $n \times n$ complex matrices and suppose the matrix $L(A)$ is nonsingular. Let $H_{ij}(A,B)$ $(i = 1,\ldots,n;\ j = 1,\ldots,n)$ be matrices given by (3). Then

(i) $H_{ii}(A,B) = -\left(B^*P_i + P_iB\right)$, where P_i is a solution to the Lyapunov equation $A^*P_i + P_iA = -Q_i$;
(ii) $H_{ii}(A,B) = H_{ii}^*(A,B)$;
(iii) $H_{ii}(A,A) = Q_i$;
(iv) $\left(H_{ij}(A,B) + H_{ji}(A,B)\right)^* = H_{ij}(A,B) + H_{ji}(A,B)$;
(v) for $B_1, B_2 \in \mathbb{C}^{n\times n}$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ we have:
   (a) $H_{ij}\left(A, B_1 + B_2\right) = H_{ij}\left(A, B_1\right) + H_{ij}(A, B_2)$;
   (b) $H_{ij}\left(\alpha A, \beta B\right) = \frac{\beta}{\alpha} H_{ij}(A,B)$.

Proof. The nonsingularity of the matrix $L(A)$ ensures that there exists, for each matrix Q_i $(i = 1,\ldots,n)$, the Hermitian matrix P_i satisfying the Lyapunov equation

$$A^*P_i + P_iA = -Q_i.$$

It follows from Proposition 2 that

$$B^*P_i + P_iB = -H_{ii}(A,B), \tag{7}$$

proving (i). Point (ii) follows immediately from (i).

Taking $B = A$ in Proposition 2 Eq. (6) yields the following

$$A^*P_i + P_iA = -H_{ii}(A,A). \tag{8}$$

It proves (iii). To justify point (iv) let introduce, for fixed $i,j \in \{1,\ldots,n\}$, an $n \times n$ matrix $Q^{(i,j)} = [q_{k,l}^{(i,j)}]$ defined as follows

$$q_{k,l}^{(i,j)} = \begin{cases} 1 & \text{for} \quad (k,l) \in \{(i,j),(j,i)\}, \\ 0 & \text{otherwise.} \end{cases}$$

By applying Proposition 2 for $Q = Q^{(i,j)}$ we get from (6)

$$B^*P + PB = -\left(H_{ij}(A,B) + H_{ji}(A,B)\right).$$

Since matrices $Q^{(i,j)}$ $(i,j=1,\ldots,n)$ are Hermitian we get that P is also Hermitian and hence (iv) holds. Finally, one can easily seen that for $\alpha,\beta\in\mathbb{R}$, $\alpha\neq 0$ we have

$$L(\beta A)=\beta L(A)\quad\text{and}\quad L^{-1}(\alpha A)=\alpha^{-1}L^{-1}(A).$$

Combining these observations with the linearity of the operator $B\to L(B)L^{-1}(A)$ we get point (v). It completes the proof.

3. Main results

Let $\mathcal{A}=\{A_1,\ldots,A_m\}$, where $A_i\in\mathbb{C}^{n\times n}$ $(i=1,\ldots,m)$. The main aim of this section is to obtain sufficient and necessary and sufficient conditions for the existence of a common solution P to the Lyapunov equations

$$A_i^*P+PA_i=-Q_i\quad(i=1,\ldots,m),$$

where $Q_1,\ldots,Q_m$ are some unknown positive definite matrices. If such a solution P exists we shall say that the set $\mathcal{A}=\{A_1,\ldots,A_m\}$ has *a common Lyapunov solution*.

3.1. Sufficient conditions for the existence of a common Lyapunov solution. The following theorem gives a sufficient condition for the existence of a common Lyapunov solution for $\mathcal{A}=\{A_1,\ldots,A_m\}$.

Theorem 4. Let $\mathcal{A}=\{A_1,\ldots,A_m\}\subset\mathbb{C}^{n\times n}$, let A_1 be a stable matrix and suppose there exists a positive semidefinite diagonal matrix $D=\operatorname{diag}(d_1,\ldots,d_n)$ such that

$$\sum_{i=1}^{n}H_{ii}(A_1,A_k)d_i>0\quad(k=2,\ldots,m).\tag{9}$$

Then the set $\mathcal{A}$ has a common Lyapunov solution.

