OPERATIONS RESEARCH AND DECISIONS

No. 2 2018
DOI: 10.5277/ord180206

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OPINION FORMATION IN SOCIAL NETWORKS

A number of selected works on the dynamics of opinions and beliefs in social networks has been discussed. Both Bayesian and non-Bayesian approaches to social learning have been considered, but the analysis has been focused on a simple, tractable and widely used model of updating beliefs – the DeGroot model. The author studied the dynamics of opinions based on the DeGroot model from different points of view. First, its attractive features and shortcomings were discussed and then some of its extensions have been presented. These models are based on the DeGroot updating rule, but additionally incorporate the possibility of improvements and enrichments of the framework.

Keywords: *social network, opinion formation, consensus, DeGroot model*

1. Introduction

Nowadays, almost everything: technology, trends, fashion, prices, the political atmosphere are changing so fast. It is very important to update and have the latest data. We get information from various sources: e.g. the Internet, mass media, friends and peers, colleges. All of this can be composed into a social network, because we are connected and interact day by day. Networking is a significant part in a person's life, because the resulting information flows play a role in a range of phenomena, including job search, financial planning, product choice [26, 16], speaking languages [2], becoming criminals [3] and voting decisions. Every day we face situations where a choice has to be made. People decide based on their own experience, their observations or the beliefs and opinions they have formed on different issues. Information can be obtained through different channels: as a result of the observation of others' actions, through our own experience and communication with others about their beliefs and behaviour, news from

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media sources, propaganda from political leaders, etc. In other words, an individual can adopt the behaviour or beliefs of others that she knows or has observed, or alternatively, she may use a combination of their behaviour or beliefs to fashion her own views.

A key part of the process of opinion formation is how an individual will combine her own opinion and the information she receives. A significant amount of work in the literature on social networks has been assigned to modelling and analysing these processes [1, 6, 9, 29, 34]. There are two prominent approaches of modelling social learning through networks: (i) Bayesian learning (e.g., [4, 22]) and (ii) non-Bayesian models (e.g., [16, 20, 21, 26]). According to Bayesian models, individuals use Bayes' rule to form the best mathematical estimate of the relevant unknowns given their initial beliefs, in other words, to form a reliable "model of the world". Non-Bayesian models are defined as all those that are not Bayesian.

There is a lot of literature considering non-Bayesian approaches. The simplest ones include some type of imitation. This means that an agent adopts the opinions or behaviour of her neighbours' or someone she has observed, by combining her own and other agents' beliefs. A well-known example of this type of learning is the DeGroot model [14]. It is a model based on repeated communication, where people can keep talking to each other and taking weighted averages of information that they get from their friends [15, 17, 18]. The key issue according to this model concerns reaching a consensus in a society, i.e., a situation in which all the members of a society end up with the same opinion.

In parallel, Lehrer and Wagner [31, 41, 42] developed a model of rational consensus formation in a society, which in both its implications and mathematical structure is very similar to the DeGroot model. These authors propose a method of combining group opinion by weighted arithmetic averaging. This model states that when individuals differ in expertise, a weighted arithmetic mean with unequal weights may be the appropriate way to reflect such differences. Arriving at consensual weights is achieved by iterated weighted averaging, i.e., by the repeated multiplication of weight matrices. Some researchers later referred to these models with skepticism. Forrest [19] defines the zero weight problem and shows counter-examples where Lehrer's solution fails. It is claimed that weights should be allowed to have infinitesimal values, distinguishing between very little credit for an agent's opinion and reliance on a randomisation device. Bradley [8] says that these models are unsatisfactory from the point of view of the revision of rational beliefs, and hence cannot be used to support the claim that linear averaging is robust in the face of the revision of individual judgments. There is a vast literature that proposes various modifications and refinements that complement this method [35, 39, 43] and the DeGroot model for which similar criticisms might be applied.

There are also other forms of non-Bayesian approaches. One of them is replicator dynamics. This concept states that actions or beliefs, which have performed better in the past, are more likely to be adopted or receive greater attention (weight). In other words, an individual will imitate actions that were successful [37, 38]). A more general approach (called case-based decision) is presented by Gilboa and Schmeidler [23, 24], where the main idea is that individuals form beliefs about a situation based on their experiences from similar situations in the past. Also several models related to interacting particle systems have been developed [32]. In these frameworks, opinions are represented by either finitely many discrete values or continuous values [33].

In this paper the author discusses research on the dynamics of opinions and beliefs based on a model of network influence and consensus – the DeGroot model of learning, which was introduced by the American statistician, Morris H. DeGroot in 1974. It is a simple, tractable and widely used model of updating beliefs. Several experimental studies show that people tend to behave according to the DeGroot model rather than according to Bayesian updating (see, e.g., [12], also [11]). Agents have initial beliefs about a common question of interest – for instance, the probability of some event and are embedded in a social network. At successive moments of time, they communicate with their neighbours in the network and update their beliefs simply by taking weighted averages of their neighbours' opinions from the previous period, possibly placing some weight on their own previous beliefs. Over time, provided the network satisfies some basic assumptions (more precisely, when the network is strongly connected, i.e., there is a directed path from any agent to any other, and an aperiodicity condition is satisfied), the beliefs converge to a limit belief. The agents in this scenario are boundedly rational. This means that they fail to adjust correctly for repetitions and dependencies in information that they hear multiple times, continuing to average things, even though they might end up over-weighting the things that they hear. This feature may be interpreted as duplication of information and as a shortcoming of the DeGroot model, since agents do not adjust the weights they place on others' opinions over time [15]. Alternatively, the DeGroot framework is a more naïve model, where agents are not fully rational in a Bayesian way.

