Free Vibrations of Medium Thickness Microstructured Plates

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Abstract

A problem of free vibrations of medium thickness microstructured plates, which can be treated as made of functionally graded material on the macrolevel is presented. The size of the microstructure of these plates is of an order of the plate thickness. Averaged governing equations of these plates can be obtained using the tolerance modelling technique, cf. [18, 19, 9]. Because, the derived tolerance model equations have the terms dependent of the microstructure size, this model describes the effect of the microstructure size. Results can be evaluated introducing the asymptotic model. Calculated results can be compared to those from the finite element method or a similar tolerance model of thin plates, cf. [9].

Keywords: medium thickness functionally graded plates, microstructure, tolerance modelling

1. Introduction

In this paper, medium thickness functionally graded plates with microstructure are investigated. Their microstructure is in planes parallel to the plate midplane along one, i.e. the x_1-axis direction. It is assumed that plate properties along the perpendicular direction are constant. Moreover, the size of the microstructure is assumed to be of an order of the plate thickness. An example of these plates is shown in Figure 1, cf. [12].

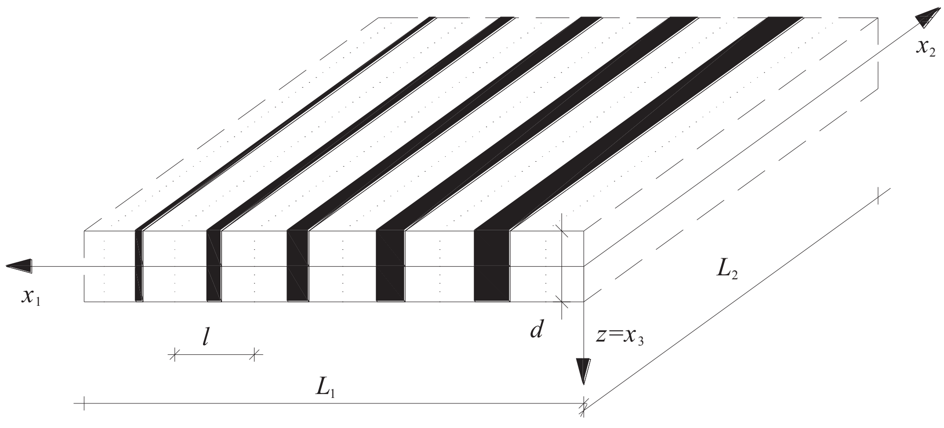

Figure 1. Fragment of a medium thickness functionally graded plate with the microstructure, cf. [12]

These plates consist of many small elements along the x_1-axis with a span equal l, cf. Figure 2, ($x \equiv x_1$). Such elements are called the cells. Their length l describes the size of the microstructure and is called the microstructure parameter.

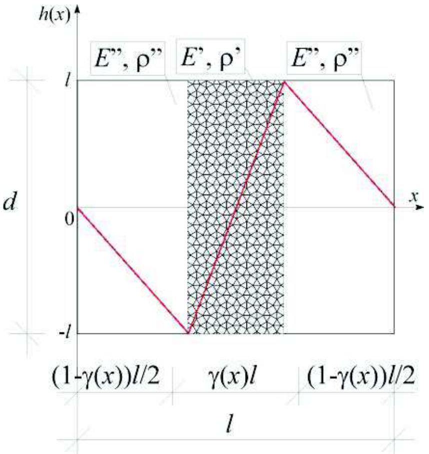

Figure 2. Element of the plate with a fluctuation shape function, cf. [12]

Thermomechanical problems of functionally graded media are often described applying various averaging approaches, which are used for macroscopically homogeneous structures, cf. Jędrysiak [7, 8]. Models of periodic plates based on the asymptotic homogenization method play a role between them, cf. Kohn and Vogelius [14]. In a series of papers there are shown applications of other methods, which describe various problems of thermoelasticity of beams, plates and shells, e.g. frequencies of functionally graded plates using a meshless method by Ferreira et al. [5], vibrations of functionally graded shells by Tornabene and Viola [17], thermoelasticity of functionally graded plates by Akbarzadeha [1], dynamics of beams with functionally graded core by Bui et al. [3], buckling of sandwich beams with variable properties of a core by Grygorowicz et al. [6]. However, the effect of the microstructure size is neglected in governing equations of these models.