Proof. It follows from the stability of A_1 that the matrix $L^{-1}(A_1)$ is invertible and hence matrices $H_{ii}(A_1,A_k)$ $(k=1,\ldots,m;\ \ i=1,\ldots,n)$ exist. Moreover, it follows from (9) that the set $N_+=\{i\in\mathbb{N}:d_i>0\}$ is nonempty. Take any $\varepsilon>0$ and consider the matrix $\widetilde{Q}_\varepsilon=\operatorname{diag}(\widetilde{q}_1(\varepsilon),\ldots,\widetilde{q}_n(\varepsilon))$ defined as follows

$$\widetilde{q}_i(\varepsilon)=\begin{cases} d_i & \text{for}\quad i\in N_+,\\ \varepsilon & \text{for}\quad i\notin N_+.\end{cases}$$

The matrix $\widetilde{Q}_\varepsilon$ is positive definite. Thus, there exists a positive definite Hermitian matrix P_ε satisfying the Lyapunov equation

$$A_1^*P_\varepsilon+P_\varepsilon A_1=-\widetilde{Q}_\varepsilon.$$

We now show that for some ε the matrix P_ε is a common Lyapunov solution for $\mathcal{A}$. From Proposition 2 we get that for $k=2,\ldots,m$:

$$A_k^*P_\varepsilon+P_\varepsilon A_k=-\sum_{i=1}^{n}H_{ii}(A_1,A_k)\,\widetilde{q}_i(\varepsilon)$$
$$=-\sum_{i\in N_+}H_{ii}(A_1,A_k)\,d_i-\varepsilon\sum_{i\notin N_+}H_{ii}(A_1,A_k).$$

Assumption (9) implies that there exists an $\varepsilon_0>0$ for which

$$\sum_{i\in N_+}H_{ii}(A_1,A_k)d_i+\varepsilon_0\sum_{i\notin N_+}H_{ii}(A_1,A_k)>0,$$

for $k=2,\ldots,m$. It follows from Proposition 2 matrix P_{ε_0} is a common Lyapunov solution for $\mathcal{A}$.

From Theorem 4 one can immediately draw two following conclusions.

Conclusion 1. If for some $i\in\{1,\ldots,n\}$ the matrices $H_{ii}(A_1,A_k)$ are all positive definite for $k=2,\ldots,m$ then $\mathcal{A}$ has a common Lyapunov solution.

Conclusion 2. Let A and B be two $n\times n$ complex matrices. Suppose the matrix A is stable. If for some $i\in\{1,\ldots,n\}$ the matrix $H_{ii}(A,B)$ is positive definite then the pair $\{A,B\}$ has a common Lyapunov solution.

3.2. Necessary and sufficient conditions for the existence of a common Lyapunov solution. Our next two theorems give necessary and sufficient conditions for the existence of a common Lyapunov solution for $\mathcal{A}=\{A_1,\ldots,A_m\}$.

Theorem 5. Let $\mathcal{A}=\{A_1,\ldots,A_m\}\subset\mathbb{C}^{n\times n}$ and suppose the matrix A_1 is stable. The set $\mathcal{A}$ has a common Lyapunov solution if and only if there exists an $n\times n$ positive definite matrix $Q=[q_{ij}]$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}H_{ij}(A_1,A_k)\,q_{ij}>0,\tag{10}$$

for $k=2,\ldots,m$.

Proof. It follows from the assumptions that the matrix $L^{-1}(A_1)$ is invertible and, hence, all the matrices $H_{ij}(A_1,A_k)$ exist.

Sufficiency. Suppose there exists a positive definite matrix $Q=[q_{ij}]$ satisfying inequalities (10). Let P be a solution to the Lyapunov equation

$$A_1^*P+PA_1=-Q.\tag{11}$$

It follows from the Lyapunov's theorem that P is positive definite. We now show that P is a common Lyapunov solution for $\mathcal{A}$. From (11) and from assumption (10) by applying Proposition 2 we obtain that, for $k=2,\ldots,m$,

$$A_k^*P+PA_k=-\sum_{i=1}^{n}\sum_{j=1}^{n}H_{ij}(A_1,A_k)\,q_{ij}<0.$$

Necessity. Suppose there exists a positive definite matrix P being a Lyapunov solution for $\mathcal{A}$, i.e.

$$A_k^*P+PA_k<0\quad(k=1,\ldots,m).\tag{12}$$

Letting $Q:=-(A_1^*P+PA_1)$, it follows from (12) that such defined matrix Q is positive definite. Moreover, combining (12) and Proposition 2, we get for $k=2,\ldots,m$ that

$$A_k^*P+PA_k=-\sum_{i=1}^{n}\sum_{j=1}^{n}H_{ij}(A_1,A_k)\,q_{ij}<0.$$

It means that (10) holds completing the proof.