One attractive feature of this model lies in the fact that the analysis of convergence is quite straightforward. When convergence is achieved, the consensus belief is a weighted average of the agents' initial beliefs and these weights provide a measure of the agents' social influence or importance. These weights are given by the unique left hand side eigenvector of the social network matrix that has eigenvalue 1. This is what makes the DeGroot model so tractable, and we take advantage of this well-known feature to trace how influential different agents are as a function of the structure of the social network.

Society is a complicated mechanism, which has been the subject of extensive research for many years. Developing and studying models of social interactions can bring us crucial and interesting insights that can explain many historic and contemporary collaborations, agreements, partnerships and alliances, as well as polarization and, what is crucial, can give us tools to enhance or prevent them in future. With these goals, we will combine interesting results from the past decades on learning information. By presenting various models in turn, we will show the intuition, the extensions and the scope of their results. We want to highlight their importance and use in today's information age.

Our motive is to spread the ideas resulting from these models beyond the scientific sphere and attract attention to the topic of opinion formation, which is familiar to everyone.

In this paper, we proceed as follows. First, we recall some preliminaries on networks, and then we introduce the initial model, together with specific assumptions and conditions for the convergence of opinions in society. Next, we present several variants of this model with a range of useful improvements. We consider some selected papers presenting various extensions of the DeGroot model. We start with the work of Golub and Jackson [25], who introduce a definition of wisdom, "large" societies and specific properties ensuring efficient learning in such societies. Furthermore, we study a model where a society contains heterogeneous agents, such that there exist conforming, counter conforming or simply honest individuals [10]. Moreover, we consider modelling the spread of misinformation, in other words, the presence of "prominent agents" (community leaders, politicians, mass media sources) [27]. Obviously, there is a long list of papers that consider other aspects of DeGroot learning, e.g., papers by DeMarzo et al. [15], and Friedkin and Johnsen [20, 21], who consider a time-varying weight on one's own beliefs, Krause [30], where agents weigh only those with similar beliefs, Banerjee et al. [5] present a model that can deal with a sparse initial signal, and many other generalized models that have been studied extensively in the literature. At the end, we conclude the paper by presenting a discussion highlighting the main features of the basic model, its advantages and disadvantages, the extensions presented and a comparison of their features.

Preliminaries on networks

Before diving more deeply into the DeGroot model and its extensions, we introduce some intuition of the formalization of social networks, as well as definitions and properties used in the later sections of the paper. First of all, for analytical purposes, a social network is represented as a graph. The two most basic parameters of a social network represented by a graph are the number of vertices and the number of arcs or links, depending on whether we are dealing with directed or undirected graphs. In a graph, we call a unit – whether an individual, a family, a household, or a village – a vertex or node. A direct path between two nodes indicates the presence of a relationship between them. The absence of such a path indicates the absence of a relationship. A direct path with a defined direction in a directed graph is called an arc, and such a path without a given direction in an undirected graph is called a link or an edge.

Figure 1 presents a directed graph with four vertices. It can be interpreted as a society with four agents (individuals, players, actors) $n = 4$, $N = \{A, B, C, D\}$, interacting with each other. A directed arc from x to y may be, e.g., interpreted as x asks y for advice. The arcs show that this social relationship is not symmetric (for example, A asks C for advice, but C never asks A).

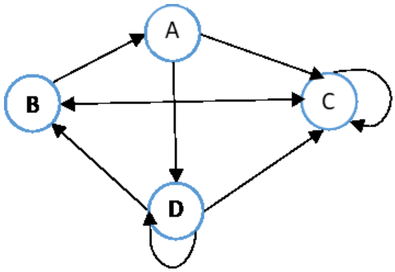

Fig. 1. Social network with 4 agents

The intensity of social interaction is reported in a very simple square matrix with as many rows and columns as there are actors in the society. The elements, or scores, in the cells of the matrix record information about the relationship between each pair of actors.

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & w_{AC} & w_{AD} \\ w_{BA} & 0 & w_{BC} & 0 \\ 0 & w_{CB} & w_{CC} & 0 \\ 0 & w_{DB} & w_{DC} & w_{DD} \end{bmatrix}$$

Here, the matrix **W** is a 4×4 matrix (n = 4) where $w_{ij} > 0$, with $i, j \in N$, means that there is an arc from agent i to agent j, and, consequently, 0 indicates the absence of any interaction. In our example, w_{BA} corresponds to the arc from B to A and $w_{AB} = 0$ indicates that A does not seek advice from B.

Agent i is a neighbour of agent j if $w_{ij} > 0$. This can be interpreted as agent i listens to agent j. In an undirected network, if A is a neighbour of B, then B is a neighbour of A and the number of neighbours of an agent determines its degree. Since we are here considering a directed network, we should distinguish between in-degree and out-degree (the in-degree of node A is the number of agents who listens to A and the out-degree of node A is the number of agents whom agent A listen to). For instance, agent A has out-degree 2, but in-degree 1 (A listens to C and D, while B listens to A).

Assuming that agent B can hold important information and transfer it to his neighbours, and after that his neighbours transfer this information to their neighbours, we can record the passage of information from one agent to another one inside the network. Such a passage is called a path if any node appears at most once.

Summarizing:

- A network is represented by a graph $g = (N, W)$, where

– $N = \{1, 2, ..., n\}$ is a set of nodes (agents).

– $\mathbf{W} = \left[W_{ij}\right]$ is a real-valued $n \times n$ matrix (adjacency matrix, interaction matrix).

