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Reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues

Abstract

The problem of reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues is analyzed. Methods for finding the transformation matrix are presented. Considerations are illustrated by numerical examples.

Keywords: linear, electrical circuit, complex, real, eigenvalues, transformation matrix.

1. Introduction

Matrix calculus is a basic mathematical tool in linear systems theory [1, 2, 3, 7, 10]. Elementary matrix operations like transposition and similarity are considered in [5, 6, 8]. Problems of symmetric matrices are analyzed in [4, 5, 9].

It is easy to show that it is impossible to reduce every matrix to diagonal form using similarity transformation with constant matrix. In the monograph [5] the problem of diagonalization of matrices with time-dependent transformation matrix is analyzed.

In this paper, the problem of reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues is analyzed. Methods for finding the transformation matrix with time-dependent entries are presented. Considerations are illustrated by numerical examples.

The paper is organized as follows. In Section 2, the problem of reduction of linear systems with complex eigenvalues to linear systems with real eigenvalues is presented. A method of diagonalization of a matrix is given in Section 3. Analysis of electrical circuits and examples based on RLC circuit are presented in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ - set of real numbers, $\Re^{n\times m}$ - set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$, I_n - $n\times n$ identity matrix, $\mathcal{L}^{-1}$ - inverse Laplace transform.

2. The problem of reduction of linear systems with complex eigenvalues to linear systems with real eigenvalues

Consider a continuous-time linear system described by the equation

$$\dot{x}(t)=Ax(t), \tag{1}$$

where $x(t)\in\Re^n$ is the state vector and $A\in\Re^{n\times n}$.

We assume that the matrix A has at least one pair of complex eigenvalues $s_k=-\alpha_k+\mathrm{j}\beta_k$, $k=1,2,...,n$. We have to find the linear transformation

$$x(t)=L(t)z(t),\ z(t)=L^{-1}(t)x(t) \tag{2}$$

so that

$$\dot{z}(t)=Bz(t), \tag{3}$$

where the matrix $B\in\Re^{n\times n}$ has only real eigenvalues $\tilde{s}_k=-\alpha_k$, $k=1,2,...,n$ and $L(t)$ is continuously differentiable. Substituting (2) into (1) we obtain

$$\dot{x}(t)=\dot{L}(t)z(t)+L(t)\dot{z}(t)=\dot{L}(t)z(t)+L(t)Bz(t)= \\ [\dot{L}(t)+L(t)B]z(t)=Ax(t)=AL(t)z(t) \tag{4}$$

and therefore

$$\dot{L}(t)=AL(t)-L(t)B, \tag{5}$$

Theorem 1. Let A be a matrix with at least one pair of complex eigenvalues $s_k=-\alpha_k+\mathrm{j}\beta_k$, $k=1,2,...,n$. There exists the linear transformation (2) which reduces the equation (1) to the form (3) with the matrix B with only real eigenvalues $\tilde{s}_k=-\alpha_k$. The solution to the equation (5) has the form

$$L(t)=e^{At}e^{-Bt},\ L^{-1}(t)=e^{Bt}e^{-At}, \tag{6}$$

Proof. Differentiating (6) with respect to time we obtain

$$\dot{L}(t)=Ae^{At}e^{-Bt}+e^{At}(-B)e^{-Bt}=Ae^{At}e^{-Bt}-e^{At}e^{-Bt}B= \\ AL(t)-L(t)B, \tag{7}$$

which is equivalent to (5). □

3. Diagonalization of a matrix

Consider the continuous-time linear system (1) with the matrix A with at least one pair of complex eigenvalues. We have to find the linear transformation (2) so that

$$\dot{z}(t)=Dz(t), \tag{8}$$

where $D\in\Re^{n\times n}$ is a diagonal matrix. From (8) and (1) we obtain

$$\dot{x}(t)=Ax(t)=AL(t)z(t)=\dot{L}(t)z(t)+L(t)\dot{z}(t)= \\ [\dot{L}(t)+L(t)D]z(t) \tag{9}$$