In order to take into account this effect the tolerance averaging technique can be applied, cf. [18, 19, 7]. Various periodic structures are modelled by using this method in many papers, e.g. medium thickness periodic plates by Baron [2], higher order vibrations of thin periodic plates by Jędrysiak [7], nonlinear thin periodic plates by Domagalski and Jędrysiak [4], vibrations of periodic three-layered plates by Marczak and Jędrysiak [15]. The tolerance method is also adopted and successfully used to analyse different functionally graded structures, e.g. thermoelastic problems of laminates, plates and shells with functionally graded structure by Jędrysiak [8] or Michalak [16], stability of thin functionally graded plates by Jędrysiak and Michalak [11], vibrations of thin transversally graded plates by Kaźmierczak and Jędrysiak [13], vibrations of thin functionally graded plates with the size of the microstructure of an order of the plate thickness by Jędrysiak [9, 10], Jędrysiak and Pazera [12].

2. Modelling foundations

By $Ox_1x_2x_3$ the orthogonal Cartesian coordinate system is denoted and t is the time coordinate. Let us introduce arguments: $\mathbf{x}\equiv(x_1,x_2)$, $z\equiv x_3$; and p as a loading along z-axis. The region of the undeformed plate is described by $\Omega\equiv\{(\mathbf{x},z):-d/2\leq z\leq d/2,\mathbf{x}\in\Pi\}$, with the midplane Π and the plate thickness d. The "basic cell" $\Delta\equiv[-l/2,l/2]$ is defined in the interval $\Lambda=(-L_1/2,L_1/2)$ on the x_1-axis, with l as the span of cell Δ, called *the microstructure parameter*. Parameter l is assumed to satisfy the conditions $d\sim l<<L_1$.

Let properties of the plate: a mass density μ, a rotational inertia ϑ and bending stiffnesses $d_{\alpha\beta\gamma\delta}$, be tolerance-periodic functions in x defined as:

$$\begin{aligned}
&\mu(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)dz, && \vartheta(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)z^2dz,\\
&b_{\alpha\beta\gamma\delta}(x)\equiv\int_{-d/2}^{d/2}c_{\alpha\beta\gamma\delta}(x,z)z^2dz, && d_{\alpha\beta}(x)\equiv\tfrac{5}{6}\int_{-d/2}^{d/2}c_{\alpha3\beta3}(x,z)dz.
\end{aligned}\tag{1}$$

Under assumptions of the Hencky-Bolle-type plates theory equations for deflection $u(\mathbf{x},t)$ and rotations $\phi_\alpha(\mathbf{x},t)$, $\alpha=1,2$, of functionally graded plates under consideration can be written in the following form:

$$\begin{aligned}
&\partial_\beta(b_{\alpha\beta\gamma\delta}\partial_\delta\phi_\gamma)-d_{\alpha\beta}(\partial_\beta u+\phi_\beta)-\vartheta\ddot{\phi}_\alpha=0,\\
&\partial_\alpha[d_{\alpha\beta}(\partial_\beta u+\phi_\beta)]-\mu\ddot{u}=-p.
\end{aligned}\tag{2}$$

The above equations have highly oscillating, tolerance-periodic, non-continuous coefficients being functions in x.

3. Tolerance modelling

3.1. Modelling concepts

In the tolerance averaging technique there are used some basic concepts, defined in books, cf. [18, 19, 8].

Denote $\Delta(x)\equiv x+\Delta$, $\Lambda_\Delta=\{x\in\Lambda\colon \Delta(x)\subset\Lambda\}$, as a cell at $x\in\Lambda_\Delta$. The first concept is *the averaging operator* for an arbitrary integrable function f, defined by

$$<f>(x,x_2)=\tfrac{1}{l}\int_{\Delta(x)}f(y,x_2)dy,\quad x\in\Lambda_\Delta.\tag{3}$$

For function f being tolerance-periodic in x, averaged value by (3) is a slowly-varying function in x.

Following the aforementioned books other introductory concepts are denoted as: a set of tolerance-periodic functions by $TP_\delta^\alpha(\Lambda,\Delta)$, a set of slowly-varying functions by $SV_\delta^\alpha(\Lambda,\Delta)$, a set of highly oscillating functions by $HO_\delta^\alpha(\Lambda,\Delta)$, where $\alpha\geq0$, δ is a tolerance parameter. Very important concept is *the fluctuation shape function* $g(\cdot)$, called of the 1-st kind of that function, if it is: a continuous highly oscillating function, $g\in FS_\delta^1(\Lambda,\Delta)$, with a piecewise continuous and bounded gradient $\partial^1 g$; and it depends on l as a parameter and satisfies conditions: $\partial^k g\in O(l^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\partial^k g\equiv g$, and $<\mu g>(x)\approx0$ for every $x\in\Lambda_\Delta$, $\mu>0$, $\mu\in TP_\delta^1(\Lambda,\Delta)$.