- W_{ij} is the strength of the relationship between i and j (possibly weighted and/or directed), represented by a link or arc ij if the value is positive.
- A network is directed if $W_{ij} \neq W_{ji}$ for some $i, j \in N$, and undirected otherwise.
- $N_i(g)$ is the neighbourhood (set of neighbours) of i in g

$$N_i(g) = \left\{ j \in N : W_{ij} > 0 \right\}$$

- $d_i(g)$ is the degree of i in g, i.e., the number of i's neighbours in g

$$d_i(g) = | N_i(g) |$$

- A walk between nodes i and j in g is a sequence $i_1, i_2, ..., i_{K-1}, i_K$ such that $W_{i_k i_{k+1}} > 0$ for each $k \in \{1, ..., K-1\}$ with $i_1 = i$ and $i_k = j$.
- A directed path from node i to node j is a walk between i and j such that each node in the sequence $i_1, i_2, ..., i_{K-1}, i_K$ is distinct.
- A simple cycle is a walk such that each node in the sequence is distinct, except the initial and the end nodes that are the same.
- A strongly connected network g (matrix $\mathbf{W}$) is a network (matrix) in which each node can be reached from any other node via a directed path.
- An adjacency matrix is aperiodic if the greatest common divisor of all of the lengths of the simple cycles in the corresponding network is 1.
- A closed set of agents is a set $C \subset N$ such that there is no arc from any agent in C to any agent outside C, i.e., there is no pair $i \in C$ and $j \notin C$ such that $W_{ij} > 0$.
- A consensus is a vector of opinions $\mathbf{x} = (x_1, ..., x_n) \in [0, 1]^n$ such that for all $i, j \in N$, $x_i = x_j$. We say that a group of agents reaches a consensus under a given interaction matrix for an initial vector of beliefs if, in the long run, the limiting beliefs of all of the agents in the group are the same.

Armed with this knowledge, we begin our acquaintance with the DeGroot model.

2. The DeGroot model

We now place ourselves in the world of social networks with information transmission, opinion formation and consensus reaching. As a starting point, we analyze the DeGroot model [14], since it is a very simple and natural model of social interaction. Its main objective is to investigate how the structure of a network influences the spread of information and the formation of the final opinion among the agents. Particular focus is placed on the issue of reaching a consensus among all of the players, that is, a situation where all of the individuals share the same opinion in the long run.

Consider a group of agents $N=\{1, 2, ..., n\}$ who interact with each other. At time $t = 0$ each agent i has his own initial belief (opinion) $x_i(t) \in [0,1]$ concerning some topic (for instance, his belief that a given candidate is the appropriate one to be selected) and updates it at discrete moments in time $\in \{1, 2, ..., n\}$. This opinion might simply be interpreted as the intensity or inclination of agent i to say "yes" on the topic. Agents exchange information about their beliefs with their neighbours, that is, with people that listen to them. We capture this interaction through an $n \times n$ nonnegative matrix $\mathbf{W} = [W_{ij}]$, where every element $W_{ij} > 0$ means that i pays attention or listens to j. We refer to $\mathbf{W}$ as the interaction matrix. It is row stochastic, so that for each i we have $\sum_j W_{ij} = 1$. This widely used feature can be interpreted as follows: each agent decides how to split his time (attention) among his peers, so that W_{ij} determines the portion of i's total time that he devotes to j. Note that we do not assume that the interaction is symmetric, i.e., that $W_{ij} = W_{ji}$, but we allow agents to rate each other asymmetrically. Also, we cannot have $W_{ij} < 0$, since a negative share of attention does not make sense. It is assumed that the matrix $\mathbf{W}$ is strongly connected, that is, there is a directed path from any agent to any other one. In other words, information can be transmitted between any two individuals. At any time $t \geq 1$ agent i updates her belief according to:

$$x_i(t) = \sum_j W_{ij} x_i(t-1) \tag{1}$$

In other words, agents update their beliefs by repeatedly taking the weighted average of the beliefs of those they listen to with W_{ij} being the weight or trust that agent i places on the current belief of agent j in forming her own belief for the next period. The column vector of beliefs at time t is denoted by $\mathbf{x}(t)$. So, we have:

The initial opinions of individuals:

$$\mathbf{x}(0) = (x_1(0),\, x_2(0),\, ...,\, x_n(0))^T$$

where the upper index T indicates vector transposition.

The weights placed on others' opinions – the interaction matrix:

$$\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} & \cdots & \\ W_{21} & W_{22} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ W_{n1} & W_{n2} & \cdots & \end{bmatrix}$$

Updated opinions[2]:

$$\mathbf{x}(t) = \mathbf{W}\mathbf{x}(t-1) = \mathbf{W}^t\mathbf{x}(0)$$

In other words, we consider a society consisting of individuals that have their own initial opinions on a given issue. At each time period, the individuals talk to each other, update their own beliefs and influence their neighbours' beliefs according to the interaction matrix **W**. Now the following questions arise: When do the beliefs of all of the agents in the network converge to well-defined limits? After a number of periods, is there a consensus in the society since the individuals discuss and know opinions of others?

The definition of convergence means that a limiting belief (i.e., the set of beliefs when time t tends to infinity) exists for all initial beliefs. So, no matter what initial beliefs the individuals start with, they always end up converging to some belief. Due to the assumption of strong connectedness, a well-known convergence result for Markov chains can be used, and one obtains that the crucial condition ensuring convergence in a strongly connected stochastic matrix is its aperiodicity. The full necessary and sufficient condition for convergence is that every set of nodes that is strongly connected and closed is aperiodic.

Regarding the study of consensus reaching, in the literature there exist several characterizations of consensus. First of all, it is worth noting that if beliefs converge, then a strongly connected and closed group of individuals will reach a consensus. Also, a necessary and sufficient condition for reaching a consensus under the DeGroot model is that there exists exactly one strongly connected and closed group of individuals and the interaction matrix is aperiodic on that group. Golub and Jackson [25] show that the limiting beliefs are equal to a combination of the individuals' initial beliefs and the so called influence weights v_i, for each $i \in N$, where $\mathbf{v}$ is the unique left eigenvector of the interaction matrix corresponding to the eigenvalue 1. More precisely, the influence of agent i is a weighted sum of the influences of various agents j who pay attention to i, with the influence v_j weighted by W_{ij}, which represents the trust that j places in i. This

[2]We consider a Markov process, which is a stochastic (random) process with the property that the probability of any future behaviour of the process depends only on the current state, not on its past behaviour.

property gives a measure of influence and indicates that the more influential the people listening to an agent, the more influential that agent is. A number of important results on convergence and consensus reaching under the DeGroot model can be found e.g., in the article of Golub and Jackson [25] mentioned above, and recalled in Jackson [29]. Also, Berger [7] is interested in characterizations of consensus. Let us consider some illustrative examples presented by Jackson [29].