and therefore

$$\dot{L}(t)=AL(t)-L(t)D. \tag{10}$$

Theorem 2. Let A be a matrix with at least one pair of complex eigenvalues $s_k=-\alpha_k+\mathrm{j}\beta_k$, $k=1,2,...,n$. There exists the linear transformation (2) which reduces the equation (1) to the form (8) with the diagonal matrix D with only real eigenvalues $\tilde{s}_k=-\alpha_k$. The solution to the equation (10) has the form

$$L(t)=e^{At}e^{-Dt}. \tag{11}$$

Proof. The proof is similar to the proof of Theorem 1.

Remark 1. If the matrix D is a scalar matrix, i.e. $D=cI_n$ for some $c\in\Re$, then the solution to the equation (10) has the form

$$L(t)=e^{(A-D)t}, \tag{12a}$$

since

$$L(t)D=DL(t) \text{ and } \dot{L}(t)=(A-D)e^{(A-D)t}. \tag{12b}$$

4. Analysis of linear electrical circuits

Consider the RLC circuit shown in Fig. 1, where $e(t)$ is the input and $u_R(t)$, $u_L(t)$, $u_C(t)$ are voltage drops on elements of the circuit.

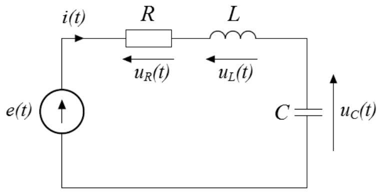

Fig. 1. RLC circuit diagram

From the Kirchhoff's laws we have

$$e(t) = RC\frac{du_C(t)}{dt} + LC\frac{d^2u_C(t)}{dt^2} + u_C(t)\,, \tag{13a}$$

$$i(t) = C\frac{du_C(t)}{dt}\,. \tag{13b}$$

Let $e(t) = u(t)$, $u_C(t) = x_1(t)$, $\dot{u}_C(t) = x_2(t)$. The state equation of the circuit has the form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\,. \tag{14}$$

With $u(t) = 0$ the state equation of the circuit has the form (1), where

$$A = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}. \tag{15}$$

The considered system has two coupled complex eigenvalues if and only if

$$R < 2\sqrt{\frac{L}{C}}\,. \tag{16}$$

We have to find the transformation matrix $L(t)$ which reduces the system (14) with complex eigenvalues to the form (3) or (8).

Example 1.

Consider the RLC electrical circuit with $u(t) = 0$ and $R = 2\ \Omega$, $L = 1\ \mathrm{H}$, $C = 0{,}2\ \mathrm{F}$. The system has two coupled complex eigenvalues since the condition (16) is satisfied. The system matrix has the form

$$A = \begin{bmatrix}0 & 1\\ -5 & -2\end{bmatrix}. \tag{17}$$

There are two eigenvalues $s_{1,2} = -1 \pm \mathrm{j}2$. We have to find the linear transformation (2) which reduces the system matrix (17) to the form

$$B = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \tag{18}$$

with two real eigenvalues $\tilde{s}_{1,2} = -1$.

We compute the transition matrix of the system (17) using the Sylvester method

$$e^{At} = Z_1e^{s_1t} + Z_2e^{s_2t}\,, \tag{19}$$

where

$$Z_k = \prod_{\substack{i=1\\ i\neq k}}^{n}\frac{A - I_ns_i}{s_k - s_i}\,, \tag{20}$$