3.2. Fundamental assumptions of the tolerance modelling

Two fundamental modelling assumptions stand a base of the tolerance modelling, cf. the books edited by Woźniak et al. [18, 19] and for thin functionally graded plates [8, 9, 13].

The first assumption of them is *the micro-macro decomposition*, where the plate displacements are decomposed as:

$$\begin{aligned} &u_3(\mathbf{x},z,t)=u(\mathbf{x},t)=w(\mathbf{x},t),\\ &u_\alpha(\mathbf{x},z,t)=z[\varphi_\alpha(\mathbf{x},t)+g(x)\theta_\alpha(\mathbf{x},t)], \qquad \alpha=1,2; \end{aligned} \tag{4}$$

with new basic kinematic unknowns: *macrodeflection* $w(\cdot,x_2,t)\in SV^1_\delta(\Lambda,\Delta)$, *macrorotations* $\varphi_\alpha(\cdot,x_2,t)\in SV^1_\delta(\Lambda,\Delta)$, and *the fluctuation amplitudes* $\theta_\alpha(\cdot,x_2,t)\in SV^1_\delta(\Lambda,\Delta)$; $g(\cdot)$ as the known *fluctuation shape function*, having the form of a saw-type function of x, cf. Figure 2.

The tolerance averaging approximation is the second assumption, in which it is assumed that terms of an order of $O(\delta)$ are treated as negligibly small, cf. [18, 19, 8], e.g. for $f\in TP^1_\delta(\Lambda,\Delta)$, $g\in FS^1_\delta(\Lambda,\Delta)$, $F\in SV^a_\delta(\Lambda,\Delta), a=1,2,$ in: $<f>(x)=<\bar{f}>(x)+O(\delta)$, $<fF>(x_1)=<f>(x_1)F(x_1)+O(\delta)$, $<f\partial_\alpha(gF)>(x)=<f\partial_\alpha g>(x)F(x)+O(\delta)$.

3.3. The outline of the tolerance modelling procedure

The tolerance modelling procedure is shown in the books: for composites in [18, 19], for plates in [8]. Here, an outline of this method is shown.

In the tolerance modelling a few basic steps can be distinguished. In the first step micro-macro decomposition (4) is used. Than averaging operator (3) is applied to the resulting formula, and the tolerance averaged lagrangean $<\Lambda_g>$ is derived:

$$\begin{aligned} <\Lambda_g>=&\tfrac{1}{2}(<\mu>\dot{w}\dot{w}+<\vartheta>\dot{\varphi}_\alpha\dot{\varphi}_\beta\delta_{\alpha\beta}+\underline{<\vartheta gg>\dot{\theta}_\alpha\dot{\theta}_\beta\delta_{\alpha\beta}})-\\ &-\tfrac{1}{2}(<b_{\alpha\beta\gamma\delta}>\partial_\alpha\varphi_\beta\partial_\gamma\varphi_\delta+2<b_{\alpha\beta1\delta}\partial_1 g>\partial_\alpha\varphi_\beta\theta_\delta+<b_{1\beta1\delta}\partial_1 g\partial_1 g>\theta_\beta\theta_\delta+\underline{<b_{2\beta2\delta}gg>\partial_2\theta_\beta\partial_2\theta_\delta})-\\ &-\tfrac{1}{2}(<d_{\alpha\beta}>\partial_\alpha w\partial_\beta w+<d_{\alpha\beta}>\partial_\alpha w\varphi_\beta+<d_{\alpha\beta}>\varphi_\alpha\varphi_\beta+\underline{<d_{\alpha\beta}gg>\theta_\alpha\theta_\beta})+<p>w \end{aligned} \tag{5}$$

In the next step using the principle of stationary to (5) the Euler-Lagrange equations for $w(\cdot,x_2,t)$, $\varphi_\alpha(\cdot,x_2,t)$, $\theta_\alpha(\cdot,x_2,t)$ can be obtained:

$$\begin{aligned} &-\frac{\partial}{\partial t}\frac{\partial<\Lambda_g>}{\partial\dot{w}}-\partial_\alpha\frac{\partial<\Lambda_g>}{\partial\,\partial_\alpha w}+\frac{\partial<\Lambda_g>}{\partial w}=0,\\ &-\frac{\partial}{\partial t}\frac{\partial<\Lambda_g>}{\partial\dot{\varphi}_\alpha}-\partial_\alpha\frac{\partial<\Lambda_g>}{\partial\,\partial_\alpha\varphi_\beta}+\frac{\partial<\Lambda_g>}{\partial\varphi_\alpha}=0,\\ &-\frac{\partial}{\partial t}\frac{\partial<\Lambda_g>}{\partial\dot{\theta}_\alpha}-\partial_2\frac{\partial<\Lambda_g>}{\partial\,\partial_2\theta_\alpha}+\frac{\partial<\Lambda_g>}{\partial\theta_\alpha}=0. \end{aligned} \tag{6}$$