Example 2.1. Suppose that there are three individuals embedded in a network with the following interaction matrix:

$$\mathbf{W} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

The interaction matrix shows that individual 1 listens to everybody equally (including himself), agent 2 trusts agent 1 and himself equally, but does not take into account agent 3's opinion when forming his own belief, while agent 3 does not listen to agent 1, and trusts himself much more than agent 2. We have a strongly connected network, and the interaction matrix is stochastic and aperiodic. Suppose that the three individuals have the following initial vector of beliefs on a certain issue:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This means that agent 1 is 100% in favour of saying yes, while agents 2 and 3 have the opposite view, that is, definitely say no. According to the DeGroot framework, the agents' beliefs are updated as follows:

$$\mathbf{x}(1) = \mathbf{W}\mathbf{x}(0) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\mathbf{x}(2)=\mathbf{W}\mathbf{x}(1)=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\ \frac{1}{2}&\frac{1}{2}&0\\ 0&\frac{1}{4}&\frac{3}{4}\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{2}\\ 0\end{bmatrix}=\begin{bmatrix}\frac{5}{18}\\ \frac{5}{12}\\ \frac{1}{8}\end{bmatrix}$$

Iteratively:

$$\mathbf{x}(t)=\mathbf{W}\mathbf{x}(t-1)=\mathbf{W}^t\mathbf{x}(0)\to\begin{bmatrix}\frac{3}{11}\\ \frac{3}{11}\\ \frac{3}{11}\end{bmatrix}$$

which means that the individuals reach consensus and their limiting opinion is equal to 3/11. We are considering a strongly connected network, the interaction matrix is stochastic and we have aperiodicity. As we stated before, given all these conditions beliefs converge over time.

Example 2.2. Suppose now that the interaction matrix is the following:

$$\mathbf{W}=\begin{bmatrix}0&\frac{1}{2}&\frac{1}{2}\\ 1&0&0\\ 0&1&0\end{bmatrix}$$

This means that no agent trusts himself, as all the entries on the leading diagonal of the matrix are equal to 0. Moreover, agent 1 listens equally to the remaining two agents, agent 2 only trusts agent 1, and agent 3 only listens to agent 2. Updated beliefs are based on the matrices:

$$\mathbf{W}=\begin{bmatrix}0&\frac{1}{2}&\frac{1}{2}\\ 1&0&0\\ 0&1&0\end{bmatrix},\qquad \mathbf{W}^2=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\ 0&\frac{1}{2}&\frac{1}{2}\\ 1&0&0\end{bmatrix}$$

$$\mathbf{W}^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{W}^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Iteratively,

$$\mathbf{W}^t \to \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

If we denote the limit belief (consensus) by $x(\infty)$ then we obtain:

$$x(\infty) = \frac{2}{5} x_1(0) + \frac{2}{5} x_2(0) + \frac{1}{5} x_3(0)$$

We have a strongly connected network and the interaction matrix is stochastic. Again, all of the conditions for convergence are satisfied. We obtain the limit belief, $x(\infty)$, as a weighted average of agents' initial beliefs and the influence weights given by the eigenvector of the matrix **W** corresponding to the eigenvalue 1. This indicates the influence of each agent in the society. In our example, agents 1 and 2 have twice as much influence over the limiting belief as agent 3 does. Hence, the influence weights are $(2/5, 2/5, 1/5)$.

Example 2.3. Suppose that there is a social network with 3 individuals, and we make a small change to Example 2.2:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Updating beliefs are based on the matrices:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{W}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{W}^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{W}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

This society will not converge to limit beliefs, since there are always periodically changing opinions.

3. Obtaining wisdom in a society

Naïve learning in social networks and the wisdom of crowds

There exist numerous variations and extensions of this model. We present some examples that focus on different aspects of the DeGroot model. We start with the paper of Golub and Jackson [25], whose main objective is to examine whether large societies whose agents update their beliefs in a naïve way can aggregate dispersed information efficiently. The authors extend the framework of the DeGroot model by considering the possibility of convergence to the correct limit in large societies. Groups of agents that receive a substantial amount of attention can undermine efficient learning. In addition, Golub and Jackson [25] provide sufficient structural conditions for the emergence of wisdom. In what follows, we give more details about the framework and the main messages of the paper.

When individuals in a society reach a consensus, a related question is whether the consensus beliefs are correct. Regarding this issue of the correct aggregation of beliefs, Golub and Jackson [25] provide a definition of wise societies. They also expand the notion of a society and consider large societies. This can find a wider range of application in real life and, moreover, a large number of agents can provide enough diversity of opinions for a society to be able to eliminate errors and discover the truth.

A large society is presented by a sequence of networks $(W(n))_{n=1}^{\infty}$, where n is the number of agents in each network. We focus on situations, where there is some true state $\mu \in [0, 1]$ that agents are trying to learn. The process goes like this:

At $t = 0$, agent i in network n has belief $x_i^{(0)}(n) \in [0, 1]$ (initial belief). This belief comes from some distribution and $\left\{x_1^{(0)}(n), x_2^{(0)}(n), ..., x_n^{(0)}(n)\right\}$ are independent for each n.

Let $\mathbf{v}(n)$ be the influence vector corresponding to the interaction matrix $\mathbf{W}(n)$. For any given n and realization of $\mathrm{x}^{(0)}(n)^2$, the belief of each agent i in network n approaches a limit which is denoted by $x_i^{(\infty)}(n)$. We say that the sequence of networks is wise when the limiting beliefs converge jointly in probability to the true state μ as n tends to infinity.