From (17), (19) and (20) we obtain

$$e^{At} = \begin{bmatrix}0.5-\mathrm{j}0.25 & -\mathrm{j}0.25\\ \mathrm{j}1.25 & 0.5+\mathrm{j}0.25\end{bmatrix}e^{(-1+\mathrm{j}2)t} + \begin{bmatrix}0.5+\mathrm{j}0.25 & \mathrm{j}0.25\\ -\mathrm{j}1.25 & 0.5-\mathrm{j}0.25\end{bmatrix}e^{(-1-\mathrm{j}2)t} = \begin{bmatrix}e^{-t}[\cos(2t)+0.5\sin(2t)] & e^{-t}[0.5\sin(2t)]\\ e^{-t}[-2.5\sin(2t)] & e^{-t}[\cos(2t)-0.5\sin(2t)]\end{bmatrix}. \tag{21}$$

We compute the transition matrix of the system (18) for matrix $-B$ using the inverse Laplace transform method

$$e^{-Bt} = \mathcal{L}^{-1}\{[I_ns-(-B)]^{-1}\} = \mathcal{L}^{-1}\{[I_ns+B]^{-1}\}\,. \tag{22}$$

From (18) and (22) we obtain

$$e^{-Bt} = \mathcal{L}^{-1}\left\{\begin{bmatrix}\frac{1}{(s-1)} & \frac{-1}{(s-1)^2}\\ 0 & \frac{1}{(s-1)}\end{bmatrix}\right\} = \begin{bmatrix}e^t & -te^t\\ 0 & e^t\end{bmatrix}. \tag{23}$$

Using (6) we obtain

$$L(t) = e^{At}e^{-Bt} = \begin{bmatrix}\cos(2t)+0.5\sin(2t) & -t\cos(2t)+(-0.5t+0.5)\sin(2t)\\ -2.5\sin(2t) & \cos(2t)+(2.5t-0.5)\sin(2t)\end{bmatrix}. \tag{24}$$

We also compute matrices $L^{-1}(t)$ and $\dot{L}(t)$

$$L^{-1}(t) = \begin{bmatrix}\cos(2t)+(2.5t-0.5)\sin(2t) & t\cos(2t)+(0.5t-0.5)\sin(2t)\\ 2.5\sin(2t) & \cos(2t)+0.5\sin(2t)\end{bmatrix}, \tag{25}$$

$$\dot{L}(t) = \begin{bmatrix}\cos(2t)-2\sin(2t) & -t\cos(2t)+(2t-0.5)\sin(2t)\\ -5\cos(2t) & (5t-1)\cos(2t)+0.5\sin(2t)\end{bmatrix}. \tag{26}$$

Verification of the correctness of calculations will be done by transforming the equation (5) to the form

$$B = L^{-1}(t)[AL(t) - \dot{L}(t)]\,. \tag{27}$$

Substituting (17) and (24-26) into (27) we obtain

$$AL(t) = \begin{bmatrix}-2.5\sin(2t) & \cos(2t)+(2.5t-0.5)\sin(2t)\\ -5\cos(2t)+2.5\sin(2t) & (5t-2)\cos(2t)+(-2.5t-1.5)\sin(2t)\end{bmatrix}, \tag{28}$$

$$AL(t) - \dot{L}(t) = \begin{bmatrix}-\cos(2t)-0.5\sin(2t) & (t+1)\cos(2t)+0.5t\sin(2t)\\ 2.5\sin(2t) & -\cos(2t)+(-2.5t-2)\sin(2t)\end{bmatrix}. \tag{29}$$

$$L^{-1}[AL(t) - \dot{L}(t)] = \begin{bmatrix}-[\sin^2(2t)+\cos^2(2t)] & \sin^2(2t)+\cos^2(2t)\\ 0 & -[\sin^2(2t)+\cos^2(2t)]\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} = B\,. \tag{30}$$

The computed transformation matrix (24) reduces the matrix (17) to the form (18).

Example 2.

Consider the RLC electrical circuit from Example 1. The system matrix has the form (17) with two eigenvalues $s_{1,2} = -1 \pm \mathrm{j}2$. We have to find the linear transformation (2) which reduces the system matrix (17) to the diagonal (scalar) form

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad (31)$$

with two real eigenvalues $\tilde{s}_{1,2} = -1$.