Theorem 6. Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{C}^{n\times n}$ and let the matrix A_1 be stable. Suppose P is an $n \times n$ positive definite matrix such that

$$A_1^* P + P A_1 = -\operatorname{diag}(d_1, \ldots, d_n) < 0. \tag{13}$$

Then P is a common Lyapunov solution for $\mathcal{A}$ if and only if

$$\sum_{i=1}^{n} H_{ii}(A_1, A_k)\, d_i > 0,$$

for $k = 2, \ldots, m$.

Proof. It follows from the stability of the matrix A_1 that there exists a positive definite matrix P satisfying condition (13). Thus, from Proposition 2, we obtain for $k = 2, \ldots, m$ that:

$$A_k^* P + P A_k = -\sum_{i=1}^{n} H_{ii}(A_1, A_k)\, d_i < 0.$$

It ends the proof.

From Theorem 6 we get the following

Conclusion 3. Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{C}^{n\times n}$ and let the matrix A_1 be stable. Suppose P is an $n \times n$ positive definite matrix such that

$$A_1^* P + P A_1 = -I. \tag{14}$$

Then P is a common Lyapunov solution for $\mathcal{A}$ if and only if

$$\sum_{i=1}^{n} H_{ii}(A_1, A_k) > 0,$$

for $k = 2, \ldots, m$.

3.3. A common Lyapunov solution for A and A^*. Given two $n \times n$ matrices A and B. One can easily show that between conditions

(a) A and B have a common Lyapunov solution;
(b) the convex combination of A and B is stable;
(c) $A + B$ is stable;

the following relations hold: (a) implies (b) and (b) implies (c); it can be also easily seen that (c) does not imply neither (b) nor (a). We now show that (b) does not imply (a) in general. To see this, take two matrices in a companion form:

$$A = \begin{pmatrix} 0 & 1 \\ -9 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.$$

One can easily check that the matrix $\alpha A + (1-\alpha) B$ is stable for $\alpha \in [0,1]$. On the other hand, the matrix AB has a negative eigenvalue. It means, according to Theorem 3.1 in [5], that A and B do not have a common Lyapunov solution.

Consider now a pair of matrices $\{A, A^*\}$, where A^* is a transpose conjugate of A. It is well known that matrices A and A^* are always both stable or unstable. In case of stability, however, it does not mean they have a common Lyapunov solution. For example, the matrix

$$A = \begin{pmatrix} -1 & 2i \\ 0 & -1 \end{pmatrix}$$

is stable whereas the matrix $\frac{1}{2}(A + A^*)$ is singular and hence unstable. It follows that, in general, matrices A and A^* do not have to possess a common Lyapunov solution.

Cohen and Lewkowicz considered in [4, 7] convex cones of the form

$$\mathcal{H}_X = \bigcap_{A\in X} \{P > 0 : A^* P + P A < 0\},$$

where $X \subset \mathbb{C}^{n\times n}$ is a matrix family, providing some properties of sets of matrices possessing common Lyapunov solutions (see Section 3 in [4]). It follows from their observation $\mathcal{H}_{X^*} = (\mathcal{H}_X)^{-1}$ (see Eq. (3.6) in [4]) that if some matrix A has a Lyapunov solution being an involution (recall that a matrix is an involution if it is its own inverse), then that involution is also a Lyapunov solution for A^*. The next theorem answers the question when such an involution exists.

Theorem 7. For a stable matrix A, the following conditions are equivalent:

(i) the matrices A and A^* have a common Lyapunov solution;
(ii) the convex combination of A and A^* is stable;
(iii) the matrix $A + A^*$ is negative definite.

Moreover, if any of the above conditions holds, then the identity matrix is an involution being a common Lyapunov solution for A and A^*.

Proof. (i)⇒(ii) This condition is obvious and holds for any pair of matrices.

(ii)⇒(iii) By (ii) the matrix $\dfrac{1}{2}(A + A^*)$ is stable and, as a Hermitian matrix, must be negative definite.

(iii)⇒(i) By the negative definiteness of $A + A^*$ one can write

$$A^* I + I A = A I + I A^* < 0,$$

where I is the identity matrix. It means that $P = I$ is a common Lyapunov solution for A and A^* proving (i). The proof is completed.