4. Model governing equations

Substitute the tolerance averaged lagrangean (5) to the Euler-Lagrange equations (6). Than, the system of equations for $w(\cdot,x_2,t)$, $\varphi_\alpha(\cdot,x_2,t)$, $\theta_\alpha(\cdot,x_2,t)$ is derived in the following form:

$$\begin{aligned} &\partial_\beta(<b_{\alpha\beta\gamma\delta}>\partial_\delta\varphi_\gamma)+\partial_\beta(<b_{\alpha\beta\gamma1}\partial_1 g>\theta_\gamma)-<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta)-<\vartheta>\ddot{\varphi}_\alpha=0,\\ &\partial_\alpha(<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta))-<\mu>\ddot{w}=-p,\\ &-<b_{\alpha1\gamma\delta}\partial_1 g>\partial_\delta\varphi_\gamma-(<b_{\alpha1\beta1}\partial_1 g\partial_1 g>+\underline{\underline{<d_{\alpha\beta}gg>}})\theta_\beta+\underline{\underline{<b_{\alpha2\gamma2}gg>}}\partial_{22}\theta_\gamma-\underline{<\vartheta gg>}\ddot{\theta}_\alpha=0. \end{aligned} \tag{7}$$

Equations (7) together with micro-macro decomposition (4) determine *the tolerance model of dynamics of medium thickness functionally graded plates with the microstructure size of an order of the plate thickness*. The underlined terms depend on the microstructure parameter l. Hence, the effect of the microstructure size on dynamic problems of these plates is taken into account. All coefficients of equations (7) are slowly-varying functions in $x \equiv x_1$ in contrast to equations (2), which have non-continuous, highly oscillating and tolerance-periodic coefficients. The basic unknowns w, φ_α, θ_α, $\alpha=1,2$, are slowly-varying functions in x. It can be observed that boundary conditions have to be formulated for *the macrodeflection* w and *the macrorotations* φ_α on all edges, and for *the fluctuation amplitudes* θ_α only for edges normal to the x_2-axis.

Using the asymptotic modelling procedure, shown in [19, 8, 13], or neglecting the underlined terms in equations (7), the following equations of *the asymptotic model* are derived:

$$\begin{aligned}
&\partial_\beta(<b_{\alpha\beta\gamma\delta}>\partial_\delta\varphi_\gamma)+\partial_\beta(<b_{\alpha\beta\gamma1}\partial_1 g>\theta_\gamma)-<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta)-<\vartheta>\ddot{\varphi}_\alpha=0,\\
&\partial_\alpha(<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta))-<\mu>\ddot{w}=-p,\\
&-<b_{\alpha1\gamma\delta}\partial_1 g>\partial_\delta\varphi_\gamma-<b_{\alpha1\beta1}\partial_1 g\partial_1 g>\theta_\beta=0.
\end{aligned} \tag{8}$$

These equations have smooth, slowly-varying coefficients in the contrast to equations (2). The asymptotic model equations describe vibrations of medium thickness plates under consideration on the macrolevel only.

5. Final remarks

In this contribution there are derived two systems of averaged equations of medium thickness plates with functionally graded macrostructure, which have the microstructure size of an order of the plate thickness. These equations are obtained using two modelling procedures – the tolerance modelling and the asymptotic modelling. These modelling approaches are based on the known Hencky-Bolle-type plates assumptions. Using these procedures the governing equations with non-continuous, tolerance-periodic functional coefficients of x_1 can be replaced by the systems of differential equations with slowly-varying, continuous coefficients of x_1 for each model.

The tolerance model, which governing equations take into account the effect of the microstructure size, makes it possible to analyse not only macrovibrations, but also microvibrations, related to the microstructure of the functionally graded plates.

Equations of the tolerance model have a physical sense for unknowns w, φ_α, θ_α, being slowly-varying functions in x_1. It can be treated as a certain *a posteriori* condition of physical reliability for the model.

On the other side, *the asymptotic model*, because its governing equations neglect the aforementioned effect, describes only macrovibrations of these plates under consideration.

Some applications to special dynamic problems of medium thickness functionally graded plates, which have the microstructure size of an order of the plate thickness will be presented in forthcoming papers.

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