Golub and Jackson [25] focus on characterizing wise societies. Their first result concerns a characterization of wisdom in terms of the influence weights, $\mathrm{v}(n)$. It shows that a society can be misled if there is an agent with too much influence. In other words, all of the opinions in a large society converge to the truth if and only if the influence of the most influential agent vanishes as the society grows. Formally, if the agents are arranged by influence in decreasing order such that $v_i(n) \geq v_{i+1}(n) \geq 0$ for each i, so that $v_1(n)$ is the agent with the highest influence, then a sequence of convergent stochastic matrices $(\mathbf{W}(n))_{n=1}^{\infty}$ is wise if and only if the associated influence vectors satisfy $v_1(n) \to 0$.

This result gives us a characterization of wisdom in terms of influence that is easily tractable, but for analysis, it is better to have a characterization in terms of the social structure, as it could be more useful to know the consequences of the dynamics of opinion by looking at the interaction matrix. Golub and Jackson [25] discuss the existence of prominent groups that receive a disproportionate share of attention, which might be an obstacle to wisdom. To study this concept, the authors introduce the concept of prominent families (groups). Let B and C be two groups where

$$W_{B,\,C} = \sum_{i \in B,\, j \in C} W_{ij} \tag{2}$$

is the weight that group B places on group C. We say that a group is prominent within t steps if every agent outside the group is influenced by at least someone in that group within t steps of updating. For example, if $t = 1$, then there is a direct link between agents in the prominent group and the agents outside the prominent group.

Adapting this definition to the notion of a family, i.e., a sequence of groups (B_n) such that $B_n \subset \{1, ..., n\}$ for each n, we introduce the notion of uniform prominence.

The family (B_n) is uniformly prominent if for each n the group B_n is prominent with respect to $\mathbf{W}(n)$ in some number of steps, without the prominence growing too small. It is clear that at least one uniformly prominent family always exists, namely $\{1, ..., n\}$. Furthermore, we say that a family is finite if it stops growing.

Golub and Jackson [25] state a necessary condition for wisdom in terms of prominence. If there is a finite, uniformly prominent family with respect to $(\mathrm{W}(n))$, then the sequence is not wise. In other words, wisdom rules out finite, uniformly prominent families and disproportionate popularity is an obstacle to wisdom. The crucial part of their paper concerns two sufficient structural conditions for a society to be wise.

The first one is the balance condition, which states that the received influence weights of a family of size less than or equal to $j(n)$ cannot be higher than it gives to outsiders. The size $j(n)$ converges to infinity as n tends to infinity, which means that the balance property is required for families of any size. This result is consistent with the one saying that the presence of prominent families and wisdom are incompatible, since wisdom cannot be guaranteed when we have groups of any size with imbalanced flows of information.

The second condition is the minimal out-dispersion condition which rules out situations where agents can ignore the majority of society. This condition says that any sufficiently large finite family has to pay at least minimal attention to any other family. This rules out situations where agents ignore the vast majority of society.

Golub and Jackson [25] provide the conditions for a wise society. They show that if $(\mathbf{W}(n))_{n=1}^{\infty}$ is a sequence of convergent stochastic matrices satisfying the balance and minimal out-dispersion conditions, then it is wise. This result suggests that avoiding the emergence of extreme imbalances in the interaction matrix and of small families that interact with a very narrow part of the outside world can lead to a wise society.

The dynamics of opinion and wisdom under conformity

So far, we have considered the formation of opinions through communication in a given social network such that individuals are influenced by the opinions stated by others. However, in real life there are situations where people want to hide their true motives and personal opinions, so they do not straightforwardly state what they truly think. Buechel et al. [10] consider a situation where agents can misrepresent their beliefs. Honest people behave in a way we already know: they update their opinion in a naive way by taking a weighted average of others' stated opinions, while dishonest people conform or counter conform to their neighbours. Now the question is as follows: what is the optimal level of conformity in order to increase wisdom in society?

In this framework, agent $i \in N$ expresses some opinion $s_i(t) \in \mathbb{R}$ which need not coincide with her true opinion $x_i(t)$. A central assumption of this approach is that an agent cannot observe the true opinions of the others, but only their stated opinions. Since each agent knows her own true opinion $x_i(t)$, agent *i*'s opinion in the next period is given by

$$x_i(t+1) = W_{ii}x_i(t) + \sum_{j \neq i} W_{ij}s_j(t) \tag{3}$$

where the weights W_{ij} are the individual learning weights as in the classical model by DeGroot [14]. This holds for all agents $i \in N$ and, thus, the updating process becomes:

$$\mathbf{x}(t+1) = \mathbf{D}\mathbf{x}(t) + (\mathbf{W} - \mathbf{D})\mathbf{s}(t) \tag{4}$$

where **D** is the $n \times n$ diagonal matrix containing the leading diagonal of **W**.

When expressing her own opinion on some topic, an agent faces the stated opinion of the group, q_i. Agent *i* has the utility function:

$$u_i(s_i \mid x_i) := -(1-\delta_i)(s_i - x_i)^2 - \delta_i(s_i - q_i)^2 \tag{5}$$

This expression is divided into two parts: the first part represents the incentive to behave truly, while the second part corresponds to conforming or counter conforming. Thus, the utility of the agent depends on the distance between her own opinion and the stated one, and between her stated opinion and the stated opinion of the group. The parameter $\delta_i \in (-1, 1)$ represents the relative importance of the agent's inclination for (counter) conformity compared to the importance placed on honesty.