4. Examples

We close this paper with three examples illustrating and completing the results.

Example 1. In this example we deal with the problem of the existence of a common quadratic Lyapunov function for a pair of linear time-invariant continuous-time systems:

$$\dot{x}(t) = A x(t), \quad \dot{x}(t) = B x(t), \tag{15}$$

where

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

One can easily check that the matrix A is stable. Thus, according to notations introduced in Sec. 2, we have (matrices

$L^{-1}(A)$ and $L(B)L^{-1}(A)$ are presented in Appendix):

$$H_{11}(A,B) = \frac{1}{4}\begin{pmatrix} 6 & -3 & 3 \\ -3 & 2 & -2 \\ 3 & -2 & 2 \end{pmatrix},$$

$$H_{22}(A,B) = \frac{1}{16}\begin{pmatrix} 14 & 1 & 9 \\ 1 & 14 & -2 \\ 9 & -2 & 6 \end{pmatrix},$$

$$H_{33}(A,B) = \frac{1}{16}\begin{pmatrix} 22 & -11 & 29 \\ -11 & 6 & -10 \\ 29 & -10 & 30 \end{pmatrix}.$$

None of three matrices $H_{11}(A,B)$, $H_{22}(A,B)$, $H_{33}(A,B)$ is positive definite, and hence, we cannot use Conclusion 2. However, since

$$H_{11}(A,B) + H_{22}(A,B) + H_{33}(A,B) = \frac{1}{8}\begin{pmatrix} 30 & -11 & 25 \\ -11 & 14 & -10 \\ 25 & -10 & 22 \end{pmatrix} > 0,$$

letting $Q = I_3$ in Theorem 5 we get that the matrices A and B have a common Lyapunov solution. Moreover, it follows from the proof of Theorem 5 that the matrix P being a solution to the equation $A^T P + PA = -I_3$ is a common Lyapunov solution for A and B. In our case, according to (2), we have

$$P = -\operatorname{matr}\left(L^{-1}(A)\operatorname{vec}(I_3)\right) = \operatorname{matr}\begin{pmatrix} 1 & -\frac{3}{8} & \frac{7}{8} & -\frac{3}{8} & \frac{7}{8} & -\frac{5}{8} & \frac{7}{8} & -\frac{5}{8} & \frac{11}{8} \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} 8 & -3 & 7 \\ -3 & 7 & -5 \\ 7 & -5 & 11 \end{pmatrix}.$$

Indeed, the matrix P is positive definite,

$$A^T P + PA = -I_3 < 0$$

and

$$B^T P + PB = \frac{1}{8}\begin{pmatrix} -30 & 11 & -25 \\ 11 & -14 & 10 \\ -25 & 10 & -22 \end{pmatrix} < 0.$$

It means that P is a common Lyapunov solution for A and B and the function $V(x) = x^T P x$ is a common quadratic Lyapunov function for systems (15).

Example 2. This example illustrates some new method of finding a common Lyapunov solution. It is based on the idea used in the proof of Theorem 4.

Consider now two 2×2 stable matrices of the form

$$A = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}.$$

We have

$$L(A) = \begin{pmatrix} -2 & -2 & -2 & 0 \\ 1 & -1 & 0 & -2 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$L(B) = \begin{pmatrix} -2 & -2 & -2 & 0 \\ 2 & -2 & 0 & -2 \\ 2 & 0 & -2 & -2 \\ 0 & 2 & 2 & -2 \end{pmatrix}$$

and, after necessary calculations, we get

$$L^{-1}(A) = \frac{1}{4}\begin{pmatrix} -2 & 0 & 0 & -4 \\ 0 & -2 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ -1 & -1 & -1 & -3 \end{pmatrix}$$

$$L(B)L^{-1}(A) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & -1 & -3 \\ -1 & -1 & 3 & -3 \\ 1 & 1 & 1 & 7 \end{pmatrix}.$$