Firstly we compute the difference of matrices (17) and (31)

$$A - D = \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix}. \quad (32)$$

The matrix (32) has two eigenvalues $\hat{s}_{1,2} = \pm \mathrm{j}2$.

We compute the transition matrix using the Sylvester method

$$e^{(A-D)t} = \tilde{Z}_1 e^{\hat{s}_1 t} + \tilde{Z}_2 e^{\hat{s}_2 t}, \quad (33)$$

where

$$\tilde{Z}_k = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{(A-D) - I_n \hat{s}_i}{\hat{s}_k - \hat{s}_i}, \quad (34)$$

Using (32-34) we obtain

$$e^{(A-D)t} = \begin{bmatrix} 0.5 - \mathrm{j}0.25 & -\mathrm{j}0.25 \\ \mathrm{j}1.25 & 0.5 + \mathrm{j}0.25 \end{bmatrix} e^{\mathrm{j}2t} + \begin{bmatrix} 0.5 + \mathrm{j}0.25 & \mathrm{j}0.25 \\ -\mathrm{j}1.25 & 0.5 - \mathrm{j}0.25 \end{bmatrix} e^{\mathrm{j}2t} = \begin{bmatrix} \cos(2t) + 0.5\sin(2t) & 0.5\sin(2t) \\ -2.5\sin(2t) & \cos(2t) - 0.5\sin(2t) \end{bmatrix}. \quad (35)$$

According to (12a) the transformation matrix $L(t)$ has the form (35). We also compute the matrices $L^{-1}(t)$ and $\dot{L}(t)$

$$L^{-1}(t) = \begin{bmatrix} \cos(2t) - 0.5\sin(2t) & -0.5\sin(2t) \\ 2.5\sin(2t) & \cos(2t) + 0.5\sin(2t) \end{bmatrix}, \quad (36)$$

$$\dot{L}(t) = \begin{bmatrix} \cos(2t) - 2\sin(2t) & \cos(2t) \\ -5\cos(2t) & -\cos(2t) - 2\sin(2t) \end{bmatrix}. \quad (37)$$

Verification of the correctness of calculations will be done by transforming the equation (10) to the form

$$D = L^{-1}(t)[AL(t) - \dot{L}(t)]. \quad (38)$$

Substituting (17) and (35-37) into (38) we obtain

$$AL(t) = \begin{bmatrix} -2.5\sin(2t) & \cos(2t) - 0.5\sin(2t) \\ -5\cos(2t) + 2.5\sin(2t) & -2\cos(2t) - 1.5\sin(2t) \end{bmatrix}, \quad (39)$$

$$AL(t) - \dot{L}(t) = \begin{bmatrix} -\cos(2t) - 0.5\sin(2t) & -0.5\sin(2t) \\ 2.5\sin(2t) & -\cos(2t) + 0.5\sin(2t) \end{bmatrix}. \quad (40)$$

$$L^{-1}[AL(t) - \dot{L}(t)] = \begin{bmatrix} -[\sin^2(2t) + \cos^2(2t)] & 0 \\ 0 & -[\sin^2(2t) + \cos^2(2t)] \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = D. \quad (41)$$

The computed transformation matrix (35) reduces the matrix (17) to the form (31).

5. Concluding remarks

The problem of reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues has been investigated. It has been shown that the linear transformation (2) based on a matrix with time-dependent entries can reduce the system matrix with complex eigenvalues to the form (3) or (8) with real eigenvalues. The use of transformation (2) has been presented on RLC electrical circuit. The considerations have been illustrated by numerical examples.

It is well known that every square matrix can be presented as a sum of symmetric and antisymmetric matrices. It is possible to use the linear transformation (2) to make the antisymmetric part equal to zero so the matrix transformed is a symmetric matrix. However, the symmetric part of the RLC circuit system matrix (15) is unstable.

The presented approach can be extended to fractional and time-varying linear systems.

6. References

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