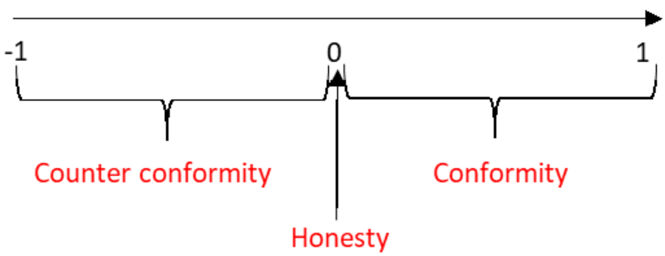

Fig. 2. Level of conformity

Figure 2 shows that a negative value of δ_i represents a preference for counter-conformity. Hence, positive values indicate conforming agents and in the case of $\delta_i = 0$, we recover the standard DeGroot model.

Using this model, we introduce a new time scale $\tau \in \mathbb{N}$ such that within each period $t \in N$ at each time step τ one or more agents speak. The set of agents who are selected to state their opinions at time τ of period t is denoted by $A^{\tau}(t)$. Let $s^{\tau}(t)$ be the vector of stated opinions at step τ of period t. Hence, there are two ways to update opinions. First, agents who do not revise their opinion, keep their stated opinion from the previous time step, i.e.,

$$s_i^{\tau}(t) = s_i^{\tau-1}(t) \quad \text{if } i \in N \backslash A^{\tau}(t) \tag{6}$$

Second, agents who are selected to speak and thereby revise their stated opinions, observe the stated opinions of those they listen to from the previous time step. These are considered as a reference opinion $q_i^{\tau-1}(t)$, which is the weighted average of the stated opinions with weights according to the interaction matrix $\mathbf{W}$, i.e.,

$$q_i^{\tau}(t) = \sum_{j \neq i} \frac{W_{ii}}{1 - W_{ii}} s_j^{\tau}(t) \tag{7}$$

This gives us the reference opinion for agent i. If agent $i \in A^{\tau}(t)$, she chooses the stated opinion which maximizes her current utility.

$$s_i^{\tau}(t) = (1 - \delta_i) x_i(t) + \delta_i q_i^{\tau-1}(t) \tag{8}$$

Hence, the stated opinion given by the best response differs from the true opinion in proportion to the difference between the reference opinion and the true opinion. The level of conformity δ_i plays a crucial role, because it determines this proportion.

Buechel et al. [10] show that the vector $\mathbf{s}^{\tau}(t)$ of stated opinions at step τ of period t converges as $\tau \to \infty$ to the vector

$$\mathbf{s}(t) := [\mathbf{I} - \mathbf{\Delta}(\mathbf{I} - \mathbf{D})^{-1}(\mathbf{G} - \mathbf{D})]^{-1}(\mathbf{I} - \mathbf{\Delta})\mathbf{x}(t) \tag{9}$$

where $\mathbf{\Delta}$ is the $n \times n$ diagonal matrix where the entries $\delta_i \in (-1, 1)$ on the leading diagonal represent the levels of conformity in society. This result, expressing the final opinions $\mathbf{s}(t)$ stated by the agents as a function of their true opinions $\mathbf{x}(t)$, is obtained when the agents revise their own opinions and conform or counter-conform to what those they listen to said last.

According to this model, the interaction of agents can be interpreted as discussion sessions. In each period $t \in N$, an agent discusses with other agents, expresses her opinions, (counter) conforms to the opinions she hears and learns from one discussion to the next one. Since we know that the stated opinions, $\mathbf{s}(t)$, determine the reference opinions, the opinion profile in period $t+1$ can be written in the following way:

$$\mathbf{x}(t+1) = \mathbf{M}\mathbf{x}(t) \tag{10}$$

where

$$\mathbf{M} := [\mathbf{D} + (\mathbf{W} - \mathbf{D})[\mathbf{I} - \mathbf{\Delta}(\mathbf{I} - \mathbf{D})^{-1}(\mathbf{W} - \mathbf{D})]^{-1}(\mathbf{I} - \mathbf{\Delta})] \tag{11}$$

The matrix **M** is independent of and plays the role of an interaction matrix. It is an extended version of **W** where **M** = **W** if $\delta_i = 0.$ In other words, if every agent is honest, then we are back to the standard case of the DeGroot model.

Buechel et al. [10] examine how the dynamics of opinion evolve in their framework. First, they consider conditions for the convergence of opinions and then study convergence for strongly connected societies and unconnected ones. They show that if we exclude counter-conformity and every agent has some self-confidence, i.e., if $W_{ii} > 0$ and $\delta_i \geq 0$ for all $i \in N$, then the dynamics of opinion converge, i.e., true opinions $\mathbf{x}(t)$, stated opinions $\mathbf{s}(t)$, and perceived opinions $\mathbf{q}(t)$ converge as $t \to \infty$.

Buechel et al. [10] are also interested in the influence of each agent's initial opinion on the long-run opinion given her position in the network **W**, and they study the impact of an agent's degree of conformity δ_i. More precisely, given convergence and the initial opinions of the agents, $\mathrm{x}(0)$, the question appears as to where opinions converge to. The authors prove that for a strongly connected society, if opinions converge, then true, stated and perceived opinions always converge to the same limit, and for every $i \in N$,

$$x_i(\infty) = s_i(\infty) = q_i(\infty) = \sum_{j \in N} l_j x_j(0) \tag{12}$$

where **l** is the influence vector corresponding to **M**, and l_j denotes the weight of agent *j*'s initial opinion on the limit opinion. Moreover, if opinions converge in a strongly connected society, then for every $i \in N$

$$l_i = \frac{(1-\delta_i)v_i}{\sum_{j \in N}(1-\delta_j)v_j} \tag{13}$$

where the vector $\mathbf{v}$ denotes the left unit eigenvector of the matrix **W**. In other words, Eq. (13) gives the connection between the power to form opinions in a heterogeneous society and the power to form opinions in an honest society. Hence, the influence l_i of agent *i* is determined by a combination of v_i and the individual's degree of conformity δ_i divided by the sum of these combinations over all of the agents.