Thus, according to (3),

$$H_{11}(A,B) = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

$$H_{22}(A,B) = \frac{1}{2}\begin{pmatrix} 0 & -3 \\ -3 & 7 \end{pmatrix}.$$

Since the matrix $H_{11}(A,B)$ is positive definite it follows from Conclusion 2 that A and B have a common solution. In order to find this solution we will adopt the method used in the proof of Theorem 4. Since the matrix $H_{22}(A,B)$ is not positive definite, it follows from the proof of Theorem 4 that a common Lyapunov solution for A and B can be obtained from the following Lyapunov equation

$$A^T P_\varepsilon + P_\varepsilon A = -Q_\varepsilon, \tag{16}$$

where $Q_\varepsilon = \operatorname{diag}(1,\varepsilon)$ for some $\varepsilon > 0$. The matrix

$$H_{11}(A,B) + \varepsilon H_{22}(A,B) = \frac{1}{2}\begin{pmatrix} 2 & -1-3\varepsilon \\ -1-3\varepsilon & 1+7\varepsilon \end{pmatrix} \tag{17}$$

is positive definite for $\varepsilon \in (-1/9, 1)$. Putting in (16) $\varepsilon = \dfrac{1}{2}$ we get the common Lyapunov solution for A and B:

$$P_{1/2} = -\operatorname{matr}\left(L^{-1}(A)\operatorname{vec}\left(Q_{1/2}\right)\right) = \operatorname{matr}\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}.$$

Indeed,

$$A^T P_{1/2} + P_{1/2} A = \frac{1}{8} \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix}^T \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$$

$$+\frac{1}{8} \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} < 0$$

(according to (16)),

$$B^T P_{1/2} + P_{1/2} B = \frac{1}{8} \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}^T \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$$

$$+\frac{1}{8} \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -4 & 5 \\ 5 & -9 \end{pmatrix} < 0,$$

as expected.

Example 3. Let

$$A = \begin{pmatrix} -3+i & -3i & 1-i \\ i & -4 & -1 \\ -1+i & -3 & -3+i \end{pmatrix}.$$

According to Theorem 7, in order to check if the matrices A and A^* have a common Lyapunov solution it is necessary and sufficient to examine the negative definiteness of the matrix $A + A^*$. In our case,

$$A + A^* = \begin{pmatrix} -6 & -4i & -2i \\ 4i & -8 & -4 \\ 2i & -4 & -6 \end{pmatrix} < 0,$$

and hence, a common Lyapunov solution exists. It follows from the proof of Theorem 7 that one of such solutions is the identity matrix I_3.

5. Concluding remarks

In this paper the existence of a common solution to a finite number of the Lyapunov equations was considered. Both sufficient and necessary and sufficient conditions for the existence of such a solution were derived. Also, it was proved that a necessary and sufficient condition for matrices A and A^* to have a common Lyapunov solution is that the matrix $A + A^*$ is negative definite.

Appendix

For matrices A and B from Example 1 we have:

$$L^{-1}(A) = \frac{1}{16} \times$$

$$\times \begin{pmatrix} -8 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 \\ 4 & -6 & 2 & 4 & -1 & 3 & 4 & -1 & 3 \\ -4 & -2 & -10 & -4 & -3 & -7 & -4 & -3 & -7 \\ 4 & 4 & 4 & -6 & -1 & -1 & 2 & 3 & 3 \\ -4 & 1 & -3 & 1 & -7 & -1 & -3 & -1 & -3 \\ 4 & 3 & 7 & -1 & 1 & -5 & 3 & 3 & 5 \\ -4 & -4 & -4 & -2 & -3 & -3 & -10 & -7 & -7 \\ 4 & -1 & 3 & 3 & 1 & 3 & 7 & -5 & 5 \\ -4 & -3 & -7 & -3 & -3 & -5 & -7 & -5 & -15 \end{pmatrix},$$

$$L(B)L^{-1}(A) = \frac{1}{16} \times$$

$$\times \begin{pmatrix} 24 & 14 & 22 & 14 & 14 & 18 & 22 & 18 & 22 \\ -12 & 13 & -7 & -11 & 1 & -9 & -15 & 7 & -11 \\ 12 & 7 & 27 & 11 & 9 & 19 & 15 & 11 & 29 \\ -12 & -11 & -15 & 13 & 1 & 7 & -7 & -9 & -11 \\ 8 & -2 & 6 & -2 & 14 & 2 & 6 & 2 & 6 \\ -8 & -6 & -14 & 2 & -2 & 10 & -6 & -6 & -10 \\ 12 & 11 & 15 & 7 & 9 & 11 & 27 & 19 & 29 \\ -8 & 2 & -6 & -6 & -2 & -6 & -14 & 10 & -10 \\ 8 & 6 & 14 & 6 & 6 & 10 & 14 & 10 & 30 \end{pmatrix}.$$

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