Furthermore, Buechel et al. [10] investigate wisdom in the presence of (counter) conforming agents. They show that the level of wisdom decreases when importance is given to only one agent whose signal is the most imprecise. This would be the case if all of the other agents were close to full conformity, i.e., δ_i close to 1. Moreover, the authors determine a necessary and sufficient condition under which the level of wisdom of a strongly connected society is increasing in the conformity level of an agent. If agents with a high ratio of influence to signal precision, as compared to the group's average, are more conformist, then this will reduce their influence within the group and thereby increase wisdom. Vice versa, agents who are not sufficiently influential enough, but possess a precise signal will increase the level of wisdom if they are less conformist, because this will increase their influence and foster wisdom.

4. Other extensions of the DeGroot model

As already mentioned, in the literature on opinion formation, there is a variety of modifications and extensions to the DeGroot model. Although we are not able to discuss them in great detail in the current paper, we will try to give the core intuition and ideas of several selected extensions.

We have seen the notion of the duplication of information in the DeGroot model. It is complicated to revise weights optimally, so agents use constant weights. However, this leads to a situation in which people tend to be unduly swayed by things they hear repeatedly. Time varying weights have been studied extensively in the literature on social models of belief formation. They capture the natural setting in which an individual changes his own or the weights of those she listens to as she gets more information, or as the underlying network of interactions changes over time.

Time-varying weight on one's own beliefs

DeMarzo et al. [15] present a model with time varying weights. The updating rule is:

$$\mathbf{x}(t) = [(1-\lambda_t)\mathbf{I} + \lambda_t \mathbf{W}]\, \mathbf{x}(t-1) \tag{14}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\lambda_t \in (0, 1]$ is an adjustment factor that enables the updating process to vary over time. In contrast to the classical DeGroot model, where we refer to the same $\mathbf{W}$ all the time, this case allows us to recalculate the trust we give to other agents, hence, to modify the interaction matrix at any time. We obtain the DeGroot model when λ_t is constant.

Also, Friedkin and Johnsen [20] and [21] study a model with a time-varying weight on one's own belief. This modified rule of opinion formation is:

$$\mathbf{x}(t) = \mathbf{D}\mathbf{W}\mathbf{x}(t-1) + (\mathbf{I} - \mathbf{D})\mathbf{x}(0) \tag{15}$$

where $\mathbf{D}$ is an $n \times n$ matrix with positive entries only along the leading diagonal, $D_{ii} \in (0, 1)$ indicates the extent to which i pays attention to others' attitudes. This model describes the process of opinion formation in a group where each agent weighs his own and other members' opinions on an issue and repeatedly modifies his opinion until a settled opinion on the issue is formed. An agent always averages his original belief and the latest belief of another agent:

$$x_i(t) = \frac{x_j(t-1)}{2} + \frac{x_i(0)}{2} \tag{16}$$

Only weighting those with similar beliefs

The literature has also considered versions of the DeGroot model with belief dependent weights that represent situations in which the underlying communication patterns are affected by the current beliefs of agents. Krause [30] and Hegselmann and Krause [28] investigate such a belief-dependent model of weights, which assumes that an agent pays attention to beliefs that do not differ too much from his own. In one version, the agent places equal weights on the opinions within some distance from her own opinion and zero otherwise. The convergence of beliefs according to such models is studied in Krause [30], who, under mild conditions on the weight matrix, shows that the set of agents can be partitioned into groups, such that each group reaches a consensus. This model is also related to Deffuant et al. [13], where at each moment two agents are randomly matched and then update their beliefs only if they are sufficiently close.

Naïve learning with an uninformed agent

The models we have considered require that everyone in the population starts with a signal (initial opinion) and the entire learning process evolves as an exchange of opinions among an already informed population. Banerjee, Breza, Chandrasekhar and Mobius [5] relax this assumption and allow signals to be sparse. In other words, they allow the possibility that many or even most members of a network may start by having absolutely no views on a particular issue, and only start having an opinion after someone else shares their opinion with them. Learning uses the following rule:

$$x_i(t+1) = \begin{cases} \varnothing & \text{if} \quad J_i(t) = \varnothing \\ \dfrac{\sum_{j \in J_i(t)} x_j(t)}{|J_i(t)|} & \text{if} \quad J_i(t) \neq \varnothing \end{cases} \tag{17}$$

where $J_i(t)$ denotes the set of individuals that agent i listens to who are informed at time t. Following this rule, an uninformed agent i remains so as long as those he listens to are also uninformed. When a set of individuals that agent i listens to receives information, agent i will adopt the mean opinion of that set. When everyone in the population is informed, the model reduces to the classical DeGroot model. The authors show that an agent's social influence is essentially proportional to the number of uninformed nodes who will hear about an event for the first time via this agent. This result then allows us to relate the structure of the network to information aggregation.

The models presented above are examples of generalized DeGroot models. They have various objectives and study different aspects of social interaction. Consequently, when choosing a model we should have a clear picture of the society that we are modelling. Hence, it depends on who listens to whom and what the network structure looks like. There are also hybrid models in the literature. Another research direction based on the DeGroot model is the presence of strategic players in a society. Several important contributions consider extensions where a subset of agents are "stubborn", i.e., their opinion is fixed to one of two values (see, e.g., [1, 40]).

5. Presence of strategic players in society

So far, we have considered a society with non-strategic agents. We assume now that there exist individuals who try to impose their ideas on a society. We recall a variation of the DeGroot model, developed in Grabisch et al. [27], who investigate a model of strategic targeting in a social network and study the effect of prominent agents on the beliefs of the society. The authors consider a similar setup as in the previous section with a set $N = \{1, ..., n\}$ of agents, more precisely, non-strategic agents (players, individuals), who have initial beliefs $x_i(0) \in [0, 1], i \in N,$ about some underlying state. They interact and update their opinions as in the DeGroot model. Additionally, it is assumed that there are two stubborn external (strategic) players, denoted by a_1 and a_2 with "fixed" and opposite opinions, 1 and 0, respectively. Both of the strategic players form exactly one link with a non-strategic agent in order to influence the formation of opinions in the network. This framework is suitable for modelling many social, economic, and political situations. For example, the non-strategic agents might represent decision-makers or experts who have to make a certain decision by repeated discussions, while the strategic agents try to influence the final decision by "targeting" the opinions of these experts. One can think of lobbying and targeting voters in political campaigns, influence by media sources, targeting consumers in processes of product promotion, among many others.

Grabisch et al. [27] introduce a non-cooperative (constant-sum) game played by two external agents. Strategic agent a_i uses a strategy s_i that chooses one non-strategic agent among *N*. There are also two additional parameters, denoted by λ and μ, which can be interpreted as the impact of the external players. They can have different magnitudes. If $i \in N$ is targeted by one or two external player(s), then, when updating his new opinion on the given issue, besides the other non-strategic agents that *i* was listening to before, he now also takes into account the strategic agent(s). Without going into detail, the $n \times n$ matrix $\mathbf{B}$ representing the interaction among the non-strategic agents is extended to a $(n+2)\times(n+2)$ matrix $\mathbf{P}_{\lambda,\mu}(s)$ with $\mathrm{s} = (s_1, s_2)$, that takes into account the presence of the strategic players and their impact on the non-strategic agents. The opinion updating rule now becomes:

$$\mathbf{x}(t+1) = \mathbf{P}_{\lambda,\mu}(s)\mathbf{x}(t) \tag{18}$$

where, as before, $\mathbf{x}(t)$ denotes the vector of opinions in time period *t*. Grabisch et al. [27] prove that in this extended framework the convergence of opinions is preserved. They characterize the asymptotic opinions of the non-strategic agents, which are independent of their initial opinions. Moreover, their limit belief depends on **s**, that is, it is determined by the positions of the strategic agents. This is a consequence of the presence of the external players in the model.

Regarding consensus, such a society does not necessarily reach a consensus among the non-strategic agents when the strategic external agents enter into the interaction. However, if a_1 and a_2 target the same individual, i.e., if $s_1 = s_2$, then the agents in *N* reach a consensus, which is determined by the relative impacts of the external agents, and is equal to $\alpha = \lambda/(\lambda + \mu)$. In particular, if $\lambda = \mu$, i.e., if the external players are equally influential, then the consensus is $\alpha = 1/2$. This is highly intuitive: since the strategic agents are equally strong and have the extreme opinions 1 and 0. In the long run, the society ends up with an opinion midway between the two strategic ones.

Besides an analysis of convergence and consensus reaching, Grabisch et al. [27] study the existence and characterization of a pure strategy Nash equilibrium in the extended framework with strategic agents. The existence of a pure Nash equilibrium depends on the network structure. The equilibrium is characterized by the so called influenceability and centrality of the targeted non-strategic players.

Rusinowska and Taalaibekova [36] extend the model of Grabisch et al. [27] by introducing one more strategic player, a central persuader with an opinion equal to 1/2.

Their model can often provide a more realistic explanation of the political and socioeconomic spectrum.

6. Conclusion

In this paper, we have provided a short discussion of several papers examining different frameworks of opinion formation and learning in social networks. We have mentioned both Bayesian and non-Bayesian approaches to social learning, but our focus was on non-Bayesian (imitation) models – the DeGroot model and its variations. We have shown some shortcomings, as well as advantages, of that model. We have presented in more detail several generalizations of the DeGroot model and emphasized the main insights resulting from some other extensions. The idea was to present different models based on the DeGroot model of learning that have various research applications and ask different questions. For example, Golub and Jackson [25] show that the existence of prominent agents or groups of opinion leaders can prevent efficient learning. On the other hand, Buechel et al. [10] find that even if wisdom cannot be ensured due to prominent agents, it is still possible to achieve wisdom if these agents are sufficiently conformist. Moreover, Golub and Jackson [25] provide sufficient structural conditions for attaining wisdom. The results obtained show that the efficiency of learning can depend on the way in which the social network is organized. Hence, the main question of Golub and Jackson [25] is whether large societies with naïve agents can come to the correct aggregated information. The authors give a positive answer, but only if the society spreads attention uniformly enough and does not pay excessive attention to some particular groups. In such a situation, society can learn efficiently.

As we have already mentioned, Buechel et al. [10] introduce heterogeneous agents with different levels of conformity into the model of DeGroot [14]. We have learned from their work that convergence in such societies can be achieved when all of the agents are conformist or honest.

Furthermore, we have recalled a model with external (strategic) players that have the goal of imposing their ideas on a society [10]. The external players possess opinions that do not change over time, while the non-strategic players update their beliefs as in the DeGroot model. The structure of the interaction matrix differs from the classical one, and consequently, the asymptotic opinions do not depend on the vectors of initial opinions. The answer to the question regarding whom should we target, usually depends on the relative strengths of the strategic players. If the external agents are equally powerful, the relative importance of centrality is increasing in the impact of the strategic agents and decreasing in the size of the network. When there is a significant inequality between the powers of the external players, then there is no pure strategy Nash equilibrium, and the low impact player uses a mixed strategy to "hide" his targeted non-strategic agent from the influence of his opponent.

Obviously, our short discussion of various models of imitation did not include all of the related articles. A large amount of work on learning and opinion formation is available in the literature, since social networks are a powerful tool for modelling and predicting humans' opinions and behaviour, and their various results show the intuitive consequences of interactions in real life.

Acknowledgements

The author thanks Agnieszka Rusinowska for excellent assistance in this research and the Innovative Training Network *Expectations and Social Influence Dynamics in Economics* (ExSIDE) for financial support.

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Received 9 January 2018
Accepted 26 April 2018